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Ondřej Došlý

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ON SOME PROPERTIES OF TRIGONOMETRIC MATRICES

ONDŘEJ DOŠLÝ, Brno

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Summary. By investigation of selfadjoint linear differential systems of the second order the trigonometric matrices seem to be very useful tool. These matrices are defined as the matrix solutions of differential system

$$Y' = Q(x)Z, \quad Y(0) = 0$$
$$Z' = -Q(x)Y, \quad Z(0) = E,$$

where $Q(x)$ is a symmetric $n \times n$ matrix and E is the n-dimensional identity matrix. The trigonometric matrices satisfy some identities which are in the case $n = 1$ identical with the elementary trigonometric identities.

In the present paper there are proved another properties of trigonometric matrices which generalize the properties of the sine and cosine functions and the sets of singularities of these matrices are investigated.

Keywords: Trigonometric matrices, self-adjoint systems.
AMS classification: 34C10.

1. INTRODUCTION

Let us consider the matrix differential system

$$\text{(1.1)} \qquad \begin{aligned} Y' &= Q(x)Z \\ Z' &= -Q(x)Y, \end{aligned}$$

where $Q(x)$ is an $n \times n$ symmetric matrix of continuous functions on an interval $I = [a, \infty)$. We denote by $\{S(x), C(x)\}$ the pair of $n \times n$ matrices satisfying (1.1) and the initial condition $S(a) = 0$, $C(a) = E$, where E is the $n \times n$ identity matrix. This pair of the so called trigonometric matrices was introduced by Barrett [2] and used by himself, Reid [7], Etgen [5], [6] and others in order to study the selfadjoint matrix differential system

$$\text{(1.2)} \qquad (F(x)Y')' + G(x)Y = 0$$

by means of the generalized Prüfer transformation. Another method of investigation of systems (1.2) involving trigonometric matrices was introduced by the author in [3] and [4].

Etgen [5], [6] showed that the pair of matrices $\{S(x), C(x)\}$ has many of the properties of the sine and cosine functions. He also proved that if the matrix $Q(x)$ is positive definite, then the oscillatory behaviour of $\{S(x), C(x)\}$ is analogous to that of $\left\{\sin \int_a^x q(s)\, ds, \cos \int_a^x q(s)\, ds\right\}$, where $q(x)$ is a positive continuous function.

The aim of the present paper is to establish further oscillation properties of trigonometric matrices and to extend the well known formulae

$$(1.3) \qquad \begin{aligned} \sin(x \pm y) &= \sin x \cos y \pm \sin y \cos x\,, \\ \cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \end{aligned}$$

to the trigonometric matrices.

2. THE SUM FORMULAE

Barrett [2] has shown that the matrices $S(x)$, $C(x)$ satisfy the identities

$$(2.1) \qquad \begin{aligned} S^T(x)\, S(x) + C^T(x)\, C(x) &= E\,, \quad S^T(x)\, C(x) = C^T(x)\, S(x)\,, \\ S(x)\, S^T(x) + C(x)\, C^T(x) &= E\,, \quad S(x)\, C^T(x) = C(x)\, S^T(x)\,, \end{aligned}$$

where (^{T}) denotes the transpose of the matrix indicated. Etgen [6] introduced the following double angle formulae.

Theorem A. *Let a pair of matrices $\{Y(x), Z(x)\}$ be the solution of the matrix differential system*

$$(2.2) \qquad \begin{aligned} Y' &= \quad Q(x)\, Z + Z\, Q(x)\,, \quad Y(a) = 0 \\ Z' &= -Q(x)\, Y - Y\, Q(x)\,, \quad Z(a) = E\,, \end{aligned}$$

where $Q(x)$ is a symmetric $n \times n$ matrix. Then the following identities hold:

$$\begin{aligned} Y(x) &= 2\, S(x)\, C^T(x)\,, \quad Z(x) = C(x)\, C^T(x) - S(x)\, S^T(x)\,, \\ Y^2(x) &+ Z^2(x) = E\,, \quad Y(x)\, Z(x) = Z(x)\, Y(x)\,, \end{aligned}$$

where $\{S(x), C(x)\}$ is the solution of (1.1). *Moreover, both $Y(x)$ and $Z(x)$ are symmetric on I.*

Proof. See [6].

In the sequel, let $\{S_i(x), C_i(x)\}$, $i = 1, 2$, be solutions of the matrix systems

$$(2.3) \qquad \begin{aligned} S_i' &= \quad Q_i(x)\, C_i\,, \quad S_i(a) = 0\,, \\ C_i' &= -Q_i(x)\, S_i\,, \quad C_i(a) = E\,, \end{aligned}$$

where $Q_i(x)$, $i = 1, 2$, are symmetric matrices of continuous functions. We set

$$
(2.4) \quad \begin{aligned}
S_+(x) &= S(x, Q_1, Q_2, +) = S_1(x)\, C_2^{\mathrm{T}}(x) + C_1(x)\, S_2^{\mathrm{T}}(x)\,,\\
S_-(x) &= S(x, Q_1, Q_2, -) = S_1(x)\, C_2^{\mathrm{T}}(x) - C_1(x)\, S_2^{\mathrm{T}}(x)\,,\\
C_+(x) &= C(x, Q_1, Q_2, +) = C_1(x)\, C_2^{\mathrm{T}}(x) - S_1(x)\, S_2^{\mathrm{T}}(x)\,,\\
C_-(x) &= C(x, Q_1, Q_2, -) = C_1(x)\, C_2^{\mathrm{T}}(x) + S_1(x)\, S_2^{\mathrm{T}}(x)\,.
\end{aligned}
$$

These formulae imply

$$
\begin{aligned}
S(x, Q_1, Q_2, +) &= S^{\mathrm{T}}(x, Q_2, Q_1, +)\,,\\
C(x, Q_1, Q_2, +) &= C^{\mathrm{T}}(x, Q_2, Q_1, +)\,,\\
S(x, Q_1, Q_2, -) &= -S^{\mathrm{T}}(x, Q_2, Q_1, -)\,,\\
C(x, Q_1, Q_2, -) &= C^{\mathrm{T}}(x, Q_2, Q_1, -)\,.
\end{aligned}
$$

Hence we see that the matrix $S(x, Q_1, Q_2, -)$ is "odd" and the matrix $C(x, Q_1, Q_2, -)$ is "even".

Setting $Q_1(x) = Q_2(x) = Q(x)$ in the part "+" of the following sum formulae for trigonometric matrices, we obtain Theorem A.

Theorem 1. *The pairs of matrices* $\{S_+(x), C_+(x)\}$ *and* $\{S_-(x), C_-(x)\}$ *are solutions of the differential systems*

$$
(2.5)_+ \quad \begin{aligned}
Y' &= Q_1(x)\, Z + Z\, Q_2(x)\,, \quad Y(a) = 0\,,\\
Z' &= -Q_1(x)\, Y - Y\, Q_2(x)\,, \quad Z(a) = E\,,
\end{aligned}
$$

$$
(2.5)_- \quad \begin{aligned}
Y' &= Q_1(x)\, Z - Z\, Q_2(x)\,, \quad Y(a) = 0\,,\\
Z' &= -Q_1(x)\, Y + Y\, Q_2(x)\,, \quad Z(a) = E\,,
\end{aligned}
$$

respectively, and satisfy the following identities:

$$
(2.6)_+ \quad \begin{aligned}
S_+(x)\, S_+^{\mathrm{T}}(x) + C_+(x)\, C_+^{\mathrm{T}}(x) &= E\,, \quad S_+(x)\, C_+^{\mathrm{T}}(x) = C_+(x)\, S_+^{\mathrm{T}}(x)\,,\\
S_+^{\mathrm{T}}(x)\, S_+(x) + C_+^{\mathrm{T}}(x)\, C_+(x) &= E\,, \quad S_+^{\mathrm{T}}(x)\, C_+(x) = C_+^{\mathrm{T}}(x)\, S_+(x)\,,
\end{aligned}
$$

$$
(2.6)_- \quad \begin{aligned}
S_-(x)\, S_-^{\mathrm{T}}(x) + C_-(x)\, C_-^{\mathrm{T}}(x) &= E\,, \quad S_-(x)\, C_-^{\mathrm{T}}(x) = C_-(x)\, S_-^{\mathrm{T}}(x)\,,\\
S_-^{\mathrm{T}}(x)\, S_-(x) + C_-^{\mathrm{T}}(x)\, C_-(x) &= E\,, \quad S_-^{\mathrm{T}}(x)\, C_-(x) = C_-^{\mathrm{T}}(x)\, S_-(x)\,.
\end{aligned}
$$

Proof. All relations and identities of the theorem can be verified by a direct computation which we will not carry out here.

3. OSCILLATORY PROPERTIES

Let $n = 1$. Then $S(x) = \sin \int_a^x Q(s)\, \mathrm{d}s$, $C(x) = \cos \int_a^x Q(s)\, \mathrm{d}s$, $S(x, Q_1, Q_2, \pm) = \sin \int_a^x (Q_1(s) \pm Q_2(s))\, \mathrm{d}s$, $C(x, Q_1, Q_2, \pm) = \cos \int_a^x (Q_1(s) \pm Q_2(s))\, \mathrm{d}s$. If $Q(x) >$

> 0, the oscillatory properties of these functions follow from the well known properties of the sine and cosine functions.

To establish the oscillatory behavior of $S_+(x), S_-(x), C_+(x), C_-(x)$ in the case $n > 1$, we shall use the technique similar to that used by Atkinson in [1]. The matrices $Q_i(x)$, $i = 1, 2$, are supposed to be positive definite on I.

Let us denote $G(x) = (C_+(x) + iS_+(x))(C_+^T(x) + iS_+^T(x))$, $H(x) = (C_-(x) + iS_-(x))(C_-^T(x) + iS_-^T(x))$, $i^2 = -1$. According to (2.6) both matrices are unitary, i.e. $G^{-1}(x) = G^*(x)$, $H^{-1}(x) = H^*(x)$, where $(^*)$ denotes the conjugate transpose of the matrix indicated. Consequently, their eigenvalues lie on the unit circle in the complex plane.

Lemma 1. *The eigenvalues of $G(x)$ move along the unit circle in the positive direction as x increases.*

Proof. Let $x_0 \in I$ and let t be such that $\det(e^{it} - G(x_0)) \neq 0$. As the matrix $G(x)$ is continuous, there exists $\varepsilon > 0$ such that e^{it} is not an eigenvalue of $G(x)$ for $x \in I_0 = (x_0 - \varepsilon, x_0 + \varepsilon)$. Let $A(x) = i(e^{it} + G(x))(e^{it} - G(x))^{-1}$. By a direct calculation we can verify that $A(x)$ is hermitian, i.e. $A^*(x) = A(x)$, and $A(x) = 2ie^{it}(e^{it} - G(x))^{-1} - iE$. Calculating $A'(x)$, we have $A' = 2ie^{it}(e^{it} - G)^{-1} G'(e^{it} - G)^{-1} = -2ie^{it}(e^{it} - G)^{-1} G'G^*e^{-it}(e^{it} - G)^{*-1} = -2i(e^{it} - G)^{-1} G'G^*(e^{it} - G)^{*-1}$.

Denote $G_1 = (C_+ + iS_+)$, $G_2 = (C_+^T + iS_+^T)$. Then G_1, G_2 are unitary,

$$G_1' = i(Q_1(x) G_1 + G_1 Q_2(x)), \tag{3.1$_1$}$$

$$G_2' = i(Q_2(x) G_2 + G_2 Q_1(x)), \tag{3.1$_2$}$$

and $G'G^* = (G_1'G_2 + G_1G_2') G_2^*G_1^* = i(Q_1G_1 + G_1Q_2) G_2G_2^*G_1^* + iG_1(Q_2G_2 + G_2Q_1) G_2^*G_1^* = i(Q_1 + 2G_1Q_2G_1^* + G_1G_2Q_1G_2^*G_1^*)$. Hence $A' = 2(e^{it} - G)^{-1} \cdot (Q_1 + 2G_1Q_2G_1^* + G_1G_2Q_1G_2^*G_1^*)(e^{it} - G)^{*-1}$. As the matrices $Q_1(x)$, $Q_2(x)$ are positive definite, $A'(x)$ is also positive definite for $x \in I_0$. Now, let $a_i(x)$, $i = 1, \ldots, n$ denote the eigenvalues of $A(x)$. Since $A(x)$ is differentiable and $A'(x)$ is positive definite, the functions $a_i(x)$ are continuous and increasing, see [5, Lemma B]. Let $g(x)$ be an eigenvalue of $G(x)$ with the associated eigenvector v. Then $Av = i(e^{it} + G)(e^{it} - G)^{-1} v = i(e^{it} + g)/(e^{it} - g) v$, i.e. $i(e^{it} + g(x))/(e^{it} - g(x))$ is an eigenvalue of $A(x)$ with the associated vector v. Consider the mapping $w = i(e^{it} + z)/(e^{it} - z)$. This mapping maps the positively oriented unit circle onto the positively oriented real axis. Consequently, as w increases, z moves along the unit circle in the positive sense. Now, since the eigenvalues of $A(x)$ are increasing on I_0, the eigenvalues of $G(x)$ move positively along the unit circle if $x \in I_0$. As $x_0 \in I$ was arbitrary, the eigenvalues of $G(x)$ move positively along the unit circle on I and the proof is complete.

Example 1. Let $g_1(x) = 2 + \cos x$, $g_2(x) = 2$, $a = 0$. Then $C_-(x) = \cos(\sin x)$, $S_-(x) = \sin(\sin x)$, $H(x) = \cos(\sin x) + i \sin(\sin x)$. This example shows that the eigenvalues of the matrix $H(x)$ need not move on the unit circle in the positive direction.

Lemma 2. *Let* $g(x) = \det G(x)$, $h(x) = \det H(x)$. *Then*

$$g(x) = \exp\left\{2i \int_a^x \operatorname{tr}(Q_1(s) + Q_2(s))\, ds\right\}, \tag{3.2$_+$}$$

$$h(x) = \exp\left\{2i \int_a^x \operatorname{tr}(Q_1(s) - Q_2(s))\, ds\right\}. \tag{3.2$_-$}$$

Proof. We shall prove only $(3.2)_+$ since the proof of $(3.2)_-$ is analogous. Let $G_1(x)$, $G_2(x)$ be the same as in the proof of Lemma 1. Then $G_1(x) = T_1(x)\, T_2^{\mathrm{T}}(x)$, $G_2(x) = T_2(x)\, T_1^{\mathrm{T}}(x)$, where $T_j(x)$ are solutions of $T_j' = i\, Q_j(x)\, T_j$, $T_j(a) = E, j = 1, 2$. The Jacobi formula yields $\det T_j(x) = \exp\left\{i \int_a^x \operatorname{tr} Q_j(s)\, ds\right\}$. Since $G(x) = G_1(x) \,.\, G_2(x) = T_1(x)\, T_2^{\mathrm{T}}(x)\, T_2(x)\, T_1^{\mathrm{T}}(x)$, we have $\det G(x) = (\det T_1(x))^2 (\det T_2(x))^2 = \exp\left\{2i \int_a^x \operatorname{tr}(Q_1(s) + Q_2(s))\, ds\right\}$ which was to be proved.

Now, let $g_j(x), j = 1, \ldots, n$ be the eigenvalues of the matrix $G(x)$. Since this matrix is unitary, $g_j(x) = \exp\{i\, \alpha_j(x)\}$, where $\alpha_j(x)$ are increasing real functions. Then

$$g(x) = \det G(x) = \prod_{j=1}^n g_j(x) = \prod_{j=1}^n \exp\{i\, \alpha_j(x)\} = \exp\{i \sum_{j=1}^n \alpha_j(x)\} =$$
$$= \exp\left\{2i \int_a^x \operatorname{tr}(Q_1(s) + Q_2(s))\, ds\right\}$$

and hence

$$\sum_{j=1}^n \alpha_j(x) = 2 \int_a^x \operatorname{tr}(Q_1(s) + Q_2(s))\, ds. \tag{3.3$_+$}$$

Similarly, if $\exp\{i\, \beta_j(x)\}$, $j = 1, \ldots, n$ are the eigenvalues of $H(x)$, then

$$\sum_{j=1}^n \beta_j(x) = 2 \int_a^x \operatorname{tr}(Q_1(s) - Q_2(s))\, ds. \tag{3.3$_-$}$$

Lemma 3. *Let $x \in I$ be fixed. The number* 1 *is an eigenvalue of the matrix $G(x)$ if and only if the matrix $S_+(x)$ is singular, and the number* -1 *is an eigenvalue of $G(x)$ if and only if $C_+(x)$ is singular.*

Proof. Let $\exp\{i\, \alpha(x)\}$ be an eigenvalue of $G(x)$ with the associated eigenvector v, i.e. $Gv = (C_+ + iS_+)(C_+^{\mathrm{T}} + iS_+^{\mathrm{T}})\, v = e^{i\alpha} v$. Denote $(C_+^{\mathrm{T}} + iS_+^{\mathrm{T}})\, v = z$. Then

$$(C_+ + iS_+)\, z = e^{i\alpha}(C_+^{\mathrm{T}} + iS_+^{\mathrm{T}})^{-1}\, z,$$
$$e^{-i(\alpha/2)}(C_+ + iS_+)\, z = e^{i(\alpha/2)}(C_+ - iS_+)\, z,$$

$$\left(\cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\right)(C_+ + iS_+)\, z = \left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)(C_+ + iS_+)\, z\,,$$

$$S_+ z\cos\frac{\alpha}{2} = C_+ z\sin\frac{\alpha}{2}\,.$$

Setting $\alpha = 0 \pmod{2\pi}$ or $\alpha = \pi \pmod{2\pi}$ in the last equality, we obtain the statement.

Remark 1. By the same method we prove that the number 1 is an eigenvalue of $H(x)$ if and only if $S_-(x)$ is singular, and -1 is an eigenvalue of $H(x)$ if and only if $C_-(x)$ is singular.

Theorem 1. *Let $S_+(x)\,(C_+(x))$ be nonsingular on an interval $[c, d] \subset I$. Then $C_+(x)\,(S_+(x))$ has at most n singularities on $[c, d]$.*

Proof. If $S_+(x)$ is nonsingular on $[c, d]$, no eigenvalue of the matrix $G(x)$ can be equal to 1 for $x \in [c, d]$. Since the eigenvalues of $G(x)$ move on the unit circle in the positive direction, each of them can pass at most once through the point -1 on the unit circle, hence $C_+(x)$ has at most n singularities on $[c, d]$. If $C_+(x)$ is nonsingular on $[c, d]$ the proof is analogous.

Theorem 2. *Neither $C_+(x)$, nor $S_+(x)$ can be identically singular on any subinterval of I. Moreover, neither the singularities of $C_+(x)$, nor the singularities of $S_+(x)$ can have a finite cluster point.*

Proof. Let $C_+(x)$ be singular for $x \in [c, d] \subset I$. Then there exists an eigenvalue of $G(x)$, say $\exp\{i\,\alpha(x)\}$, for which $\exp\{i\,\alpha(x)\} = -1$, i.e. $\alpha(x) = \pi \pmod{2\pi}$ for $x \in [c, d]$. Since the eigenvalues of $G(x)$ move on the unit circle in the positive direction, the function $\alpha(x)$ is increasing on I, which is a contradiction.

Now, let $x_n \to x_0$ be a sequence of singularities of $C_+(x)$. Without loss of generality we can suppose that this sequence is increasing. Let $\exp\{i\,\alpha_j(x)\}$, $j = 1, \ldots, n$, be the eigenvalues of $G(x)$. As $G(a) = E$, we can suppose that $\alpha_j(a) = 0$. The matrix $C_+(x)$ has infinitely many singularities on $[a, x_0]$, hence at least one eigenvalue of $G(x)$, say $\exp\{i\,\alpha_{j_0}(x)\}$, must pass infinitely many times through the point -1 on the unit circle. Since the functions $\alpha_j(x)$ are positive and increasing on I, $\lim\limits_{x\to x_0} \alpha_{j_0}(x) = \infty$.

Consequently,

$$\lim_{x\to x_0} \int_a^x \mathrm{tr}(Q_1(s) + Q_2(s))\, ds = 1/2 \lim_{x\to x_0} \sum_{j=1}^{n} \alpha_j(x) > 1/2 \lim_{x\to x_0} \alpha_{j_0}(x) = \infty\,.$$

This is a contradiction, since the matrices $Q_1(x)$, $Q_2(x)$ are continuous on I. The part of the proof which concerns the distribution of singularities of $S_-(x)$ is analogous.

Definition. *A solution $Y(x)$ of (2.5) is said to be oscillatory on I if there exists* *a sequence* $x_n \to \infty$ *such that* $\det Y(x_n) = 0$. *In the opposite case this solution is said to be non-oscillatory.*

Theorem 3. *A necessary and sufficient condition for* $C_+(x)$ *and* $S_+(x)$ *to be nonoscillatory on* I *is* $\int_a^\infty \operatorname{tr} Q_j(s)\, ds < \infty$, $j = 1, 2$.

Proof. From Theorem 1 it follows that $S_+(x)$ and $C_+(x)$ are simultaneously oscillatory or nonoscillatory on I. Let these matrices be nonoscillatory on I, i.e. the eigenvalues of $G(x)$ pass only finitely many times through the points -1 and 1 on the unit circle. Hence there exists a constant K such that $\sum_{j=1}^{n} \alpha_j(x) \leqq K$ on I, where $\alpha_j(x)$, $j = 1, \ldots, n$, are arguments of the eigenvalues of $G(x)$. From $(3.3)_+$ it follows that $\int_a^x \operatorname{tr}(Q_1(s) + Q_2(s))\, ds \leqq K/2$ for $x \in I$. Since the matrices $Q_j(x)$, $j = 1, 2$, are positive definite on I, the last inequality implies $\operatorname{tr} \int_a^\infty Q_j(s)\, ds < K/2$. As all our arguments can be reversed the proof is complete.

Recall that in the preceding theorems it is essential that the eigenvalues of $G(x)$ move on the unit circle in the positive direction if x increases. Since this statement is not generally valid for the matrix $H(x)$, the analogues of Theorems 1–3 cannot be stated for matrices $C_-(x)$ and $S_-(x)$. However, in this case we have the following result.

Theorem 4. *If the matrix* $S_-(x)$ *is nonsingular on an interval* $[c, d] \subset I$, *then there exists an integer* k *such that*

$$k\pi < \frac{1}{n} \int_a^x \operatorname{tr}(Q_1(s) - Q_2(s))\, ds < (k + 1)\, \pi$$

for $x \in [c, d]$.

Proof. Let $\exp\{i\, \beta_j(x)\}$, $j = 1, \ldots, n$, be the eigenvalues of $H(x)$. From Remark 1 it follows that no eigenvalue of $H(x)$ can pass through the point 1 on the unit circle, i.e. there exists an integer k such that $2k\pi < \beta_j(x) < 2(k + 1)\, \pi$ for $x \in [c, d]$. Hence $2kn\pi < \sum_{j=1}^{n} \beta_j(x) < 2(k + 1)\, n\pi$ and from $(3.3)_-$ we obtain

$$k\pi < 1/n \int_a^x \operatorname{tr}(Q_1(s) - Q_2(s))\, ds < (k + 1)\, \pi\,.$$

Remark 2. Similarly we prove: If the matrix $C_-(x)$ is nonsingular on an interval $[c, d] \subset I$, then there exists an integer k such that

$$(2k - 1)\frac{\pi}{2} < \frac{1}{n} \int_a^x \operatorname{tr}(Q_1(s) - Q_2(s))\, ds < (2k + 1)\, \frac{\pi}{2}$$

for $x \in [c, d]$.

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Souhrn

O NĚKTERÝCH VLASTNOSTECH TRIGONOMETRICKÝCH MATIC

ONDŘEJ DOŠLÝ

Při vyšetřování samoadjungovaných lineárních diferenciálních systémů 2. řádu jsou velmi užitečným nástrojem tzv. trigonometrické matice, což jsou maticová řešení diferenciálního systému

$$Y' = Q(x)\,Z\,, \quad Y(0) = 0$$
$$Z' = -Q(x)\,Y\,, \quad Z(0) = E\,,$$

kde $Q(x)$ je symetrická matice typu $n \times n$ a E je n-dimensionální jednotková matice. Trigonometrické matice splňují některé identity, které jsou v případě $n = 1$ totožné s elementárními trigonometrickými identitami.

V předložené práci jsou dokázány další vlastnosti trigonometrických matic, které zobecňují vlastnosti funkcí $\sin x$, $\cos x$ a jsou studovány množiny bodů, ve kterých jsou tyto matice singulární.

Резюме

О НЕКОТОРЫХ СВОЙСТВАХ ТРИГОНОМЕТРИЧЕСКИХ МАТРИЦ

ONDŘEJ DOŠLÝ

При исследовании самосопряженных линейных дифференциальных систем второго порядка очень полезным средством являются так называемые тригонометрические матрицы, представляющие собой решения дефференциальной системы

$$Y' = Q(x)\,Z\,, \quad Y(0) = 0$$
$$Z' = -Q(x)\,Y\,, \quad Z(0) = E$$

где $Q(x)$ — симметрическая матрица вида (n, n) и E — единичная матрица. Тригонометричес-

кие матрицы удовлетворяют некоторым тождествам, которые в случае $n = 1$ совпадают с элементарными тригонометрическими тождествами.

В работе доказываются некоторые дальнейшие свойства тригонометрических матриц, которые обобщают свойства функций $\sin x$, $\cos x$, и исследуются множества точек, в которых эти матрицы являются особыми.

Author's address: 662 95 Brno, Janáčkovo nám. 2a (katedra matematické analýzy PF UJEP).