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# BETWEENNESS SPACES AND TREE ALGEBRAS

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By a *betwenness space* we mean a pair $(X, \beta)$, where $X$ is a nonvoid set, and $\beta \subset X^3$ is a ternary relation on $X$ subject to the following axioms:[1])

β1: $\beta abb$,

β2: $\beta aba \Rightarrow a = b$,

β3: $\beta abc \Rightarrow \beta cba$,

β4: $\beta abc \wedge \beta acd \Rightarrow \beta bcd$,

β5: $\beta abc \wedge \beta bcd \wedge b \neq c \Rightarrow \beta abd$.

Here, $\beta xyz$ means that $y$ lies between $x$ and $z$. If, for every $a$, $b$, there is only a finite number of elements between $a$ and $b$, we call the space $(X, \beta)$ discrete.

The betweenness relation $\beta$ may be called connected, or linear, whenever the additional condition

L: $\beta abc \vee \beta bca \vee \beta cab$

is also fulfilled. The axiom system β2–β5, L for linear betweenness has appeared already in [1], [4], where it is proved that each of these axioms is independent of the others and that β1 follows from L and β2. Clearly, β1 cannot be derived from the conditions β2–β5 alone, hence, our axiom system is also independent. In [2] we considered betweenness spaces in which $\beta$ fulfils, instead of L, two weaker conditions of smoothness

IS: $\beta acb \wedge \beta adb \Rightarrow \beta acd \vee \beta adc$,

OS: $\beta abc \wedge \beta abd \Rightarrow \beta acd \vee \beta adc$,

both being consequences of β2–β5, L (cf. [4]). Here we shall deal with spaces in which, in addition to β1–β5, the following axiom for $\beta$ is valid:

M: $\exists x(\beta axb \wedge \beta bxc \wedge \beta cxa)$.

We call these spaces M-*spaces*. Obviously, β1 is a consequence of M and β2. It will be shown that the notion of an M-space is equivalent to that of a tree algebra [3], [5], and, using a result from [5], a one-to-one correspondence between discrete M-spaces and trees will be established.

[1]) Here, as well as throughout the whole paper, we omit the universal quantifiers which might be placed in front of a formula to bound the free variables occurring in it.

In what follows, let $(X, \beta)$ be a fixed betweenness space, if not otherwise stated. Elements of $X$ will be referred to as points. For brevity, we write $xyz$ for $\beta xyz$. Those properties of $\beta$ that will be needed below are summarized in the following lemma, where $\mu \subset X^4$ is a relation on $X$ defined by

$$\mu abcp \Leftrightarrow apb \wedge bpc \wedge cpa .$$

**Lemma.** *For arbitrary $a, b, c, p, q \in X$ we have:*

μ1: $\mu abcp \Leftrightarrow \mu bacp \Leftrightarrow \mu acbp,$

μ2: $\mu abcb \Leftrightarrow abc,$

μ3: $\mu abcp \wedge cdp \Rightarrow \mu abdp,$

μ4: $\mu abcp \wedge bcd \wedge c \neq p \Rightarrow \mu abdp,$

μ5: $\mu abcp \wedge cqp \wedge bqd \wedge p \neq q \Rightarrow \mu abdp,$

μ6: $\mu abcp \wedge abd \wedge acd \Rightarrow p = b \vee p = c.$

*If $\beta$ fulfils the condition* M, *then* IS *holds and, moreover, the following implications are valid*:

μ7: $\mu abcp \wedge bdc \Rightarrow apd,$

μ8: $\mu abcp \wedge \mu abcq \Rightarrow p = q,$

μ9: $\mu abcp \wedge \mu abdq \wedge apd \Rightarrow \mu acqp,$

μ10: $\mu abcp \wedge \mu abdq \wedge p \neq q \Rightarrow \mu bcdp \vee \mu bcdq.$

Proof. μ1 and μ2 are obvious.

| | | |
|---|---|---|
| μ3: | $cpa \wedge cdp \Rightarrow dpa$ | [β4] |
| | $cpb \wedge cdp \Rightarrow dpb$ | [β4] |
| | $apb \wedge dpa \wedge dpb \Rightarrow \mu abdp.$ | [μ1] |
| μ4: | $bpc \wedge bcd \Rightarrow pcd$ | [β4] |
| | $apc \wedge pcd \wedge c \neq p \Rightarrow apd$ | [β5] |
| | $bpc \wedge pcd \wedge c \neq p \Rightarrow bpd$ | [β5] |
| | $apb \wedge bpd \wedge apd \Rightarrow \mu abdp.$ | [μ1] |
| μ5: | $\mu abcp \wedge cqp \Rightarrow \mu abqp$ | [μ3] |
| | $\mu abqp \wedge bqd \wedge p \neq q \wedge \mu abdp.$ | [μ4] |
| μ6: | $abd \wedge apb \Rightarrow pbd$ | [β4] |
| | $cpb \wedge pbd \Rightarrow p = b \vee cpd$ | [β5] |
| | $apc \wedge acd \Rightarrow pcd$ | [β4] |
| | $cpd \wedge pcd \Rightarrow p = c \vee dpd$ | [β3, β5] |
| | $dpd \wedge pcd \Rightarrow p = c.$ | [β2, β2] |
| IS: | $\mu acdx_0$ | [M] |
| | $\mu acdx_0 \wedge acb \wedge adb \Rightarrow x_0 = c \vee x_0 = d$ | [μ6] |
| | $\mu acdx_0 \wedge (x_0 = c \vee x_0 = d) \Rightarrow acd \vee adc.$ | [μ1, μ2] |
| μ7: | $bpc \wedge bdc \Rightarrow bpd \vee bdp$ | [IS] |
| | $bdp \wedge bpa \Rightarrow apd$ | [β4, β3] |
| | $bpd \wedge bdc \Rightarrow pdc$ | [β4] |
| | $pdc \wedge apc \Rightarrow apd.$ | [β3, β4] |

μ8: $\mu abcp \wedge bqc \Rightarrow apq$ [μ7]
$\mu abcq \wedge bpc \Rightarrow aqp$ [μ7]
$apq \wedge aqp \Rightarrow p = q \vee apa$ [β3, β5]
$apa \wedge aqp \Rightarrow p = q.$ [β2, β2]
μ9: $\mu badq \wedge apd \Rightarrow bqp$ [μ7]
$\mu acbp \wedge bqp \Rightarrow \mu acqp.$ [μ3]
μ10: $apb \wedge aqb \Rightarrow apq \vee aqp$ [IS]
$\mu dbaq \wedge apq \wedge bpc \wedge p \neq q \Rightarrow \mu bcdq$ [μ5, μ1]
$\mu cbap \wedge aqp \wedge bqp \wedge p \neq q \Rightarrow \mu bcdp.$ [μ5, μ1]

We say that $p$ is the *median* of the points $a$, $b$, $c$, if $p$ is the unique point that satisfies the condition $\mu abcp$. From μ8 we get

**Corollary.** *$(X, \beta)$ is an* M-*space if and only if every three points of $X$ have the median.*

Following [3], we call a pair $(X, m)$ a *tree algebra*, if $m: X \to X$ is a ternary operation on $X$ which satisfies the following axioms (we write $(xyz)$ for $m(xyz)$):
m1: $(aab) = a$,
m2: $(abc) = (bac) = (acb)$,
m3: $((abc)\,bd) = (ab(cbd))$,
m4: $(abd) \neq (bcd) \neq (acd) \Rightarrow (abd) = (acd)$.
Then the operation $m$ is said to be a *median operation.* As in [5], we omit the condition (explicit in [3]) that $X$ must be finite. Note that m4 may be rewritten in the form
m4′: $(abd) = (bcd) \vee (bcd) = (acd) \vee (abd) = (acd)$.

Any median operation $m$ has the following properties:
m5: $((abc)\,bc) = (abc)$, [m3, m2, m1]
m6: $(acd) = (bcd) \Rightarrow (abc) = (abd)$.
For m6 see [3], Theorem 1.3. Now we shall prove the main

**Theorem.** *Let $m$ be a ternary operation, and let $\beta$ be a ternary relation on $X$. Then*

a) *if $(X, m)$ is a tree algebra, and if $\beta$ is defined by*

(*) $$\beta abc \Leftrightarrow m(abc) = b\,,$$

*then $(X, \beta)$ is an* M-*space, and the condition*

(**) $$m(abc) = p \Leftrightarrow \beta apb \wedge \beta bpc \wedge \beta cpa$$

*holds;*

b) *if $(X, \beta)$ is an* M-*space, and if $m$ is defined by* (**), *then $(X, m)$ is a tree algebra, and $\beta$ fulfils* (*).

Proof. (a) Assume $m$ is a median operation and $\beta$ fulfils (*). Then β1 – β3 easily follow from μ1 and μ2. Furthermore, if $(abc) = b$ and $(acd) = c$, then

$$(bcd) = ((abc)\,cd) = (bc(acd)) = (bcc) = c\,;$$

hence, β4 is valid. To prove β5, assume that $(abc) = b$, $(bcd) = c$, $b \neq c$. Then $(abd) \neq c$, for otherwise, owing to m5, we should have

$$b = (abc) = (ab(abd)) = (abd) = c\,.$$

Hence, $(acb) \neq (cdb) \neq (adb)$, and, in virtue of m4, $(abd) = b$. To prove M, let $x_0 = (abc)$. Then by m5, $(ax_0b) = x_0$, $(bx_0c) = x_0$, $(cx_0a) = x_0$. Finally, (**) now means that

$$(abc) = p \Leftrightarrow (apb) = (bpc) = (cpa) = p\,.$$

By m5 the left hand equality implies the right hand ones. The converse follows from m6: if $(apb) = (bpc)$, then $(abc) = (cpa) = p$.

(b) Assume $\beta$ is a betweenness and $m$ fulfils (**). Let us check that m1 – m4 and (*) are valid. By μ1, μ2 we have $\mu abbb$, hence, by μ8, $\mu abbp$ implies $p = b$, and m1 follows. m2 means that

$$\mu abcp \wedge \mu bacq \wedge \mu acbr \Rightarrow p = q = r\,,$$

and this is true in virtue of μ1 and μ8. To prove m3, we need to show that

$$\mu abcp \wedge \mu pbdq \wedge \mu cbdr \wedge \mu abrs \Rightarrow q = s\,.$$

If $p = q$, then

$\mu abcp \Rightarrow \mu bcaq$ [μ1]

$\mu bcaq \wedge \mu bcdr \wedge bqd \Rightarrow \mu barq$ [μ9]

$\mu barq \wedge \mu abrs \Rightarrow q = s.$ [μ8]

If $r = s$, then

$\mu cbdr \Rightarrow \mu dcbs$ [μ1]

$\mu bcds \wedge \mu bcap \wedge bsa \Rightarrow \mu bdps$ [μ9]

$\mu bdps \wedge \mu pbdq \Rightarrow q = s.$ [μ8]

If $p \neq q$ and $r \neq s$, then

$\mu dbpq \wedge bpa \Rightarrow \mu dbaq$ [μ4]

$\mu abrs \wedge brd \Rightarrow \mu abds$ [μ4]

$\mu dbaq \wedge \mu abds \Rightarrow q = s.$ [μ1, μ8]

Furthermore, m4 means that

$$\mu abdp \wedge \mu bcdq \wedge \mu acdr \wedge p \neq q \wedge q \neq r \Rightarrow p = r\,.$$

But we have

$\mu adbp \wedge \mu adcr \wedge p \neq r \Rightarrow \mu dbcp \vee \mu dbcr$ [μ10]

$\mu dbcp \wedge \mu bcdq \Rightarrow p = q$ [μ1, μ8]

$\mu dbcr \wedge \mu bcdq \Rightarrow r = q.$ [μ1, μ8]

Finally, (*) coincides with μ2.

Therefore, there is a one-to-one correspondence between M-spaces and tree algebras. In [5], such a correspondence is established between the so called discrete tree algebras and trees. This result includes the finite as well as infinite case, and is a generalization of a result in [3] for finite trees. The resulting correspondence between discrete M-spaces and trees may be explicitly described as follows. Let $(X, E)$ be a tree, where $X$ is the set of its vertices and $E$ is the set of edges. Let $\beta abc$ mean that there is a path in the tree from $a$ to $c$ passing through $b$. Then $(X, \beta)$ is a discrete M-space. Vice versa, if $(X, \beta)$ is such a space and

$$E = \{(a, b) \in X^2 : a \neq b \wedge \forall x(\beta axb \Rightarrow a = x \vee x = b)\}$$

then $(X, E)$ is a tree.

Added November 5, 1984. In the meantime, several papers, in which ternary spaces and/or ternary algebras are discussed, have appeared. We comment here three of them being more or less closely connected with our main subject. The class of ternary spaces considered in [6] includes our betweenness spaces and, hence, M-spaces as well. Furthermore, every tree algebra is a medium in the sense of [6]. Theorem 2.1 [6] asserts that any medium is a ternary space, and Proposition 3.5 shows when a discrete ternary space is the ternary space of a medium. Some results on tree algebras are contained in Sect. 6 of [7]; this paper has also a valuable bibliography. In [8], a theorem from [5] is disproved concerning independence of a certain system of conditions on segments in tree algebras.

The author is indebted to the referee for indicating two inaccuracies in the proof of Lemma.

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