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ANTI-NETS I

(BASIC NOTATION AND PROPERTIES – CLOSURE CONDITIONS)

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Summary. In this paper, special parallel structures called anti-nets are introduced and investigated. By an anti-net we mean the affine reduction of the dual structure of a projective net with all singular points lying on a line. As the coordinate algebra of the anti-net the admissible algebra is used.

Suitable closure conditions — minor Desargues, Reidemeister and diagonal — for the anti-nets are formulated, and some properties of the coordinate algebra of any anti-net satisfying the closure conditions are derived.

In the final part, seven examples of anti-nets (finite and infinite) are given.

Keywords: Parallel structure, anti-net, coordinate algebra, closure condition, minor Desargues condition, diagonal condition, Reidemeister condition.

Classification AMS: 51A15.

INTRODUCTION

The present paper deals with a special parallel structure called an anti-net. It may be obtained by a suitable affine reduction of a projective parallel structure in which all principal lines are going through one point. Just so, a projective parallel structure is a dual one to a projective net in which all singular points lie on the singular line. The axioms of this net are presented in [4] or [6]. The fundamental properties of nets may be found, e.g., in [7].

The projective parallel structure is defined in [1]. It is also shown there that if any such projective structure is given then putting away one principal line **h** as well as its points we get an incidence structure $(\mathbf{P}, \mathscr{L})$ satisfying

a) each two different points from $\mathbf{P}$ are joinable by (exactly one) line from $\mathscr{L}$;

b) there exists a decomposition of the set $\mathscr{L}$ of lines into classes of "parallel" lines so that for each point $\mathsf{B} \in \mathbf{P}$ and for each line $\mathbf{g} \in \mathscr{L}$ there exists exactly one line $\mathbf{g}' \in \mathscr{L}$ parallel to $\mathbf{g}$ through the point B;

c) there exist three non-collinear points in $\mathbf{P}$;

d) every line $\mathbf{g} \in \mathscr{L}$ contains at least one point.

A structure $(\mathbf{P}, \mathscr{L})$ with the above properties is called an affine parallel structure.

Conversely, let an affine parallel structure $(\mathbf{P}, \mathscr{L})$ be given. If we define improper points of lines in a suitable way and the improper line, and if we add these points to the set $\mathbf{P}$ and this line to the set $\mathscr{L}$ we obtain an extension of the affine parallel structure satisfying the axioms of a projective parallel structure.

An anti-net is an affine parallel structure with one pencil of preferred lines (so called-principal lines), each of which intersects any of the remaining lines (so-called ordinary lines) in just one point. The anti-net is not a dual structure to the affine net (for affine nets, see e.g. [2]) which may be obtained from a projective net by removing its singular line and all of its singular points.

The definition of an anti-net $\mathscr{A}$, its order and degree, and some of its fundamental properties are introduced in Section 1 of this work. In a natural way the improper point of a line, the improper line of $\mathscr{A}$ and an extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ are defined.

Section 2 deals with the coordinate algebra of an anti-net. To that purpose it appeared to be suitable to use the admissible algebra $\mathfrak{A} = (S, 0, (\sigma_\iota)_{\iota \in J}, (+_\iota)_{\iota \in J})$ (for an admissible algebra see [5]).

In the next section the closure conditions in anti-nets are studied. The consequence of fulfilling these conditions in the given anti-net $\mathscr{A}$ is the validity of certain properties of the coordinate algebra $\mathfrak{A}$ of this anti-net. Explicitly, we investigate: the minor Desargues condition (Sec. 3), the diagonal condition (Sec. 4), and the Reidemeister condition (Sec. 5). For the sake of simplicity of formulation and notation, these conditions are defined for the extended anti-net. The properties of the coordinate algebra $\mathfrak{A}$ fulfilling the minor Desargues condition of a certain type are expressed in Theorem 3.1 – 3.4. To fulfil the diagonal condition is equivalent to the fact that addition "$\overset{\circ}{+}$" (defined by (7)) is a group operation (Theorem 4.1). Similarly, to fulfil the Reidemeister condition is equivalent to the property that the composition of permutations σ_ι of the coordinate algebra $\mathfrak{A}$ is a loop operation (Theorem 5.1). Analogously we define also the addition "$+_\circ$" and the multiplication "$\circ$" in the index set J' of the coordinate algebra $\mathfrak{A}$. Theorem 4.2 provides sufficient conditions for the additions "$+$", "$\overset{\circ}{+}$", "$+_\circ$" in $\mathfrak{A}$ to be commutative group operations. Moreover, sufficient conditions for the index set J' of the algebra $\mathfrak{A}$ (with respect to the operations "$+_\circ$", "$\circ$" to be a skewfield (Theorem 5.3) and for the support S of the algebra $\mathfrak{A}$ (together with the operation "$+$") to be a vector space over this skewfield (Theorem 5.4) are established.

In Section 6 seven examples of anti-nets (finite as well as infinite) are presented. In Examples 3 – 7, also the coordinate algebras of the anti-nets constructed are determined. Finally, from the properties of these algebras, conclusions about the validity of the particular closure conditions in these anti-nets are drawn.

The present paper is immediately followed by its continuation, the paper Anti-nets II. There we study some automorphisms of the anti-net $\mathscr{A}$ and describe the groups of certain collineations of this anti-net, as the group of translations, homotheties, perspective affinities and all collineations.

1. DEFINITION AND SOME BASIC PROPERTIES OF AN ANTI-NET

Definition 1.1. Let an ordered triple $\mathscr{A} = (\mathsf{P}, \mathscr{L}, (\mathbf{h}_\iota)_{\iota\in I})$ where P is a nonempty set, $\mathscr{L}$ is a set of certain subsets of P, I is a nonempty index set and $\mathbf{h}_\iota (\iota \in I)$ is element of $\mathscr{L}$ for all $\iota \in I$, be given. We shall call the elements of P points, the elements of $\mathscr{L}$ lines. The lines $\mathbf{h}_\iota \in \mathscr{L}$, $\iota \in I$ are the so-called principal lines, the remaining lines of $\mathscr{L}$ are the ordinary ones. The ordered triple $\mathscr{A} = (\mathsf{P}, \mathscr{L}, (\mathbf{h}_\iota)_{\iota\in I})$ will be called an anti-net if the following conditions are fulfilled:

(a1) for any two distinct points $\mathsf{A}, \mathsf{B} \in \mathsf{P}$ there exists exactly one line $\mathbf{g} \in \mathscr{L}$ containing both of them;

(a2) every principal line $\mathbf{h}_\iota \in \mathscr{L}$ intersects each ordinary line $\mathbf{g} \in \mathscr{L}$ in exactly one point,
for any two distinct ordinary lines there exists at most one common point;

(a3) there exists a decomposition of $\mathscr{L}$ into classes of the so-called parallel lines with the following property: for any point $\mathsf{B} \in \mathsf{P}$ and for any line $\mathbf{g} \in \mathscr{L}$ there exists axactly one line $\mathbf{p} \in \mathscr{L}$ parallel to $\mathbf{g}$ and containing B;

(a4) any two principal lines are parallel,
if $\mathbf{p}$ is an ordinary line and $\mathbf{h}_\iota$ is a principal line then $\mathbf{p}, \mathbf{h}_\iota$ are not parallel;

(a5) there exist three points in P which no line contains.

We shall call the line $\mathbf{g}$ from (a1) the join line of points A, B and denote it by $\mathbf{g} = \mathsf{AB}$. The common point B of two different lines $\mathbf{a}, \mathbf{b} \in \mathscr{L}$ (if it exists) will be called the point of intersection of $\mathbf{a}, \mathbf{b}$ and denoted by $\mathsf{B} = \mathbf{a} \cap \mathbf{b}$.

Let points $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n \in \mathsf{P}$ be given. The points $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ will be said to be collinear if there exists a line $\mathbf{g} \in \mathscr{L}$ such that $\mathsf{A}_i \in \mathbf{g}$, $i = 1, 2, \ldots, n$. In this case we shall write "$\mathsf{A}_1\mathsf{A}_2 \ldots \mathsf{A}_n$". Analogously, let lines $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathscr{L}$ be given. The lines $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ will be said to be concurrent if there exists a point $\mathsf{B} \in \mathsf{P}$ such that $\mathsf{B} \in \mathbf{a}_i$, $i = 1, 2, \ldots, n$. In this case we shall write "$\mathbf{a}_1\mathbf{a}_2 \ldots \mathbf{a}_n$". Two parallel (non-parallel) lines $\mathbf{a}, \mathbf{b} \in \mathscr{L}$ will be denoted by $\mathbf{a} \parallel \mathbf{b}$ ($\mathbf{a} \nparallel \mathbf{b}$, respectively). The fact that a point B is contained in a line $\mathbf{g}$ will be also expressed in the form "the point B lies on the line $\mathbf{g}$" or "the line $\mathbf{g}$ goes through the point B". The line $\mathbf{p}$ from (a3) going through the point $\mathsf{B} \in \mathsf{P}$ and being parallel to the line $\mathbf{g} \in \mathscr{L}$ will be denoted by $\mathbf{p} = [\mathsf{B}, \mathbf{g}]$.

The following lemma is an immediate consequence of the definition of the anti-net $\mathscr{A} = (\mathsf{P}, \mathscr{L}, (\mathbf{h}_\iota)_{\iota\in I})$:

Lemma 1.1. *Let an anti-net $\mathscr{A}$ be given. Then the following statements are true*:

a) *for any point* $\mathsf{B} \in \mathsf{P}$ *there exists exactly one principal line* $\mathbf{h}_\iota$ *with* $\mathsf{B} \in \mathbf{h}_\iota$;

b) *for any principal line* $\mathbf{h}_\iota$, $\iota \in I$ *there exists at least one point* $\mathsf{B} \in \mathsf{P}$ with $\mathsf{B} \notin \mathbf{h}_\iota$;

c) *there are at least two distinct principal lines, i.e.* $\#I \geqq 2$;

d) *if* $\mathbf{a}, \mathbf{b} \in \mathscr{L}$ *are two ordinary lines then*

$$\#\{\mathsf{X} \mid \mathsf{X} \in \mathbf{a}\} = \#\{\mathsf{Y} \mid \mathsf{Y} \in \mathbf{b}\} = \#I;$$

e) *if* $\mathbf{h}_\iota, \mathbf{h}_\varkappa \in \mathscr{L}$, $\iota, \varkappa \in \boldsymbol{I}$ *are two principal lines then*

$$\#\{\mathsf{X} \mid \mathsf{X} \in \mathbf{h}_\iota\} = \#\{\mathsf{Y} \mid \mathsf{Y} \in \mathbf{h}_\varkappa\} \geqq 2 .$$

Proof. The properties a), b), c) follow directly from the axioms (a1)–(a5) of an anti-net $\mathscr{A}$. If we denote $\mathsf{X}_\iota = \mathbf{a} \cap \mathbf{h}_\iota$, $\mathsf{Y}_\iota = \mathbf{b} \cap \mathbf{h}_\iota$ for any index $\iota \in \boldsymbol{I}$ then the bijectivity of the mapping $\mathsf{X}_\iota \mapsto \mathsf{Y}_\iota$ implies d). Let $\mathsf{B} \in \mathbf{P}$ be the point not lying on any of the two principal lines $\mathbf{h}_\iota, \mathbf{h}_\varkappa$, $\iota \neq \varkappa$; $\iota, \varkappa \in \boldsymbol{I}$. For any point $\mathsf{X} \in \mathbf{h}_\iota$ let us put $\mathsf{Y} = \mathsf{BX} \cap \mathbf{h}_\varkappa$. Then e) follows from the bijectivity of the mapping $\mathsf{X} \mapsto \mathsf{Y}$ and from the axiom (a5). ∎

Definition 1.2. a) The cardinality $k = \# \boldsymbol{I}$ will be called the *degree* of the anti-net $\mathscr{A}$.
b) The cardinality $n = \# \mathbf{h}_\iota$, $\iota \in \boldsymbol{I}$, will be called the *order* of the anti-net $\mathscr{A}$.

Lemma 1.2. *For each anti-net* $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbf{h}_\iota)_{\iota \in \boldsymbol{I}})$ *of order n and degree k the following assertions hold*:

f) *for any point* $\mathsf{B} \in \mathbf{P}$ *there exist exactly n distinct ordinary lines from* $\mathscr{L}$ *containing* B;

g) $n \geqq k$.

Proof. The validity of f) follows from the definition of an anti-net $\mathscr{A}$, its order n and degree k, and from Lemma 1.1. The same is true for the validity of g) for k, n finite. Let us suppose that k, n are infinite cardinal numbers. In this case we shall prove part g) as follows: let $\mathbf{g} \in \mathscr{L}$ be an ordinary line and let $\mathsf{B} \in \mathbf{P}$ be a point not lying on $\mathbf{g}$. Further, let $\mathbf{h}_\iota (\iota \in \boldsymbol{I})$ be the principal line not containing the point B (the existence of the lines $\mathbf{g}$ and $\mathbf{h}_\iota \in \mathscr{L}$ as well as of the point $\mathsf{B} \in \mathbf{P}$, $\mathsf{B} \notin \mathbf{g}$, $\mathsf{B} \notin \mathbf{h}_\iota$ follows from the axioms (a1)–(a5) and Lemma 1.1). Let us put $\mathbf{p} = [\mathsf{B}, \mathbf{g}]$ and let us denote by $\mathbf{h}_\varkappa$ the principal line through the point B. Finally, let us put $\mathsf{X}_0 = \mathbf{h}_\varkappa \cap \mathbf{g}$, $Y_0 = \mathbf{p} \cap \mathbf{h}_\iota$. Then we get

$$\{\mathsf{Y}_0\} \cup \{\mathsf{Y} \mid \mathsf{Y} = \mathbf{h}_\iota \cap \mathsf{BX},\ \mathsf{X} \in \mathbf{g} \setminus \{\mathsf{X}_0\}\} \subseteq \{\mathsf{Z} \mid \mathsf{Z} \in \mathbf{h}_\iota\} ,$$

hence

$$1 + \#\{\mathsf{Y} \mid \mathsf{Y} = \mathbf{h}_\iota \cap \mathsf{BX},\ \mathsf{X} \in \mathbf{g} \setminus \{\mathsf{X}_0\}\} \leqq \#\{\mathsf{Z} \mid \mathsf{Z} \in \mathbf{h}_\iota\}$$

and consequently,

$$1 + k = k \leqq n . \quad \blacksquare$$

Definition 1.3. The ani-net $\mathscr{A}$ whose set of points is finite will be called a *finite anti-net*. In the opposite case the anti-net $\mathscr{A}$ will be said to be *infinite*.

The above considerations immediately yield

Lemma 1.3. *Any finite anti-net* $\mathscr{A}$ *of order n and degree k contains*

a) *exactly* $k \,.\, n$ *distinct points*;

b) *exactly k distinct principal lines*;

c) *exactly* n^2 *distinct ordinary lines.*

Definition 1.4. Let $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbf{h}_\iota)_{\iota\in I})$ be an anti-net. By an improper point of $\mathbf{g} \in \mathscr{L}$ we shall mean the set $\langle \mathbf{g} \rangle = \{\mathbf{p} \in \mathscr{L} \mid \mathbf{p} \parallel \mathbf{g}\}$. The set of the improper points of all lines $\mathbf{g} \in \mathscr{L}$ will be called the *improper line* and denoted by $\mathbf{h}'_\infty$.

From this definition we infer

a) $\langle \mathbf{g} \rangle = \langle \mathbf{k} \rangle$ iff $\mathbf{g} \parallel \mathbf{k}$,

b) the improper line $\mathbf{h}'_\infty$ intersects any line $\mathbf{g}' = \mathbf{g} \cup \{\langle \mathbf{g} \rangle\}$, $\mathbf{g} \in \mathscr{L}$ in exactly one point – namely, in the improper point of $\mathbf{g}$ – and so it is a principal line.

Now, given an anti-net $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbf{h}_\iota)_{\iota\in I})$, let us add to the set $\mathbf{P}$ the improper points of all lines of $\mathscr{A}$, further, to any line $\mathbf{g} \in \mathscr{L}$ its improper point $\langle \mathbf{g} \rangle$ and finally, to the set $\mathscr{L}' = \{\mathbf{g}' \mid \mathbf{g}' = \mathbf{g} \cup \{\langle \mathbf{g} \rangle\}, \mathbf{g} \in \mathscr{L}\}$ the improper line $\mathbf{h}'_\infty$. In this way we obtain an incidence structure

$$\mathscr{A}' = (\mathbf{P} \cup \mathbf{h}'_\infty, \mathscr{L}' \cup \{\mathbf{h}'_\infty\}, (\mathbf{h}'_\iota)_{\iota\in I\cup\{\infty\}}) .$$

Let us remark that if we denote by L the improper point $\langle \mathbf{h}_\iota \rangle$, $\iota \in I$, then $\mathbf{h}'_\iota = \mathbf{h}_\iota \cup \cup \{\mathsf{L}\}$ for any index $\iota \in I$.

We may verify without difficulties that the structure $\mathscr{A}'$ has the following properties:

(a1′) any two distinct points $\mathsf{A}, \mathsf{B} \in \mathbf{P} \cup \mathbf{h}'_\infty$ are joinable by a unique line from $\mathscr{L}' \cup \{\mathbf{h}'_\infty\}$;

(a2′) any principal line $\mathbf{h}'_\iota$, $\iota \in I \cup \{\infty\} = I'$ intersects any ordinary line $\mathbf{g}' \in \mathscr{L}' \setminus \setminus \{\mathbf{h}'_\iota\}_{\iota\in I'}$ in exactly one point, any two distinct ordinary lines from $\mathscr{L}' \setminus \{\mathbf{h}'_\iota\}_{\iota\in I'}$ have at most one common point;

(a3′) all principal lines $\mathbf{h}'_\iota$, $\iota \in I \cup \{\infty\}$ contain the point $\mathsf{L} \in \mathbf{P} \cup \mathbf{h}'_\infty$, no ordinary line goes through the point L;

(a4′) there exists at least one principal line $\mathbf{h}'_\iota$, $\iota \in I \cup \{\infty\}$ and a triple of non-collinear points none of which lies on the line $\mathbf{h}'_\iota$.

Definition 1.5. The structure $\mathscr{A}' = (\mathbf{P} \cup \mathbf{h}'_\infty, \mathscr{L}' \cup \{\mathbf{h}'_\infty\}, (\mathbf{h}'_\iota)_{\iota\in I\cup\{\infty\}})$ with the properties (a1′)–(a4′) naturally determibed by $\mathscr{A}$ will be called an *extension of the anti-net* $\mathscr{A}$ (with respect to the improper principal line $\mathbf{h}'_\infty$). By the order and the degree of the extension $\mathscr{A}'$ we shall mean the order and degree, respectively, of the anti-net $\mathscr{A}$.

Remark 1.1. An extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ with respect to the improper principal line $\mathbf{h}'_\infty$ is the dual structure to the projective net $\mathscr{N}$ with singular points on the same line-singular line. In this duality the improper point L of the structure $\mathscr{A}'$ corresponds to the singular line of the net $\mathscr{N}$ and the principal lines $\mathbf{h}'_\iota$, $\iota \in I \cup \{\infty\}$ of the structure $\mathscr{A}'$ correspond to the singular points of the net $\mathscr{N}$.

Remark 1.2. From now on, we shall denote the ordinary lines $\mathbf{g}' \in \mathscr{L}' \setminus \{\mathbf{h}'_\iota\}$ (with the improper point $\langle \mathbf{g} \rangle$) of the extension $\mathscr{A}'$ only by $\mathbf{g}$, the principal line $\mathbf{h}'_\iota$, $\iota \in I \cup \{\infty\}$ (with the improper point L) of $\mathscr{A}'$ by $\mathbf{h}_\iota$ (same as the ordinary and principal lines of the anti-net $\mathscr{A}$). Also, the improper principal line of $\mathscr{A}'$ will be denoted only by $\mathbf{h}_\infty$. The extension $\mathscr{A}'$ of an anti-net $\mathscr{A} = (\mathbf{P}, \mathscr{L}, \mathbf{h}_\iota)_{\iota \in I})$ will be denoted always by $\mathscr{A}' = (\mathbf{P} \cup \mathbf{h}_\infty, \mathscr{L} \cup \{\mathbf{h}_\infty\}, (\mathbf{h}_\iota)_{\iota \in I \cup \{\infty\}})$.

2. COORDINATE ALGEBRA OF AN ANTI-NET

Definition 2.1. An algebra $\mathfrak{A} = (S, 0, (\sigma_\iota)_{\iota \in J}, (+_\iota)_{\iota \in J})$ where S is the support of $\mathfrak{A}$ with a preferred element 0 and J is an index set with $\#S \geqq J + 1 \geqq 2$ will be called an *admissible algebra* if the following conditions are fulfilled:

(α**1**) for any index $\iota \in J$, σ_ι is a permutation of the set S with $(0)\,\sigma_\iota = 0$;

(α**2**) there exists a significant index $\vartheta \in J$ so that $\sigma_\vartheta = \mathrm{id}_S$;

(α**3**) for any index $\iota \in J$, $(S, +_\iota)$ is a loop with the neutral element 0;

(α**4**) for any two distinct indices $\xi, \eta \in J$ and any two elements $b, c \in S$ there exists exactly one element $a \in S$ so that

$$(a)\,\sigma_\xi +_\xi b = (a)\,\sigma_\eta +_\eta c\,.$$

Definition 2.2. Let $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbf{h}_\iota)_{\iota \in I})$ be an anti-net. By a *frame* of the anti-net $\mathscr{A}$ we shall mean any triple $(\mathbf{o}; \beta, \gamma)$ where $\mathbf{o} \in \mathscr{L} \setminus \{\mathbf{h}_\iota\}_{\iota \in I}$ is an ordinary line and β, γ are mutually distinct indices from I.

Construction 2.1. Let $(\mathbf{o}; \beta, \gamma)$ be a frame of an anti-net $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbf{h}_\iota)_{\iota \in I})$. We shall construct the admissible algebra $\mathfrak{A} = (S, 0, (\sigma_\iota)_{\iota \in J}, (+_\iota)_{\iota \in J})$ with respect to the frame $(\mathbf{o}; \beta, \gamma)$. Let us denote

$$\begin{aligned} & S := \{x \mid x \in \mathbf{h}_\beta\}\,, \\ & J := I \setminus \{\beta\}\,, \\ (1) \quad & 0 := \mathbf{o} \cap \mathbf{h}_\beta\,, \\ & \sigma_\iota \colon S \to S\,, \quad x \mapsto (x)\,\sigma_\iota = [\mathbf{o} \cap \mathbf{h}_\iota, x(\mathbf{o} \cap \mathbf{h}_\gamma)] \cap \mathbf{h}_\beta\,, \quad \text{for all} \quad \iota \in J\,, \\ & +_\iota \colon (x)\,\sigma_\iota +_\iota y := [[y, \mathbf{o}] \cap \mathbf{h}_\iota, x(\mathbf{o} \cap \mathbf{h}_\gamma)] \cap \mathbf{h}_\beta\,, \quad \text{for all} \quad x, y \in S\,, \quad \iota \in J\,. \end{aligned}$$

We shall prove that the algebra $\mathfrak{A} = (S, 0, (\sigma_\iota)_{\iota \in J}, (+_\iota)_{\iota \in J})$ just described is admissible:

We have $\#S = \#\mathbf{h}_\beta \geqq 2$ according to e) of Lemma 1.1. The existence of the element $0 \in S$ follows from the axiom (a2). With respect to d) of Lemma 1.1 we get $\#J = \#I - 1 \geqq 1$. Finally, according to g) of Lemma 1.2 we obtain $\#S \geqq \geqq \#J + 1$.

The mapping $\sigma_\iota \colon S \to S$ is a bijective mapping — the permutation of the set S — for each index $\iota \in J$. The statement follows from the axioms (a2), (a1), (a3); moreover, $(0)\,\sigma_\iota = 0$ for any $\iota \in J$. Hence (α**1**). Because of $\sigma_\gamma = \mathrm{id}_S$ the index γ is the preferred index of J, therefore (α**2**) is fulfilled.

In virtue of the axioms (α1), (α2), (α3), if two of elements $a, b, c \in S$ are given the remaining one is uniquely determined by the equation $(a)\,\sigma_\iota +_\iota b = c$, $\iota \in J$, hence

$+_\iota$ is a quasigroup operation. Moreover, for any $a \in S$ we have $0 +_\iota a = a +_\iota 0 = a$, hence $(S, +_\iota)$ is a loop with the neutral element 0. Thus (α3) is true.

Let $\xi, \eta \in J$, $\xi \neq \eta$ be arbitrary different indices and let $b, c \in S$ be two elements of S. If we put

$$a = \mathbf{h}_\beta \cap [\mathbf{o} \cap \mathbf{h}_\gamma, ([b, \mathbf{o}] \cap \mathbf{h}_\xi)([c, \mathbf{o}] \cap \mathbf{h}_\eta)]$$

then we have

$$(a)\,\sigma_\xi +_\xi b = (a)\,\sigma_\eta +_\eta c\,,$$

hence (α4) is true. ■

Definition 2.3. The admissible algebra $\mathfrak{A} = (S, 0, (\sigma_\iota)_{\iota \in J}, (+_\iota)_{\iota \in J})$ constructed in 2.1 will be called the *coordinate algebra* of an anti-net $\mathscr{A}$ with respect to its frame $(\mathbf{o}; \beta, \gamma)$.

The mapping $S \times S \to \mathscr{L} \setminus \{\mathbf{h}_\iota\}_{\iota \in I}$, $\{k, q\} \mapsto [q, k(\mathbf{o} \cap \mathbf{h}_\gamma)]$ will be called a *coordinate mapping* (with respect to the frame $(\mathbf{o}; \beta, \gamma)$ of $\mathscr{A}$).

Let $\mathsf{B} \in \mathbf{P}$ be an arbitrary point of an anti-net $\mathscr{A}$. If $\mathbf{h}_\xi = [\mathsf{B}, \mathbf{h}_\beta]$, $\xi \in I$ and $y = \mathbf{h}_\beta \cap [\mathsf{B}, \mathbf{o}]$, $y \in S$ then we put $\mathsf{B} = (\xi, y)$.

Lemma 2.1. *The point* $\mathsf{B} = (\xi, y)$, $\xi \in J$, $y \in S$ *is a point of the line* $\mathbf{g} = \{k, q\}$, $k, q \in S$ *if and only if* $(k)\,\sigma_\xi +_\xi y = q$.

Proof. The validity of Lemma 2.1 follows from the definition of the permutations σ_ι and operations $+_\iota$ in Construction 2.1. The following proposition is trivial: The point $\mathsf{Q} = (\beta, y)$ lies on the line $\mathbf{g} = \{k, q\}$, $k, q \in S$ if and only if $y = q$. ■

Definition 2.3 immediately implies

Lemma 2.2. *If* $\mathbf{g} = \{k, q\}$, $\mathbf{g}' = \{k', q'\}$, $k, q, k', q' \in S$ *are two ordinary lines of the anti-net* $\mathscr{A}$ *then* $\mathbf{g} \parallel \mathbf{g}' \Leftrightarrow k = k'$.

Remark 2.1. Let $\mathfrak{A}$ be the coordinate algebra of the anti-net $\mathscr{A}$ with respect to its frame $(\mathbf{o}, \beta, \gamma)$. Then we shall take the algebra $\mathfrak{A}$ also as the coordinate algebra of an extension $\mathscr{A}'$ of $\mathscr{A}$ (with the improper principal line $\mathbf{h}_\infty$) with respect to the same frame $(\mathbf{o}; \beta, \gamma)$. In this case we shall denote it by $\mathfrak{A}$. The improper point of the line $\mathbf{g} = \{k, q)$ will have the coordinate $\langle \mathbf{g} \rangle = (k)$. The common improper point L of all principal lines will be denoted by (∞) where $\infty \notin S$. Then the improper principal line satisfies $\mathbf{h}_\infty = \{(k) \mid k \in S\} \cup \{(\infty)\}$.

Construction 2.2. Let an admissible algebra $\mathfrak{A} = (S, 0, (\sigma_\iota)_{\iota \in J}, (+_\iota)_{\iota \in J})$ with the preferred element $\vartheta \in J$ be given. Further, let us suppose $\# S \geqq \# J + 1 \geqq 2$. We construct the anti-net $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbf{h}_\iota)_{\iota \in I})$ over $\mathfrak{A}$ in the following way. For this purpose let us denote

$$
(2)\quad
\begin{aligned}
&\boldsymbol{I} := \boldsymbol{J} \cup \{\omega\} \text{ where } \omega \notin \boldsymbol{J} \text{ is an arbitrary element},\\
&\mathbf{P} := \{(\iota, x) \mid \iota \in \boldsymbol{I},\ x \in S\},\\
&\mathbf{h}_\iota := \{(\iota, x) \mid x \in S\} \quad \text{for all} \quad \iota \in \boldsymbol{I},\\
&\{k, q\} := \{(\xi, y) \mid \xi \in \boldsymbol{J},\ y \in S,\ (k)\,\sigma_\xi +_\xi y = q\} \cup \{(\omega, q)\},\\
&\mathscr{L} := \{\mathbf{h}_\iota \mid \iota \in \boldsymbol{I}\} \cup \{\{k, q\} \mid k, q \in S\},\\
&\mathbf{h}_\iota \parallel \mathbf{h}_\varkappa \quad \text{for all} \quad \iota, \varkappa \in \boldsymbol{I},\\
&\text{for all} \quad \mathbf{g}, \mathbf{g}' \in \mathscr{L}, \quad \mathbf{g} = \{k, q\}, \quad \mathbf{g}' = \{k', q'\}, \quad k, q, k', q' \in S,\\
&\mathbf{g} \parallel \mathbf{g}' \Leftrightarrow k = k'.
\end{aligned}
$$

Using the axioms (α1)–(α4) of an admissible algebra we can verify without trouble that the structure $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbf{h}_\iota)_{\iota \in \boldsymbol{I}})$ just constructed fulfils the axioms (a1)–(a5). Thus, it is an anti-net and $\mathfrak{A}$ is its coordinate algebra with respect to the frame $(\mathbf{o}; \omega, \vartheta)$ where $\mathbf{o} = \{0,0\}$. ■

3. MINOR DESARGUES CONDITION IN AN ANTI-NET

As has been said in Introduction the formulation and notation of closure conditions will be more advantageous if instead of an anti-net $\mathscr{A}$ we consider its extension $\mathscr{A}'$. The re-writing of them for the original anti-net $\mathscr{A}$ consists in replacing any statement "$\mathbf{abh}_\infty$" by the statement $\mathbf{a} \parallel \mathbf{b}$.

Now let the extension $\mathscr{A}' = (\mathbf{P} \cup \mathbf{h}_\infty,\ \mathscr{L} \cup \{\mathbf{h}_\infty\},\ (\mathbf{h}_\iota)_{\iota \in \boldsymbol{I} \cup \{\infty\}})$ of an anti-net $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbf{h}_\iota)_{\iota \in \boldsymbol{I}})$ be given. In what follows we shall always suppose that the following condition is fulfilled:

(**rs**): The extension $\mathscr{A}'$ is endowed with a frame $(\mathbf{o}; \beta, \gamma)$, $\beta, \gamma \in \boldsymbol{I}$, together with a coordinate algebra $\mathfrak{A} = (S, 0, (\sigma_\iota)_{\iota \in \boldsymbol{J}}, (+_\iota)_{\iota \in \boldsymbol{J}})$, $\boldsymbol{J} = \boldsymbol{I} \setminus \{\beta\}$ with respect to this frame.

Definition 3.1. Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ of an order $\geqq 3$ be given. Let $\varkappa, \lambda, \mu, \varrho$ be mutually distinct indices from $\boldsymbol{I} \cup \{\infty\}$. We shall say that the *minor Desargues condition* (abbr. *MDC*) *of the type* $(\varkappa, \lambda, \mu, \varrho)$ is fulfilled in $\mathscr{A}'$ if for any $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{a}', \mathbf{b}', \mathbf{c}' \in \mathscr{L} \setminus \{\mathbf{h}_\iota\}_{\iota \in \boldsymbol{I} \cup \{\infty\}}$ the following implication is true:

$$
\begin{gathered}
(\text{“}\mathbf{aa'h}_\varrho\text{”} \wedge \text{“}\mathbf{bb'h}_\varrho\text{”} \wedge \text{“}\mathbf{cc'h}_\varrho\text{”} \wedge \text{“}\mathbf{abh}_\mu\text{”} \wedge \text{“}\mathbf{a'b'h}_\mu\text{”} \wedge \text{“}\mathbf{ach}_\lambda\text{”} \wedge\\
\wedge \text{“}\mathbf{a'c'h}_\lambda\text{”} \wedge \text{“}\mathbf{bch}_\varkappa\text{”}) \Rightarrow \text{“}\mathbf{b'c'h}_\varkappa\text{”}.
\end{gathered}
$$

Let $\varrho \in \boldsymbol{I} \cup \{\infty\}$ be a fixed index. We shall say *the MDC of the type* (ϱ) is fulfilled in $\mathscr{A}'$ if the *MDC* of the type $(\varkappa, \lambda, \mu, \varrho)$ is fulfilled in $\mathscr{A}'$ for any triple of mutually distinct indices $\varkappa, \lambda, \mu \in (\boldsymbol{I} \cup \{\infty\}) \setminus \{\varrho\}$. We shall say *the MDC is fulfilled in* $\mathscr{A}'$ *universally* if the *MDC* of the type (ϱ) is fulfilled in $\mathscr{A}'$ for any index $\varrho \in \boldsymbol{I} \cup \{\infty\}$.

Remark 3.1. According to Remark 1.1 and to Proposition 2 in [2] we get a true implication if we interchange in the *MDC* its conclusion "$\mathbf{b'c'h}_\varkappa$" with an arbitrary one of the statements "$\mathbf{aa'h}_\varrho$", "$\mathbf{bb'h}_\varrho$", "$\mathbf{cc'h}_\varrho$", "$\mathbf{abh}_\mu$", "$\mathbf{a'b'h}_\mu$", "$\mathbf{ach}_\lambda$", "$\mathbf{a'c'h}_\lambda$", "$\mathbf{bch}_\varkappa$", and converselly.

Theorem 3.1. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of an order $\geqq 3$ having the property* (**rs**). *Then the binary operation $+_\gamma$ of the coordinate algebra $\mathfrak{A}$ is a group operation, and it is eaqual to all operations $+_\iota$, $\iota \in \boldsymbol{J}$ if and only if $\mathscr{A}'$ has the following property*
($\mathbf{D}_\infty$): *the MDC of the type (∞) is fulfilled in $\mathscr{A}'$.*

Theorem 3.2. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of an order $\geqq 3$ having the properties* (**rs**) *and* ($\mathbf{D}_\infty$). *Then for each index $\iota \in \boldsymbol{J}$, σ_ι is an automorphism of the group $(S, +)$ (where $+ := +_\gamma = +_\iota$, $\iota \in \boldsymbol{J}$) iff $\mathscr{A}'$ has the property*
($\mathbf{D}_\beta$): *the MDC of the type (β) is fulfilled in $\mathscr{A}'$.*

Theorem 3.3. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of an order $\geqq 3$, with the properties* (**rs**), ($\mathbf{D}_\infty$) *and* ($\mathbf{D}_\beta$). *Then $(S, +)$ is an abelian group iff $\mathscr{A}'$ has the property*
($\mathbf{D}_\gamma$): *the MDC of type (γ) is true in $\mathscr{A}'$.*

Theorem 3.4. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of an order $\geqq 3$ possessing the property* (**rs**). *Then for any index $\iota \in \boldsymbol{J}$, $+_\iota = +_\gamma =: +$ (where $+_\iota$ are binary operations in the coordinate algebra $\mathfrak{A}$), $(S, +)$ is an abelian group and for any index $\iota \in \boldsymbol{J}$, σ_ι is an automorphism of the group $(S, +)$ iff $\mathscr{A}'$ has the property*
(**D**): *the MDC is fulfilled in $\mathscr{A}'$ universally.*

Remark 3.2. Theorems 3.1 – 3.4 are suitably adapted dual propositions to Theorems 2, 3, 4 and the corollary of Theorem 14 in [4] dealing with the *MDC* in nets. The properties of automorphisms of a net were not explicitly used in their proofs. For the proofs of Theorems 3.1 – 3.4 it would be sufficient to make reference to the propositions introduced in [4] and their proofs. Direct proofs of Theorems 3.1 – 3.4, not using dualisations of the proofs of the corresponding propositions in [4], are routine affairs. We present one of them.

Proof of Theorem 3.3. As $\mathscr{A}'$ is supposed to have the properties ($\mathbf{D}_\infty$) and ($\mathbf{D}_\beta$), we have $+_\iota = +_\gamma = +$ for all indices $\iota \in \boldsymbol{J}$, $(S, +)$ is a group and for any index $\iota \in \boldsymbol{J}$, σ_ι is an automorphism of $(S, +)$.

a) In addition, let the *MDC* of the type (γ) be fulfilled in $\mathscr{A}'$. We shall prove that the group $(S, +)$ is abelian.

Let us choose two arbitrary elements $x, y \in S \setminus \{0\}$ and an index $\iota \in \boldsymbol{J} \setminus \{\gamma\}$. According to Construction 2.1 we determine the sums $x + y$, $y + x$. Further, let us denote

$$A = (\beta, 0), \quad A' = (\beta, x + y), \quad A'' = (\beta, y + x), \quad M = (\gamma, x), N = (\gamma, y),$$
$$\mathbf{b} = AN, \quad \mathbf{b}' = A'N, \quad \mathbf{c} = AM, \quad \mathbf{c}' = A''M, \quad B = \mathbf{c} \cap \mathbf{h}_\iota, \quad B' = \mathbf{c}' \cap \mathbf{h}_\iota,$$
$$\mathbf{a} = [B, \mathbf{b}], \quad K = \mathbf{a} \cap \mathbf{h}_\gamma, \quad \mathbf{a}' = KB'.$$

Under this notation we have for the lines **a, b, c, a′, b′, c′**:

$$\text{"}\mathbf{aa'h}_\gamma\text{"} \wedge \text{"}\mathbf{bb'h}_\gamma\text{"} \wedge \text{"}\mathbf{cc'h}_\gamma\text{"} \wedge \text{"}\mathbf{abh}_\infty\text{"} \wedge \text{"}\mathbf{ach}_\iota\text{"} \wedge \text{"}\mathbf{a'c'h}_\iota\text{"} \wedge \text{"}\mathbf{bch}_\beta\text{"}.$$

First of all we shall prove that "$\mathbf{a'b'h}_\infty$" holds, too. By simple computation we verify that

$$\mathbf{b}' = \{x, x + y\}, \quad \mathbf{c}' = \{y, y + x\}, \quad \mathbf{b} = \{-y, 0\},$$
$$\mathbf{a} = \{-y, -(y)\sigma_\iota + (x)\sigma_\iota\}, \quad B' = (\iota, -(y)\sigma_\iota + y + x),$$
$$C' = \mathbf{b}' \cap \mathbf{h}_\iota = (\iota, -(x)\sigma_\iota + x + y), \quad K = (\gamma, y - (y)\sigma_\iota + (x)\sigma_\iota),$$
$$Q = \mathbf{a} \cap \mathbf{h}_\beta = (\beta, -(y)\sigma_\iota + (x)\sigma_\iota)$$

hold.

As the line $\mathbf{a}' = \{k, q\}$ goes through the points K, B′, according to Lemma 2.1 the following equations are fulfilled:

$$k + y - (y)\sigma_\iota + (x)\sigma_\iota = q, \quad (k)\sigma_\iota - (y)\sigma_\iota + y + x = q.$$

Therefore

$$k + (y - (y)\sigma_\iota + (x)\sigma_\iota) = (k)\sigma_\iota + (-(y)\sigma_\iota + y + x). \tag{3}$$

With respect to the axiom (α4) in Definition 2.1 there exists exactly one element $k \in S$ fulfilling (3). We shall easily show that it is just the element $k = x$.

Indeed, if $k = x$ then the assumptions of the implication resulting from the *MDC* of the type $(\iota, \beta, \gamma, \infty)$ by mutually interchanging its conclusion with "$QB'AC'\mathbf{h}_\infty$" for the lines **a**, **a′**, QB′, **b**, **b′**, AC′. According to Remark 3.1 the conclusion "$QB'AC'\mathbf{h}_\infty$" of this implication is true. Let us suppose

$$QB' = \{k_0. q_1\}, \quad AC' = \{k_0, q_2\}.$$

With respect to Lemma 2.1,

$$(k_0)\sigma_\iota - (y)\sigma_\iota + y + x = q_1 = -(y)\sigma_\iota + (x)\sigma_\iota,$$

and

$$(k_0)\sigma_\iota - (x)\sigma_\iota + x + y = q_2 = 0$$

is true.

Hence

$$(k_0)\sigma_\iota = -(y)\sigma_\iota + (x)\sigma_\iota - x - y + (y)\sigma_\iota, \quad (k_0)\sigma_\iota = -y - x + (x)\sigma_\iota.$$

By comparing the right-hand sides we get the equality

$$-(y)\sigma_\iota + (x)\sigma_\iota - x - y + (y)\sigma_\iota = -y - x + (x)\sigma_\iota$$

which being suitably arranged yields

$$x + y - (y)\sigma_\iota + (x)\sigma_\iota = (x)\sigma_\iota - (y)\sigma_\iota + y + x .$$

It follows from this relation that $k = x$ is really a (unique) solution of the equation (3). Hence

$$\mathbf{a}' = \{x, x + y - (y)\sigma_\iota + (x)\sigma_\iota\} ,$$

in other words, "$\mathbf{a}'\mathbf{b}'\mathbf{h}_\infty$".

As the *MDC* of the type (γ) is fulfilled in $\mathscr{A}'$ under our assumption the *MDC* of the type $(\beta, \iota, \infty, \gamma)$ is fulfilled in $\mathscr{A}'$, too.

As has been shown above for the lines $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{a}', \mathbf{b}', \mathbf{c}'$ the suppositions of the *MDC* of the type $(\beta, \iota, \infty, \gamma)$ are fulfilled. Hence "$\mathbf{b}'\mathbf{c}'\mathbf{h}_\beta$". However, this means that $\mathsf{A}' = \mathsf{A}''$, therefore $x + y = y + x$ for any two elements $x, y \in S \setminus \{0\}$. If either of the elements x, y equals zero then the equality $x + y = y + x$ is satisfied trivially (see axiom (α3) of Definition 2.1).

b) Now, in addition to the assumptions of Theorem 3.3 (the properties (***rs***), ($\mathbf{D}_\infty$) and ($\mathbf{D}_\beta$) of $\mathscr{A}'$), let $(S, +)$ be an abelian group. We shall prove the property ($\mathbf{D}_\gamma$). Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{a}', \mathbf{b}, \mathbf{c}' \in \mathscr{L} \setminus \{\mathbf{h}_\iota\}_{\iota \in I \cup \{\infty\}}$ be arbitrary lines such that

$$\text{“}\mathbf{aa}'\mathbf{h}_\gamma\text{”} \wedge \text{“}\mathbf{bb}'\mathbf{h}_\gamma\text{”} \wedge \text{“}\mathbf{cc}'\mathbf{h}_\gamma\text{”} \wedge \text{“}\mathbf{abh}_\mu\text{”} \wedge \text{“}\mathbf{a}'\mathbf{b}'\mathbf{h}_\mu\text{”} \wedge \text{“}\mathbf{ach}_\lambda\text{”} \wedge$$
$$\wedge \text{“}\mathbf{a}'\mathbf{c}'\mathbf{h}_\lambda\text{”} \wedge \text{“}\mathbf{bch}_\varkappa\text{”}$$

where $\varkappa, \lambda, \mu \in I \setminus \{\beta, \gamma\}$ are mutually distinct indices. If some of them is equal to ∞, β, the course of the proof would be simplified. Let us put

$$\mathbf{a} \cap \mathbf{b} = \mathsf{C} = (\mu, z) , \quad \mathbf{a}' \cap \mathbf{b}' = \mathsf{C}' = (\mu, z') , \quad \mathbf{a} \cap \mathbf{c} = \mathsf{B} = (\lambda, y) ,$$
$$\mathbf{a}' \cap \mathbf{c}' = \mathsf{B}' = (\lambda, y') ,$$
$$\mathbf{b} \cap \mathbf{c} = \mathsf{A} = (\varkappa, x) , \quad \mathbf{b}' \cap \mathbf{h}_\varkappa = \mathsf{A}' = (\varkappa, x') , \quad \mathbf{c}' \cap \mathbf{h}_\varkappa = \mathsf{A}'' = (\varkappa, x'') ,$$
$$\mathbf{a} \cap \mathbf{a}' = \mathsf{K} = (\gamma, x_0) ,$$
$$\mathbf{b} \cap \mathbf{b}' = \mathsf{N} = (\gamma, y_0) , \quad \mathbf{c} \cap \mathbf{c}' = \mathsf{M} = (\gamma, z_0)$$

where $x, y, z, x', y', z', x_0, y_0, z_0, x''$ are suitable elements of S.

Further, let

$$\mathbf{a} = \{k_1, q_1\} , \quad \mathbf{b} = \{k_2, q_2\} , \quad \mathbf{c} = \{k_3, q_3\} , \quad \mathbf{a}' = \{k'_1, q'_1\} ,$$
$$\mathbf{b}' = \{k'_2, q'_2\} , \quad \mathbf{c}' = \{k'_3, q'_3\} .$$

As the above points lie on these lines we get eighteen equalities.

By comparing their left-hand sides we obtain the following system:

$$(4)\qquad \begin{aligned}
(k_1)\,\sigma_\lambda + y &= k_1 + x_0 \quad (=q_1)\\
(k_1)\,\sigma_\mu + z &= k_1 + x_0 \quad (=q_1)\\
(k_2)\,\sigma_\varkappa + x &= k_2 + y_0 \quad (=q_2)\\
(k_2)\,\sigma_\mu + z &= k_2 + y_0 \quad (=q_2)\\
(k_3)\,\sigma_\varkappa + x &= k_3 + z_0 \quad (=q_3)\\
(k_3)\,\sigma_\lambda + y &= k_3 + z_0 \quad (=q_3)\\
(k_1')\,\sigma_\lambda + y' &= k_1' + x_0 \quad (=q_1')\\
(k_1')\,\sigma_\mu + z' &= k_1' + x_0 \quad (=q_1')\\
(k_2')\,\sigma_\varkappa + x' &= k_2' + y_0 \quad (=q_2')\\
(k_2')\,\sigma_\mu + z' &= k_2' + y_0 \quad (=q_2')\\
(k_3')\,\sigma_\varkappa + x'' &= k_3' + z_0 \quad (=q_3')\\
(k_3')\,\sigma_\lambda + y' &= k_3' + z_0 \quad (=q_3')
\end{aligned}$$

The subtraction of the second equality from the fourth gives

$$(k_2 - k_1)\,\sigma_\mu = k_2 - k_1 + y_0 - x_0$$

and similarly, subtraction of the eight equality from the tenth gives $(k_2' - k_1')\,\sigma_\mu = k_2' - k_1' + y_0 - x_0$.

From the equalities just found we conclude

$$\begin{aligned}
((k_2' - k_2) - (k_1' - k_1))\,\sigma_\mu &= ((k_2' - k_1') - (k_2 - k_1))\,\sigma_\mu =\\
&= (k_2' - k_1')\,\sigma_\mu - (k_2 - k_1)\,\sigma_\mu =\\
&= k_2' - k_1' + y_0 - x_0 - (k_2 - k_1 + y_0 - x_0) =\\
&= k_2' - k_1' - k_2 + k_1 = (k_2' - k_2) - (k_1' - k_1)\,.
\end{aligned}$$

Because of $\mu \neq \gamma$ the automorphism σ_μ of the group $(S, +)$ fixes just one element, namely the element $0 \in S$. Hence

$$k_2' - k_2 = k_1' - k_1\,.$$

In the analogous way the subtraction of the first equality from the sixth implies

$$(k_3 - k_1)\,\sigma_\lambda = k_3 - k_1 + z_0 - x_0$$

and the subtraction of the seventh equality from the twelfth gives $(k_3' - k_1')\,\sigma_\lambda = k_3' - k_1' + z_0 - x_0$.

From the above two equalities we obtain

$$((k_3' - k_3) - (k_1' - k_1))\,\sigma_\lambda = (k_3' - k_3) - (k_1' - k_1)\,.$$

Using the properties of the automorphism σ_λ, $\lambda \neq \gamma$ we find

$$k_3' - k_3 = k_1' - k_1\,.$$

Finally, if we subtract the third equality from the fifth and the ninth from the eleventh we obtain

$$(k_3 - k_2)\,\sigma_\varkappa = k_3 - k_2 + z_0 - y_0\,,$$
$$(k_3' - k_2')\,\sigma_\varkappa + x'' - x' = k_3' - k_2' + z_0 - y_0\,.$$

From these two equalities we get

$$((k_3' - k_3) - (k_2' - k_2))\,\sigma_\varkappa + x'' - x' = (k_3' - k_3) - (k_2' - k_2)\,.$$

Substitution $k_3' - k_3 = k_1' - k_1$ and $k_2' - k_2 = k_1' - k_1$ gives

$$(0)\,\sigma_\varkappa + x'' - x' = 0\,, \quad \text{i.e.} \quad x'' = x'\,.$$

This means that $\mathsf{A}'' = \mathsf{A}'$, hence "$\mathbf{b'c'h_\varkappa}$" which is the conclusion of the *MDC* of the type $(\varkappa, \lambda, \mu, \gamma)$ in $\mathscr{A}'$. Thus we have proved that the *MDC* of the type $(\varkappa, \lambda, \mu, \gamma)$ in $\mathscr{A}'$ is satisfied for any triple of mutually distinct indices $\varkappa, \lambda, \mu \in (\boldsymbol{I} \cup \{\infty\}) \setminus \{\gamma\}$. Consequently, the *MDC* of the type (γ) is satisfied in $\mathscr{A}'$. ∎

4. DIAGONAL CONDITION IN AN ANTI-NET

Let the extension $\mathscr{A}' = (\mathbf{P} \cup \mathbf{h}_\infty, \mathscr{L} \cup \{\mathbf{h}_\infty\}, (\mathbf{h}_\iota)_{\iota \in \boldsymbol{I} \cup \{\infty\}})$ of an anti-net $\mathscr{A}$ be given. As in Sec. 3 let us suppose that $\mathscr{A}'$ satisfies the condition $(\boldsymbol{rs})$, i.e. $\mathscr{A}'$ has a frame $(\mathbf{o}; \beta, \gamma)$ and the coordinate algebra $\mathfrak{A} = (S, 0, (\sigma_\iota)_{\iota \in \boldsymbol{J}}, (+_\iota)_{\iota \in \boldsymbol{J}})$ with respect to the frame $(\mathbf{o}; \beta, \gamma)$. Let us define

(5) $\sigma_\beta\colon S \to S\,, \quad x \mapsto (x)\,\sigma_\beta = 0\,,$

(6) $\Sigma := \{\sigma_\iota \mid \iota \in \boldsymbol{J}\} \cup \{\sigma_\beta\}\,,$

(7) the sum $\sigma_\varkappa \dotplus \sigma_\lambda$ of arbitrary two elements $\sigma_\varkappa, \sigma_\lambda \in \Sigma$ by $(x)(\sigma_\varkappa \dotplus \sigma_\lambda) = (x)\,\sigma_\varkappa +_\gamma (x)\,\sigma_\lambda$, for all $x \in S$.

If besides $(\boldsymbol{rs})$ the structure $\mathscr{A}'$ fulfils also $(\mathbf{D}_\infty)$ then the mapping σ_β from (5) will be called the zero endomorphism of the group $(S, +)$.

In the sequel we shall use the notation

(8) $$\boldsymbol{J}' = \boldsymbol{J} \cup \{\beta\} = \boldsymbol{I}\,.$$

Then we may write $\Sigma = \{\sigma_\iota \mid \iota \in \boldsymbol{J}'\}$.

Lemma 4.1. *For any element* $\sigma_\iota \in \Sigma$,

$$\sigma_\beta \dotplus \sigma_\iota = \sigma_\iota \dotplus \sigma_\beta = \sigma_\iota$$

holds.

Proof. According to (7) and (5) any element $x \in S$ satisfies

$$(x)(\sigma_\beta \dotplus \sigma_\iota) = (x)\,\sigma_\beta +_\gamma (x)\,\sigma_\iota = 0 +_\gamma (x)\,\sigma_\iota = (x)\,\sigma_\iota$$

and also

$$(x)(\sigma_\iota \overset{\circ}{+} \sigma_\beta) = (x)\sigma_\iota +_\gamma (x)\sigma_\beta = (x)\sigma_\iota +_\gamma 0 = (x)\sigma_\iota .$$

As $\overset{\circ}{+}$ is a binary operation on Σ we may conclude that σ_β is its neutral element. ∎

Definition 4.1. Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of an order $\geqq 3$ and let $\alpha, \beta, \lambda, \varrho$ be mutually different indices from $I \cup \{\infty\}$. We shall say that $\mathscr{A}'$ satisfies *the diagonal condition* (abbr. *DGC*) *of the type* $(\alpha, \beta, \lambda, \varrho)$ if and only if for any $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathscr{L} \setminus \{\mathbf{h}_\iota\}_{\iota \in I \cup \{\infty\}}$ the implication ("$\mathbf{abh}_\beta$" $\wedge$ "$\mathbf{cdh}_\beta$" $\wedge$ "$\mathbf{adh}_\alpha$" $\wedge$ $\wedge$ "$\mathbf{bch}_\alpha$" $\wedge$ "$\mathbf{ach}_\lambda$") $\Rightarrow$ "$\mathbf{bdh}_\varrho$" is true. Let $\alpha \neq \beta$ be two fixes indices. We shall say that *DGC of the type* (α, β) is fulfilled in $\mathscr{A}'$ if for any index $\lambda \in (I \cup \{\infty\}) \setminus \{\alpha, \beta\}$ there exists an index ϱ such that $\mathscr{A}'$ satisfies *DGC* of the type $(\alpha, \beta, \lambda, \varrho)$.

Lemma 4.2. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of an order $\geqq 3$ possessing the properties* (**rs**) *and* ($\mathbf{D}_\infty$). *Then for any $\sigma_\lambda \in \Sigma$ there exists exactly one $\sigma_\varrho \in \Sigma$ such that*

$$\sigma_\varrho \overset{\circ}{+} \sigma_\lambda = \sigma_\lambda \overset{\circ}{+} \sigma_\varrho = \sigma_\beta$$

holds if and only if $\mathscr{A}'$ satisfies DGC of the type (∞, β) with the restriction $\mathbf{a} = \mathbf{o}$, $\mathbf{b} \neq \mathbf{a}$.

Proof of Lemma 4.2 is the dual of the proof of Proposition 2.3 in [6] with respect to Construction 2.1. ∎

Remark 4.1. The element σ_ϱ from Lemma 4.2 (if it exists in Σ) will be called the opposite element to the element σ_λ (notation $\sigma_\varrho = {}^\circ{-}\sigma_\lambda$).

Under this notation, each element $x \in S$ satisfies the relation $(x)\sigma_\beta = (x)({}^\circ{-}\sigma_\lambda \overset{\circ}{+} \sigma_\lambda) = (x)({}^\circ{-}\sigma_\lambda) + (x)\sigma_\lambda = -(x)\sigma_\lambda + (x)\sigma_\lambda = 0$ as well as the relation $(x)\sigma_\beta = (x)(\sigma_\lambda {}^\circ{-}\, \sigma_\lambda) = (x)\sigma_\lambda - (x)\sigma_\lambda = 0$.

Definition 4.2. Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of an order $\geqq 4$ and let $\alpha, \beta, \varkappa, \lambda, \varrho$ be indices from $I \cup \{\infty\}$ such that $\alpha \neq \beta$ and $\varkappa, \lambda, \varrho \in (I \cup \{\infty\}) \setminus \{\alpha, \beta\}$. We shall say that $\mathscr{A}'$ satisfies *the generalized DGC of the type* $(\alpha, \beta, \varkappa, \lambda, \varrho)$ if for any $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathscr{L} \setminus \{\mathbf{h}_\iota\}_{\iota \in I \cup \{\infty\}}$ the implication

$$(\text{"}\mathbf{abh}_\beta\text{"} \wedge \text{"}\mathbf{adh}_\alpha\text{"} \wedge \text{"}\mathbf{bch}_\alpha\text{"} \wedge \text{"}\mathbf{ach}_\varkappa\text{"} \wedge \text{"}\mathbf{bdh}_\lambda\text{"}) \Rightarrow \text{"}\mathbf{cdh}_\varrho\text{"}$$

is true.

Let $\alpha \neq \beta$ be fixed indices. We shall say that $\mathscr{A}'$ satisfies *the generalized DGC of the type* (α, β) if for any two indices $\varkappa, \lambda \in (I \cup \{\infty\}) \setminus \{\alpha, \beta\}$ there exists an index ϱ such that the generalized *DGC* of the type $(\alpha, \beta, \varkappa, \lambda, \varrho)$ is fulfilled in $\mathscr{A}'$.

Lemma 4.3. *Let $\mathscr{A}'$ be the extensions of an anti-net $\mathscr{A}$ of an order $\geqq 4$ having the properties* (**rs**) *and* ($\mathbf{D}_\infty$). *Then for any two elements $\sigma_\varkappa, \sigma_\lambda \in \Sigma$ there exists* *exactly one element $\sigma_\varrho \in \Sigma$ such that $\sigma_\varrho = \sigma_\varkappa \mathring{+} \sigma_\lambda$ if and only if the generalized DGC of the type (∞, β) with the restriction $\mathbf{a} = \mathbf{o}, \mathbf{b} \neq \mathbf{a}$ is fulfilled in $\mathscr{A}'$.*

Proof of Lemma 4.3 is the dual of the proof of Proposition 2.4 in [6] with respect to Construction 2.1. ∎

Theorem 4.1. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of an order $\geqq 4$ possessing the properties $(\mathbf{rs})$ and $(\mathbf{D}_\infty)$. Let Σ be the set given in (6) and let $\mathring{+}$ be the binary relation on Σ defined by (7). Then $(\Sigma, \mathring{+})$ is a group if and only if the structure $\mathscr{A}'$ has the following property*

$(\mathbf{dg})$: *DGC of the type (∞, β) as well as the generalized DGC of the type (∞, β) with the restriction $\mathbf{a} = \mathbf{o}, \mathbf{a} \neq \mathbf{b}$ are fulfilled in $\mathscr{A}'$.*

Proof. According to Lemma 4.3 the relation $\mathring{+}$ defined by (7) is a binary operation on the set Σ (i.e., for any two $\sigma_\varkappa, \sigma_\lambda \in \Sigma$ we also have $\sigma_\varkappa \mathring{+} \sigma_\lambda \in \Sigma$) iff the generalized *DGC* of the type (∞, β) for $\mathbf{a} = \mathbf{o}, \mathbf{b} \neq \mathbf{a}$ is fulfilled in $\mathscr{A}'$. With respect to Lemma 4.1, σ_β defined by (5) is the neutral zero-element of the operation $\mathring{+}$. Finally, according to Lemma 4.2, for any element $\sigma_\varkappa \in \Sigma$ there exists exactly one opposite element $^\circ{-}\sigma_\varkappa \in \Sigma$ iff *DGC* of the type (∞, β) with the restriction $\mathbf{a} = \mathbf{o}, \mathbf{b} \neq \mathbf{a}$ is fulfilled in $\mathscr{A}'$. Thus it suffices to show that the operation $\mathring{+}$ is associative, i.e., for any three elements $\sigma_\varkappa, \sigma_\lambda, \sigma_\mu \in \Sigma$ the associative law

$$\sigma_\varkappa \mathring{+} (\sigma_\lambda \mathring{+} \sigma_\mu) = (\sigma_\varkappa \mathring{+} \sigma_\lambda) \mathring{+} \sigma_\mu$$

is fulfilled.

The fulfilment of the associative law follows immediately by virtue of

$$(x)(\sigma_\varkappa \mathring{+} (\sigma_\lambda \mathring{+} \sigma_\mu)) = (x)\sigma_\varkappa + (x)(\sigma_\lambda \mathring{+} \sigma_\mu) = (x)\sigma_\varkappa + ((x)\sigma_\lambda + (x)\sigma_\mu) =$$
$$= ((x)\sigma_\varkappa + (x)\sigma_\lambda) + (x)\sigma_\mu = (x)(\sigma_\varkappa \mathring{+} \sigma_\lambda) + (x)\sigma_\mu =$$
$$= (x)((\sigma_\varkappa \mathring{+} \sigma_\lambda) \mathring{+} \sigma_\mu)\,;$$

these equalities hold for any $x \in S$ (we shall use the definition of the operation $\mathring{+}$ in Σ as well as the condition that $(S, +)$ is a group – $\mathscr{A}'$ has the property $(\mathbf{D}_\infty)$). ∎

Let us consider the set J' determined by (8) and let us define a binary relation "$+_\circ$" on J' in the following way:

(9) for any two indices $\varkappa, \lambda \in J'$, $\varkappa +_\circ \lambda = \varrho \Leftrightarrow \sigma_\varkappa \mathring{+} \sigma_\lambda = \sigma_\varrho$.

The correctness of the definition of $+_\circ$ follows from Theorem 4.1.

Corollary of the Theorem 4.1: *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of an order $\geqq 4$ having the properties $(\mathbf{rs})$ and $(\mathbf{D}_\infty)$. Further, let J' be the set described by (8) and let $+_\circ$ be the relation given in (9). $(J', +_\circ)$ is a group iff $\mathscr{A}'$ has the property $(\mathbf{dg})$.*

Proof. The corollary follows from the fact that the mapping $\Sigma \to J'$, $\sigma_\iota \mapsto \iota$ is an isomorphism of the group $(\Sigma, \mathring{+})$ onto the groupoid $(J', +_\circ)$. ∎

Theorem 4.2. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of an order $\geqq 4$ having the properties* ($\mathbf{rs}$), ($\mathbf{D}_\infty$), ($\mathbf{D}_\beta$) *and* ($\mathbf{dg}$). *Then the groups* $(S, +)$, $(\Sigma, \mathring{+})$, $(\boldsymbol{J}', +_\circ)$ *are abelian.*

Proof. Let $x, y \in S \setminus \{0\}$ be arbitrary elements and let $\varkappa, \lambda \in \boldsymbol{J}$ be arbitrary indices. The properties ($\mathbf{D}_\infty$) and ($\mathbf{D}_\beta$) imply the existence and uniqueness of $u, v \in S \setminus \{0\}$ such that

$$x = (u)\,\sigma_\varkappa \quad \text{and} \quad y = (v)\,\sigma_\lambda\,.$$

Further, it follows from ($\mathbf{D}_\infty$) and ($\mathbf{D}_\beta$) that

$$(v + u)\,\sigma_{\varkappa +_0 \lambda} = (v)\,\sigma_{\varkappa +_0 \lambda} + (u)\,\sigma_{\varkappa +_0 \lambda} = (v)\,(\sigma_\varkappa \mathring{+} \sigma_\lambda) + (u)\,(\sigma_\varkappa \mathring{+} \sigma_\lambda) =$$
$$= (v)\,\sigma_\varkappa + (v)\,\sigma_\lambda + (u)\,\sigma_\varkappa + (u)\,\sigma_\lambda$$

and simultaneously

$$(v + u)\,\sigma_{\varkappa +_0 \lambda} = (v + u)\,(\sigma_\varkappa \mathring{+} \sigma_\lambda) = (v + u)\,\sigma_\varkappa + (v + u)\,\sigma_\lambda =$$
$$= (v)\,\sigma_\varkappa + (u)\,\sigma_\varkappa + (v)\,\sigma_\lambda + (u)\,\sigma_\lambda\,.$$

Hence we get

$$(v)\,\sigma_\varkappa + (v)\,\sigma_\lambda + (u)\,\sigma_\varkappa + (u)\,\sigma_\lambda = (v)\,\sigma_\varkappa + (u)\,\sigma_\varkappa + (v)\,\sigma_\lambda + (u)\,\sigma_\lambda$$

and using the cancellation in the group $(S, +)$ we have

$$(v)\,\sigma_\lambda + (u)\,\sigma_\varkappa = (u)\,\sigma_\varkappa + (v)\,\sigma_\lambda\,. \tag{10}$$

Substituting into (10) we obtain for any $x, y \in S \setminus \{0\}$

$$y + x = x + y$$

and, as the neutral element $0 \in S$ commutes with any elements of S, we have proved that $(S, +)$ is an abelian group.

From this fact and with respect to ($\mathbf{dg}$) we get for any $x \in S$

$$(x)\,\sigma_{\varkappa +_0 \lambda} = (x)\,(\sigma_\varkappa \mathring{+} \sigma_\lambda) = (x)\,\sigma_\varkappa + (x)\,\sigma_\lambda = (x)\,\sigma_\lambda + (x)\,\sigma_\varkappa =$$
$$= (x)\,(\sigma_\lambda \mathring{+} \sigma_\varkappa) = (x)\,\sigma_{\lambda +_0 \varkappa}\,.$$

These equalities imply

$$\sigma_\varkappa \mathring{+} \sigma_\lambda = \sigma_\lambda \mathring{+} \sigma_\varkappa \quad \text{for all} \quad \varkappa, \lambda \in \boldsymbol{J}\,,$$
$$\varkappa +_0 \lambda = \lambda +_0 \varkappa \quad \text{for all} \quad \varkappa, \lambda \in \boldsymbol{J}\,.$$

The neutral element of any two groups $(\Sigma, \mathring{+})$, $(\boldsymbol{J}', +_\circ)$ again commutes with all their elements. This fact together with the last two equalities shows that the groups $(\Sigma, \mathring{+})$, $(\boldsymbol{J}', +_\circ)$ are abelian. ■

5. REIDEMEISTER CONDITION IN AN ANTI-NET

Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbf{h}_\iota)_{\iota \in \boldsymbol{I}})$ with the property ($\mathbf{rs}$).

For any two elements $\sigma_\varkappa, \sigma_\lambda \in \Sigma$ let us define

$$(11) \qquad ((x)\,\sigma_\varkappa)\,\sigma_\lambda = (x)\,(\sigma_\varkappa\sigma_\lambda) \quad \text{for all} \quad x \in S\,.$$

The composition $\sigma_\varkappa\sigma_\lambda$ need not be an element of the set Σ.

Lemma 5.1. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ having the property* (**rs**). *Then for any permutations σ_ι, $\iota \in \boldsymbol{J}$ from the coordinate algebra $\mathfrak{A}$ the following identities hold*:

a) $\sigma_\iota\sigma_\beta = \sigma_\beta\sigma_\iota = \sigma_\beta$.

b) $\sigma_\iota\sigma_\gamma = \sigma_\gamma\sigma_\iota = \sigma_\iota$.

Proof. The first assertion follows from (5) and (11), the second follows from Construction 2.1 (which gives $\sigma_\gamma = \mathrm{id}_S$) and (11). ∎

Let us put

$$(12) \qquad \Sigma^* = \Sigma \setminus \{\sigma_\beta\} = \{\sigma_\iota \mid \iota \in \boldsymbol{J}\}\,.$$

Definition 5.1. Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ and let $\alpha, \beta, \gamma, \varkappa, \lambda, \varrho \in \boldsymbol{I} \cup \{\infty\}$ be indices such that α, β, γ are mutually different. We shall say that *the Reidemeister condition* (abbr. *RC*) *of the type* $(\alpha, \beta, \gamma, \varkappa, \lambda, \varrho)$ is fulfilled in $\mathscr{A}'$ if for any $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{r} \in \mathscr{L} \setminus \{\mathbf{h}_\iota\}_{\iota \in \boldsymbol{I} \cup \{\infty\}}$ the implication

$$(\text{“}\mathbf{rah}_\varkappa\text{”} \wedge \text{“}\mathbf{rbh}_\gamma\text{”} \wedge \text{“}\mathbf{rch}_\lambda\text{”} \wedge \text{“}\mathbf{abh}_\beta\text{”} \wedge \text{“}\mathbf{cdh}_\beta\text{”} \wedge \text{“}\mathbf{adh}_\alpha\text{”} \wedge$$
$$\wedge\ \text{“}\mathbf{bch}_\alpha\text{”}) \Rightarrow \text{“}\mathbf{rdh}_\varrho\text{”}$$

is true. Let $\alpha, \beta, \gamma \in \boldsymbol{I} \cup \{\infty\}$ be fixed mutually different indices. We shall say that $\mathscr{A}'$ satisfies *the RC of the type* (α, β, γ) if for any two indices $\varkappa, \lambda \in (\boldsymbol{I} \cup \{\infty\}) \setminus \{\alpha, \beta\}$ there exists an index $\varrho \in (\boldsymbol{I} \cup \{\infty\}) \setminus \{\alpha, \beta\}$ such that *RC* of the type $(\alpha, \beta, \gamma, \varkappa, \lambda, \varrho)$ is fulfilled in $\mathscr{A}'$.

Lemma 5.2. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ having the property* (**rs**). *Let $\varkappa, \lambda \in \boldsymbol{J}$ be two arbitrary indices. Then there exists a unique index $\varrho \in \boldsymbol{J}$ such that*

$$((x)\,\sigma_\varkappa)\,\sigma_\lambda = (x)\,\sigma_\varrho$$

holds for any element $x \in S$ if and only if RC of the type (∞, β, γ) with restriction $\mathbf{r} = \mathbf{o}$ is fulfilled in $\mathscr{A}'$.

Proof of Lemma 5.2 is the dual of the proof of Proposition 2.6 in [6] with respect to Construction 2.1. ∎

Remark 5.1. According to Lemma 5.2 we may write

$$((x)\,\sigma_\varkappa)\,\sigma_\lambda = (x)\,(\sigma_\varkappa\sigma_\lambda) = (x)\,\sigma_\varrho\,. \quad \text{for all} \quad x \in S$$

iff *RC* of the type (∞, β, γ) with the restriction $\mathbf{r} = \mathbf{o}$ is fulfilled in $\mathscr{A}$.

Lemma 5.3. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ having the property* (**rs**). *Let $\varkappa$ be an arbitrary index of J. Then there exists exactly one index $\lambda \in J$ such that the identity*

$$((x)\,\sigma_\varkappa)\,\sigma_\lambda = ((x)\,\sigma_\lambda)\,\sigma_\varkappa = x$$

holds for any $x \in S$ iff RC of the type $(\infty, \beta, \gamma, \varkappa, \lambda, \gamma)$ with restriction $\mathbf{r} = \mathbf{o}$ is fulfilled in $\mathscr{A}'$.

Proof of Lemma 5.3 is the dual of the proof of Proposition 2.7 in [6] with respect to Construction 2.1. ■

Remark 5.2. a) The permutation σ_λ from Lemma 5.3 is the inverse permutation $\sigma_\varkappa^{-1}$ of $\sigma_\varkappa$.

b) It follows directly from Lemma 5.3 that $\sigma_\lambda = \sigma_\varkappa^{-1}$ implies $\sigma_\varkappa = \sigma_\lambda^{-1}$.

Theorem 5.1. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ having the property* (**rs**). *Then the set Σ^* defined in* (12) *together with the operation "composition of permutations" is a loop if and only if $\mathscr{A}'$ has the following property*

(**R**): *RC of the type (∞, β, γ) with the restriction $\mathbf{r} = \mathbf{o}$ is fulfilled in $\mathscr{A}'$ and for any index $\varkappa \in J$ there exists exactly one index $\lambda \in J$ such that RC of the type $(\infty, \beta, \gamma, \varkappa, \lambda, \gamma)$ with the restriction $\mathbf{r} = \mathbf{o}$ is fulfilled in $\mathscr{A}'$ as well.*

Proof. According to Lemma 5.2 the composition $\sigma_\varkappa\sigma_\lambda$ of the permutations $\sigma_\varkappa, \sigma_\lambda \in \Sigma^*$ is a permutation of Σ^* iff *RC* of the type (∞, β, γ) with the restriction $\mathbf{r} = \mathbf{o}$ is fulfilled in $\mathscr{A}'$. In accordance with part b) of Lemma 5.1 the permutation $\sigma_\gamma \in \Sigma^*$ is the neutral element of $(\Sigma^*, \;)$. According to Lemma 5.3, for any permutation $\sigma_\varkappa \in \Sigma^*$ there exists a unique inverse permutation $\sigma_\varkappa^{-1} \in \Sigma^*$ such that

$$\sigma_\varkappa\sigma_\varkappa^{-1} = \sigma_\varkappa^{-1}\sigma_\varkappa = \sigma_\gamma$$

iff *RC* of the type $(\infty, \beta, \gamma, \varkappa, \lambda, \gamma)$ with the restriction $\mathbf{r} = \mathbf{o}$ is fulfilled in $\mathscr{A}'$.

Now it is sufficient to prove the existence and uniqueness of the solution of the equations

$$\sigma_\xi\sigma_\lambda = \sigma_\varrho \quad \text{and} \quad \sigma_\varkappa\sigma_\eta = \sigma_\tau$$

for any given $\sigma_\lambda, \sigma_\varrho$ or $\sigma_\varkappa, \sigma_\tau$, respectively.

For any two permutations $\sigma_\lambda, \sigma_\varrho \in \Sigma^*$ there exists a unique permutation $\sigma_\varrho\sigma_\lambda^{-1} \in \Sigma^*$ which is a solution of the first equation (since in fact $((x)(\sigma_\varrho\sigma_\lambda^{-1}))\,\sigma_\lambda = ((x)\,\sigma_\varrho)\,\sigma_\lambda^{-1})\,\sigma_\lambda = ((x)\,\sigma_\varrho)\,\sigma_\gamma = (x)\,\sigma_\varrho$ holds for all $x \in S$) iff $\mathscr{A}'$ has the property (**R**). Analogously, for any two permutations $\sigma_\varkappa, \sigma_\tau \in \Sigma^*$ there exists exactly one permutation $\sigma_\varkappa^{-1}\sigma_\tau \in \Sigma^*$ which is a solution of the second equation (since in fact $((x)\,\sigma_\varkappa)(\sigma_\varkappa^{-1}\sigma_\tau) = (((x)\,\sigma_\varkappa)\,\sigma_\varkappa^{-1})\,\sigma_\tau = ((x)\,\sigma_\gamma)\,\sigma_\tau = (x)\,\sigma_\tau$ holds for all $x \in S$) iff $\mathscr{A}'$ has the property (**R**). The proof of Theorem 5.1 is complete. ■

Let us consider the index set J' determined by (8). Let us define the binary relation "$\circ$" on J' in the following way:

(13) for any two indices $\varkappa, \lambda \in \boldsymbol{J}'$, $\varkappa \circ \lambda = \varrho \Leftrightarrow \sigma_\varkappa \sigma_\lambda = \sigma_\varrho$.

The correctness of the definition of "$\circ$" follows from Theorem 5.1.

Corollary of Theorem 5.1. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ having the property* (***rs***). *Then the structure* $(\boldsymbol{J}, \circ)$, *where $\boldsymbol{J}$ is the index set of the coordinate algebra $\mathfrak{A}$ and "$\circ$" is the binary relation on $\boldsymbol{J}' = \boldsymbol{J} \cup \{\beta\}$ defined by* (13), *is a loop if and only if $\mathscr{A}'$ has the property* (**R**).

Proof. Our assertion is a consequence of the fact that the mapping $\Sigma^* \to \boldsymbol{J}$, $\sigma_\iota \mapsto \iota$ is a loop isomorphism of $(\Sigma^*, \ )$ onto $(\boldsymbol{J}, \circ)$. ■

If the structure has the properties (***rs***), $(\boldsymbol{D}_\infty)$ and $(\boldsymbol{D}_\beta)$ then according to Theorem 3.2 each permutation $\sigma_\iota \in \Sigma^*$ is an automorphism of the group $(S, +)$. The composition of automorphisms of a group is an associative operation, therefore we have

Theorem 5.2. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of an order $\geqq 3$ having the properties* (***rs***), $(\boldsymbol{D}_\infty)$ *and* $(\boldsymbol{D}_\beta)$. *Then $(\Sigma^*, \ )$ is a group if and only if $\mathscr{A}'$ has the property* (**R**).

Corollary of Theorem 5.2. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ having the properties* (***rs***), $(\boldsymbol{D}_\infty)$, $(\boldsymbol{D}_\beta)$. *Then $(\boldsymbol{J}, \circ)$ is a group if and only if $\mathscr{A}'$ has the property* (**R**).

Proof. The corollary follows from the corollary of Theorem 5.1 and from Theorem 5.2. ■

Lemma 5.4. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of an order $\geqq 4$ having the properties* (***rs***), $(\boldsymbol{D}_\infty)$, $(\boldsymbol{D}_\beta)$, (**dg**) *and* (**R**). *Then for any three indices $\varkappa, \lambda, \mu \in \boldsymbol{J}$ the distribution laws*

a) $\varkappa \circ (\lambda +_\circ \mu) = (\varkappa \circ \lambda) +_\circ (\varkappa \circ \mu)$,

b) $(\varkappa +_\circ \lambda) \circ \mu = (\varkappa \circ \mu) +_\circ (\lambda \circ \mu)$ *hold.*

Proof. Let $x \in S \setminus \{0\}$ be an arbitrary element. According to the relations (7), (9), (11), (13) and the corollaries of Theorems 4.1 and 5.2 we necessarily have

$$(x)\,\sigma_{\varkappa\circ(\lambda+_0\mu)} = (x)\,\sigma_\varkappa\sigma_{\lambda+_0\mu} = ((x)\,\sigma_\varkappa)\,(\sigma_\lambda \mathbin{\mathring{+}} \sigma_\mu) = (x)\,\sigma_\varkappa\sigma_\lambda + (x)\,\sigma_\varkappa\sigma_\mu =$$
$$= (x)\,\sigma_{\varkappa\circ\lambda} + (x)\,\sigma_{\varkappa\circ\mu} = (x)\,(\sigma_{\varkappa\circ\lambda} \mathbin{\mathring{+}} \sigma_{\varkappa\circ\mu}) = (x)\,\sigma_{(\varkappa\circ\lambda)+_0(\varkappa\circ\mu)}$$

and also

$$(x)\,\sigma_{(\varkappa+_0\lambda)\circ\mu} = (x)\,\sigma_{\varkappa+_0\lambda}\sigma_\mu = ((x)\,(\sigma_\varkappa \mathbin{\mathring{+}} \sigma_\lambda))\,\sigma_\mu = ((x)\,\sigma_\varkappa + (x)\,\sigma_\lambda)\,\sigma_\mu =$$
$$= (x)\,\sigma_\varkappa\sigma_\mu + (x)\,\sigma_\lambda\sigma_\mu = (x)\,\sigma_{\varkappa\circ\mu} + (x)_{\delta\lambda\circ\mu} = (x)\,(\sigma_{\varkappa\circ\mu} \mathbin{\mathring{+}} \sigma_{\lambda\circ\mu}) = (x)\,\sigma_{(\varkappa\circ\mu)+_0(\lambda\circ\mu)}\,.$$

We have used here the assumption $(\boldsymbol{D}_\beta)$ implying that σ_μ is an automorphism of the group $(S, +)$. ■

Theorem 5.3. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of an order $\geqq 4$ having the properties* $(\mathbf{rs})$, $(\mathbf{D}_\infty)$, $(\mathbf{D}_\beta)$, $(\mathbf{dg})$ *and* $(\mathbf{R})$. *Then* $(J', +_0, \circ, \beta, \gamma)$ *is a skew-field.*

Proof. The validity follows from Theorem 4.2, the corollary of Theorem 5.2, Lemma 5.4 and the relations (5), (9) and (13). ■

Theorem 5.4. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of an order $\geqq 4$ having the properties* $(\mathbf{rs})$, $(\mathbf{D}_\infty)$, $(\mathbf{D}_\beta)$, $(\mathbf{dg})$ *and* $(\mathbf{R})$. *Then the group* $(S, +)$ *is a vector space over the skew-field* $(J', +_0, \circ, \beta, \gamma)$.

Proof. The assertion of our theorem follows from the above results. It is sufficient to define the product of a "vector" $x \in S$ and a "scalar" $\varkappa \in J'$ by

$$x \cdot \varkappa = (x)\,\sigma_\varkappa \quad \text{for all} \quad x \in S \quad \text{and} \quad \varkappa \in J'\,.$$

The verification that the structure $(S, +)$ endowed with the multiplication by scalars just described is a vector space is quite easy.

6. EXAMPLES OF ANTI-NETS

Example 1. Let $\mathbf{P} = \{A, B, C, D, E, F, G, H\}$ be an arbitrary set. Let us denote

$$\mathbf{h}_1 = \{A, B, C, D\}\,, \quad \mathbf{h}_2 = \{E, F, G, H\}\,,$$

$$\begin{array}{llll} \mathbf{a} = \{A, E\}\,, & \mathbf{b} = \{A, F\}\,, & \mathbf{c} = \{A, G\}\,, & \mathbf{d} = \{A, H\}\,, \\ \mathbf{e} = \{B, E\}\,, & \mathbf{f} = \{B, F\}\,, & \mathbf{g} = \{B, G\}\,, & \mathbf{i} = \{B, H\}\,, \\ \mathbf{j} = \{C, E\}\,, & \mathbf{k} = \{C, F\}\,, & \mathbf{l} = \{C, G\}\,, & \mathbf{m} = \{C, H\}\,, \\ \mathbf{n} = \{D, E\}\,, & \mathbf{o} = \{D, F\}\,, & \mathbf{p} = \{D, G\}\,, & \mathbf{q} = \{D, H\}\,. \end{array}$$

Further, let $\mathscr{L} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, \mathbf{o}, \mathbf{p}, \mathbf{q}, \mathbf{h}_1, \mathbf{h}_2\}$. Then $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbf{h}_1, \mathbf{h}_2))$ is a finite anti-net of degree 2 and of order 4 with the following pencils of mutually parallel lines: $\{\mathbf{h}_1, \mathbf{h}_2\}$, $\{\mathbf{a}, \mathbf{f}, \mathbf{l}, \mathbf{q}\}$, $\{\mathbf{b}, \mathbf{e}, \mathbf{m}, \mathbf{p}\}$, $\{\mathbf{c}, \mathbf{i}, \mathbf{j}, \mathbf{o}\}$, $\{\mathbf{d}, \mathbf{g}, \mathbf{k}, \mathbf{n}\}$. Let us take a note of the fact that the lines **a**, **e** have exactly one common point, the lines **a**, **f** are parallel lines (without common point) and finally, the lines **a**, **m** are non-parallel lines without a common point.

Example 2. In the ordinary 3-dimensional euclidean space $\mathbf{E}_3$ let a cartesian coordinate system be chosen. Let us denote by $\mathbf{P}$ the set of all points of $\mathbf{E}_3$ and let $\boldsymbol{\rho}_i$ denote the plane with the equation $x_3 = i$, $i \in \mathbf{R}$ ($\mathbf{R}$ is the set of reals). Finally, let

$$\mathscr{L} = \{\boldsymbol{\rho}_i \mid i \in \mathbf{R}\} \cup \{\mathbf{g} \in \mathbf{E}_3 \mid \mathbf{g} \nparallel \boldsymbol{\rho}_0\}$$

be the set consisting of all planes $\boldsymbol{\rho}_i$, $i \in \mathbf{R}$ together with all lines $\mathbf{g} \in \mathbf{E}_3$ not parallel to any plane $\boldsymbol{\rho}_i$.

We can easily verify that the structure $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\boldsymbol{\rho}_i)_{i\in\mathbf{R}})$ is an anti-net with the principal lines $\boldsymbol{\rho}_i$, ordinary lines $\mathbf{g} \in \mathbf{E}_3$, $\mathbf{g} \not\parallel \boldsymbol{\rho}_0$ and with the usual incidence and parallel relations. The anti-net $\mathscr{A}$ just constructed is of the infinite degree $k = \#\mathbf{R} = \mathbf{c}$ and of the order $n = \#\boldsymbol{\rho}_0 = \#\mathbf{R}^2 = \mathbf{c}$.

Example 3. Let $(\mathbf{F}, +, \cdot, 0, 1)$ be the residue-class field $\mathbf{Z}_5$. Let us put $S = \mathbf{F} \times \mathbf{F}$, $0 = (0, 0)$, $\boldsymbol{J} = \mathbf{F}^* = \mathbf{F} \setminus \{0\}$. Further, let σ_i $(i \in \boldsymbol{J})$ be the mapping $S \to S$ carrying (x_1, x_2) into $(x_1 i, x_2 i)$. Finally, for any $i \in \boldsymbol{J}$ let $+_i = +$ be the usual addition in $\mathbf{F} \times \mathbf{F}$ (i.e. $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ for all $i \in \boldsymbol{J}$). Then the structure $\mathfrak{A} = (S, 0, (\sigma_i)_{i\in\boldsymbol{J}}, (+_i)_{i\in\boldsymbol{J}})$ with $\#S = 25$ and $\#\boldsymbol{J} = 4$ is an admissible algebra. Indeed,

(*α1*) for any index $i \in \boldsymbol{J}$, σ_i is a permutation of the set S and $(0, 0)\,\sigma_i = (0, 0)$;

(*α2*) there exists a preferred index $1 \in \boldsymbol{J}$ such that $\sigma_1 = \mathrm{id}_S$;

(*α3*) clearly, for any $(a_1, a_2), (b_1, b_2) \in S$ each of the equations $(a_1, a_2) + (x_1, x_2) = (b_1, b_2)$, $(y_1, y_2) + (a_1, a_2) = (b_1, b_2)$ has the unique solution $(x_1, x_2) = (-a_1 + b_1, -a_2 + b_2)$, $(y_1, y_2) = (b_1 - a_1, b_2 - a_2)$, respectively.

Moreover, for any couple $(c_1, c_2) \in S$ we have $(0, 0) + (c_1, c_2) = (c_1, c_2) + (0, 0) = (c_1, c_2)$. Hence for any index $i \in \boldsymbol{J}$, $(\mathbf{S}, +) = (\mathbf{S}, +_i)$ is a loop with the neutral element $(0, 0)$.

(*α4*) Let x, y be two different indices from $\boldsymbol{J}$ and let $(b_1, b_2), (c_1, c_2) \in S$. Then there is a unique pair $(a_1, a_2) \in S$ satisfying $(a_1, a_2)\,\sigma_x + (b_1, b_2) = (a_1, a_2)\,\sigma_y + (c_1, c_2)$. This relation is evidently equivalent to the relations $a_1(x - y) = c_1 - b_1$, $a_2(x - y) = c_2 - b_2$.

According to Construction 2.2 in section 2 we can form an anti-net $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbf{h}_i)_{i\in\boldsymbol{I}})$ defining

$\boldsymbol{I} := \boldsymbol{J} \cup \{0\} = \mathbf{F}$,

$\mathbf{P} := \{(x, (y_1, y_2)) \mid x \in \boldsymbol{I},\ (y_1, y_2) \in S\}$,

$\mathbf{h}_i := \{(i, (y_1, y_2)) \mid (y_1, y_2) \in S\}$ for all $i \in \boldsymbol{I}$,

$\{(k_1, k_2), (q_1, q_2)\} := \{(i, (y_1, y_2)) \mid i \in \boldsymbol{J},\ (y_1, y_2) \in S,\ (k_1, k_2)\,\sigma_i + (y_1, y_2) = (q_1, q_2)\} \cup \{(0, (q_1, q_2))\}$,

$\mathscr{L} := \{\mathbf{h}_i \mid i \in \boldsymbol{I}\} \cup \{\{(k_1, k_2), (q_1, q_2)\} \mid (k_1, k_2), (q_1, q_2) \in S\}$.

The ani-net $\mathscr{A}$ just constructed has degree 5 and order 25. As the operation "+" in $(\mathbf{F}, +, \cdot, 0, 1)$ is a commutative and associative one, obviously the operation "+" in S has the same properties. Therefore $(S, +)$ is an abelian group. Let $i \in \boldsymbol{J}$ be an arbitrary index and let $(x_1, x_2), (y_1, y_2) \in S$. Then

$$((x_1, x_2) + (y_1, y_2))\,\sigma_i = (x_1 + y_1, x_2 + y_2)\,\sigma_i = (x_1 i + y_1 i, x_2 i + y_2 i) = (x_1 i, x_2 i) + (y_1 i, y_2 i) = (x_1, x_2)\,\sigma_i + (y_1, y_2)\,\sigma_i,$$

which means that the permutation σ_i is an automorphism of the group $(S, +)$.

Now, with respect to Theorem 3.4, *MDC* is fulfilled universally in the anti-net $\mathscr{A}$.

Let us define in a very natural way the zero-endomorphism $\sigma_0, \sigma_0 : S \to S, (x_1, x_2) \mapsto \mapsto (x_1, x_2)\sigma_0 = (x_1 . 0, x_2 . 0) = (0, 0)$, and let us put $\Sigma = \{\sigma_i \mid i \in \boldsymbol{I}\}$. Then we may determine the operation "$\stackrel{\circ}{+}$" on Σ according to the relation (7) of Sec. 4. Under this condition $(\Sigma, \stackrel{\circ}{+})$ is an abelian group. In fact, for any $(x_1, x_2) \in S$ we have

$$(x_1, x_2)((\sigma_i \stackrel{\circ}{+} \sigma_j) \stackrel{\circ}{+} \sigma_k) = (x_1, x_2)(\sigma_i \stackrel{\circ}{+} \sigma_j) + (x_1, x_2)\sigma_k =$$
$$= (x_1, x_2)\sigma_i + (x_1, x_2)\sigma_j + (x_1, x_2)\sigma_k =$$
$$= (x_1, x_2)\sigma_i + (x_1, x_2)(\sigma_j \stackrel{\circ}{+} \sigma_k) = (x_1, x_2)(\sigma_i \stackrel{\circ}{+} (\sigma_j \stackrel{\circ}{+} \sigma_k)) .$$

Further, the neutral element of $(\Sigma, \stackrel{\circ}{+})$ is the zero-endomorphism σ_0 and the opposite element to $\sigma_i \in \Sigma$ is the permutation $^\circ{-}\sigma_i = \sigma_{-i}$. With respect to Theorem 4.1 the anti-net $\mathscr{A}$ has the property (**dg**).

Now let us put $\Sigma^* = \Sigma \setminus \{\sigma_0\} = \{\sigma_i \mid i \in \boldsymbol{J}\}$. Any of the permutations σ_i, $i \in \boldsymbol{J}$ is an automorphism of the group $(S, +)$.

Obviously, $\sigma_i \sigma_j = \sigma_{ij}$ for any $i, j \in \boldsymbol{J}$ since $((x_1, x_2)\sigma_i)\sigma_j = (x_1 . ij,\ x_2 . ij) = = (x_1, x_2)\sigma_{ij}$.

Evidently $(\Sigma^*, \ )$ is an abelian group with the neutral element σ_1. The inverse element of $\sigma_i \in \Sigma^*$ is the permutation $\sigma_{i^{-1}}$. It follows from Theorem 5.2 that the anti-net $\mathscr{A}$ has the property (**R**).

Summarizing all the above results and using Theorem 1.3 from our paper Anti-nets II we obtain: The anti-net $\mathscr{A}$ is a central translation structure (in the sense of [1]) with the abelian translation group; this translation group contains all possible translations of the anti-net $\mathscr{A}$.

Example 4. Let $\mathbf{K}$ be a nonempty (at least two-element) set. By non-planar Cartesian group we mean a structure $(\mathbf{K}, +, \cdot, 0, 1)$ satisfying

(*K1*) $(\mathbf{K}, +)$ is a (not necessarily commutative) group with the neutral element 0;

(*K2*) $(\mathbf{K}^*, \cdot)$, where $\mathbf{K}^* = \mathbf{K} \setminus \{0\}$, is a loop with the neutral element 1;

(*K3*) $0 . x = x . 0 = 0$ holds for any element $x \in \mathbf{K}$;

(*K4*) for any three elements $a, b, c \in \mathbf{K}$, $a \neq b$ there is a unique $x \in \mathbf{K}$ satisfying $a . x = b . x + c$;

(*K5*) there exist (at least) three elements $u, v, w \in \mathbf{K}$, $u \neq v$ such that $x . u \neq x . v + + w$ holds for all $x \in \mathbf{K}$.

About the existence of a non-planar Cartesian group see e.g. [5], Constructions 8 and 9.

Now we shall construct an anti-net $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbb{H}_i)_{i \in \mathbf{K}})$ over the non-planar Cartesian group $(\mathbf{K}, +, \cdot, 0, 1)$.

As the points of the anti-net $\mathscr{A}$ we shall take the ordered pairs $(x, y) \in \mathbf{K}^2$; consequently $\mathbf{P} = \mathbf{K}^2$. As the lines of $\mathscr{A}$ we shall take all sets

$$\{(x, y) \in \mathbf{K}^2 \mid y = xk + q\}, \quad \text{for all} \quad k, q \in \mathbf{K}$$

and

$$\{(c, y) \in \mathbf{K}^2 \mid y \in \mathbf{K}\}, \quad \text{for all} \quad c \in \mathbf{K};$$

thus $\mathscr{L} = \{\{(x, y) \in \mathbf{K}^2 \mid y = xk + q\} \mid k, q \in \mathbf{K}\} \cup \{\{(c, y) \in \mathbf{K}^2 \mid y \in \mathbf{K}\} \mid c \in \mathbf{K}\}$.

The incidence relation of points and lines of $\mathscr{A}$ is interpreted in the natural way.

The line $\mathbb{G} = \{(x, y) \in \mathbf{K}^2 \mid y = xk + q\}$ of $\mathscr{A}$ will be denoted briefly by $\mathbb{G} = (y = xk + q)$. The element $k \in \mathbf{K}$ will be called the slope of the line $\mathbb{G}$. Similarly, a line $\mathbb{H} = \{(c, y) \mid y \in \mathbf{K}\}$, $c \in \mathbf{K}$ of $\mathscr{A}$ (without slope) will be denoted by $\mathbb{H} = (x = c)$. Lines without slope will be taken for principal lines, the lines with a slope for ordinary lines. The parallel relation between lines of $\mathscr{A}$ will be interpreted in the following way: any two principal lines are parallel, two ordinary lines are parallel iff they have the same slope.

It is easy to verify that the structure $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbb{H}_i)_{i \in \mathbf{K}})$, where $\mathbb{H}_i = (x = i)$, satisfies the axioms $(\boldsymbol{a1})-(\boldsymbol{a5})$ of an anti-net (see Sec. 1).

1. For any two different points (a, b), $(c, d) \in \mathbf{P}$ there is a unique line $\mathbb{G} \in \mathscr{L}$ containing them; in fact, if $a = c$ then $\mathbb{G} = \mathbb{H}_a = (x = a)$. Let $a \neq c$, then there is exactly one $k \in \mathbf{K}$ such that $(-a) k = (-c) k + (d - b)$ and if we put $q = -ak + b = -ck + d$ then the line $\mathbb{G} = (y = xk + q)$ is the uniquely determined line containing both the above points.

2. Each principal line $\mathbb{H}_c = (x = c)$ intersects any ordinary line in exactly one point, namely $\mathsf{B} = (c, ck + q)$. Two different ordinary lines have at most one common point. Moreover, two different parallel lines obviously have no common point. However, there are lines without common points which are not parallel. As an example let us consider the lines $\mathbb{G}_1 = (y = xu + q)$ and $\mathbb{G}_2 = (y = xv + w + q)$, $q \in \mathbf{K}$, $u \neq v$ where $u, v, w \in \mathbf{K}$ satisfy the condition $(\mathbf{K5})$.

3. Let an ordinary line $\mathbb{G} = (y = xk + q)$ and a point $(a, b) \in \mathbf{P}$ be given. Then there is a unique (ordinary) line $\mathbb{G}'$ parallel to $\mathbb{G}$ and containing (a, b), namely $\mathbb{G}' = (y = xk - ak + b)$. Analogously, let a principal line $\mathbb{H}_c = (x = c)$ and a point $(a, b) \in \mathbf{P}$ be given. Then the principal line $\mathbb{H}'_a = (x = a)$ is the unique parallel line to $\mathbb{H}_c$ containing (a, b).

4. The structure $\mathscr{A}$ satisfies the axiom $(\boldsymbol{a4})$, as follows immediately from the definition of the parallel relation on $\mathscr{L}$.

5. The points $(0, 0)$, $(0, 1)$, $(1, 0)$ are non-collinear, for example.

Let us remark that the anti-net $\mathscr{A}$ is finite (infinite) iff the support $\mathbf{K}$ of the non-planar Cartesian group $(\mathbf{K}, +, \cdot, 0, 1)$ is finite (infinite).

Let us put

$$\mathsf{S} := \mathbf{K}, \quad \boldsymbol{J} = \mathbf{K}^*,$$

$$+_i := + \quad \text{for all} \quad i \in \boldsymbol{J},$$

$$\sigma_i: \mathsf{S} \to \mathsf{S}, \quad x \mapsto (x)\,\sigma_i = i \,.\, x \quad \text{for every} \quad i \in \boldsymbol{J}.$$

Using this notation we get that

$$\mathfrak{A} = (S, 0, (\sigma_i)_{i \in J}, (+_i)_{i \in J})$$

is the coordinate algebra (in the sense of Sec. 2) of the anti-net $\mathscr{A}$ just constructed.

It follows from the axioms (**K1**)–(**K5**) of the non-planar Cartesian group ($\mathbf{K}$, $+$, $\cdot$, 0, 1) that for any $I \in \boldsymbol{J}$ we have $+_i = +$, and $(S, +)$ is not necessarily a commutative group with the neutral element 0. According to Theorem 3.1, the anti-net $\mathscr{A}$ has the property ($\mathbf{D}_\infty$). Since none of the distributive laws holds in a non-planar Cartesian group, no permutation σ_i, $i \in \boldsymbol{J}$ except σ_1, $1 \in \boldsymbol{J}$ is an automorphism of the group $(S, +)$. Hence the anti-net $\mathscr{A}$ has neither the property ($\mathbf{D}_\beta$) for $\beta = 0$ nor property (**dg**). The multiplication of non-zero elements of a non-planar Cartesian group is a loop operation. According to the corollary of Theorem 5.1 the anti-net $\mathscr{A}$ has the property (**R**).

Example 5. Let us consider a non-planar left quasifield $\mathscr{Q} = (\mathbf{Q}, +, \cdot, 0, 1)$, i.e. a non-planar Cartesian group satisfying left distributivity $(a + b) \,.\, c = a \,.\, c + b \,.\, c$ for all $a, b, c \in \mathbf{Q}$. In the same way as in Example 4 we construct an anti-net $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbb{H}_i)_{i \in \mathbf{Q}})$ over this quasifield $\mathscr{Q}$ and its coordinate algebra $\mathfrak{A} = (\mathbf{Q}, 0, (\sigma_i)_{i \in \mathbf{Q}^*}, (+_i)_{i \in \mathbf{Q}^*})$ where $\mathbf{Q}^* = \mathbf{Q} \setminus \{0\}$ and $+_i = +$ for each element $i \in \mathbf{Q}^*$. Now this anti-net $\mathscr{A}$ has the property ($\mathbf{D}_\infty$) since $(\mathbf{Q}, +)$ is a group and $+_i = +$ for all $i \in \mathbf{Q}^*$ (in virtue of Theorem 3.1). The validity of the left distributivity and the definition of a permutation σ_i implies that $\mathscr{A}$ has the property (**dg**) (according to the corollary of Theorem 4.1). The anti-net $\mathscr{A}$ considered has not the property ($\mathbf{D}_\beta$) for $\beta = 0$ as the right distributivity does not hold in the given quasifield $\mathscr{Q}$. The anti-net $\mathscr{A}$ has the property (**R**) since the multiplication of non-zero elements of $\mathbf{Q}^*$ is a loop operation.

Example 6. Let us construct the anti-net $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbb{H}_i)_{i \in \mathbf{S}})$ over a non-planar left nearfield $(\mathbf{S}, +, \cdot, 0, 1)$ in the same way as in Example 4 (a non-planar left nearfield is a non-planar left quasifield in which the multiplication is an associative operation). Let $\mathbf{S}^* = \mathbf{S} \setminus \{0\}$ and $+_i = +$ for all $i \in \mathbf{S}$. Further, let $\mathfrak{A} = (\mathbf{S}, 0, (\sigma_i)_{i \in \mathbf{S}^*}, (+_i)_{i \in \mathbf{S}^*})$ be a coordinate algebra of the anti-net $\mathscr{A}$. The anti-net $\mathscr{A}$ has the properties ($\mathbf{D}_\infty$), (**dg**) and (**R**) but it has not the property ($\mathbf{D}_\beta$) for $\beta = 0$. We verify it as in Example 5.

Example 7. Let V be a vector space over a field $(\mathbf{F}, +, \cdot, 0, 1)$, i.e. $(V, +)$ is an abelian additive group with a neutral element o endowed with the mapping $V \times \mathbf{F} \to V$, $(x, \lambda) \mapsto x\lambda$ fulfilling the axioms

(**v1**) $x(\lambda \,.\, \mu) = (x\lambda)\, \mu$ for all $x \in V$ and for every $\lambda, \mu \in \mathbf{F}$,

(**v2**) $x1 = x$ for all $x \in V$,

(**v3**) $(x + y)\, \lambda = x\lambda + y\lambda$ for every $x, y \in V$ and all $\lambda \in \mathbf{F}$,

(**v4**) $x(\lambda + \mu) = x\lambda + x\mu$ for all $x \in V$ and for every $\lambda, \mu \in \mathbf{F}$.

Let us put

$$\begin{aligned} &S := V\,, \\ &J := F^* = F\setminus\{0\}\,,\quad J' := F\,, \\ &\sigma_\iota := S \to S\,,\quad x \mapsto (x)\sigma_\iota = x\iota \quad \text{for all} \quad \iota \in J'\,, \\ &+_\iota := + \quad \text{for all} \quad \iota \in J\,. \end{aligned}$$

It is easily seen that $\mathfrak{A} = (S, \circ, (\sigma_\iota)_{\iota\in J}, (+_\iota)_{\iota\in J})$ is an admissible algebra with a significant index $1 \in J$. This means that the axioms $(\alpha 1)-(\alpha 4)$ of Section 2 are fulfilled. Now we shall construct an anti-net $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbb{H}_\iota)_{\iota\in I})$ over the algebra $\mathfrak{A}$. To this end it is sufficient to put

$$\begin{aligned} &I := J' = F\,, \\ &\mathbf{P} := F \times V = \{(\xi, y) \mid \xi \in F,\ y \in V\}\,, \\ &\{k, q\} := \{(\xi, y) \mid \xi \in I,\ y \in V,\ (k)\sigma_\xi + y = q\} \quad \text{for all} \quad k, q \in V\,, \\ &\mathbb{H}_\iota := \{(\iota, y) \mid y \in V\} \quad \text{for all} \quad \iota \in I\,, \\ &\mathscr{L} := \{\mathbb{H}_\iota \mid \iota \in I\} \cup \{\{k, q\} \mid k, q \in V\}\,, \\ &\mathbb{H}_\iota \parallel \mathbb{H}_\varkappa \quad \text{for all} \quad \iota, \varkappa \in I\,, \\ &(\mathbb{G} = \{k, q\},\ \mathbb{G}' = \{k', q'\}\,,\quad k, q, k', q' \in V) \Rightarrow (\mathbb{G} \parallel \mathbb{G}' \Leftrightarrow k = k')\,. \end{aligned}$$

We can verify without dificulty that this structure satisfies axioms $(\mathbf{a1})-(\mathbf{a5})$ from Section 1.

If $\dim V = 1$ then the anti-net constructed is an affine plane. If $\dim V \geqq 2$ then the anti-net is an affine parallel structure. As $(V, +)$ is an (abelian) group and for all indices $\iota \in J$ we have defined $+_\iota = +$, the just anti-net $\mathscr{A}$ constructed has the property $(\mathbf{D}_\infty)$ (according to Theorem 3.1).

It follows from the axiom $(\mathbf{v3})$ that $\mathscr{A}$ has the property $(\mathbf{D}_\beta)$ for $\beta = 0$. The group $(V, +)$ is abelian, therefore the anti-net $\mathscr{A}$ has the property $(\mathbf{D}_\gamma)$ for $\gamma = 1$ (according to Theorem 3.3). The anti-net $\mathscr{A}$ even has the property $(\mathbf{D})$ as follows from Theorem 3.4. From the axiom $(\mathbf{v4})$, Theorem 4.1 and its corollary, we get that the anti-net $\mathscr{A}$ has the property $(\mathbf{dg})$. Using axioms $(\mathbf{v1})$ and $(\mathbf{v2})$, Theorem 5.2 and its corollary we easily verify that the anti-net $\mathscr{A}$ has also the property $(\mathbf{R})$.

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Súhrn

ANTI-NETS I

(Basic notation and properties - Closure conditions)

JAROSLAV LETTRICH

V článku sú zavedené a vyšetrované špeciálne paralelné štruktúry, nazvané antisieťami. Antisieť je afinné zúženie duálnej štruktúry k projektívnej sieti so singulárnymi bodmi ležiacimi na jednej priamke. Ako súradnicová algebra antisiete je používaná prípustná algebra.

Pre antisieť sú tu formulované vhodné uzáverové podmienky (malá Desarguesova podmienka, Reidemeisterova podmienka a diagonálna podmienka). Odvodené sú vlastnosti súradnicovej algebry antisiete, ktorá splňuje jednotlivé uzáverové podmienky.

V závere článku je uvedených sedem príkladov antisiete (konečnej aj nekonečnej).

Резюме

АНТИСЕТИ I

(Основные понятия и свойства — Условия замыкания)

JAROSLAV LETTRICH

В статье вводятся и рассматриваются специальные параллельные структуры, названные автором антисетями. Антисеть — это аффинная редукция двойственной структуры к проективной сети с сингулярными точками, лежащими на одной прямой. В качестве координатной алгебры антисети употребляется допустимая алгебра.

Для антисети сформулированы подходящие условия замыкания (малое условие Дезарга, условие Реидемейстера и диагональное условие). Выведены также свойства cooринатной алгебры антисети, в которой выполняются отдельные условия замыкания.

В заключении статьи приведено семь примеров антисетей (конечных и бесконечных).

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