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EDGE ROTATIONS AND DISTANCE BETWEEN GRAPHS

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INTRODUCTION

In [1] Zelinka introduced the following definition of distance between two graphs of the same order. Let G_1 and G_2 be two graphs of order p. Then the distance $\delta(G_1, G_2)$ between G_1 and G_2 is n $(0 \leqq n \leqq p - 1)$ if $p - n$ is the order of a largest graph that is an induced subgraph of both G_1 and G_2.

Zelinka showed that on the family of graphs having a fixed order, the above distance function δ produces a metric space. He further showed for graphs G_1 and G_2 of order p that $\delta(G_1, G_2) \leqq p - 1$ and $\delta(G_1, G_2) = \delta(\bar{G}_1, \bar{G}_2)$, where $\bar{G}$ denotes the complement of G.

In this paper, we introduce a new distance function defined on graphs having the same order and the same size (number of edges).

EDGE ROTATIONS AND TRANSFORMATIONS

We say that a graph G can be *transformed into a graph H by an edge rotation* if G contains distinct vertices u, v and w such that $uv \in E(G)$, $uw \notin E(G)$ and $H \cong G - uv + uw$. In this case, G is transformed into H by "rotating" the edge uv of G into uw. Observe that a graph G can be transformed into some graph H by an edge rotation if and only if G is neither complete nor empty.

Figure 1 shows graphs G, H_1 and H_2. Note that G can be transformed into H_1 by an edge rotation (xy is rotated into xz). Also, G can be transformed into H_2 by an edge rotation (xw is rotated into xz). Further observe that $G \ncong H_1$ and $G \cong H_2$.

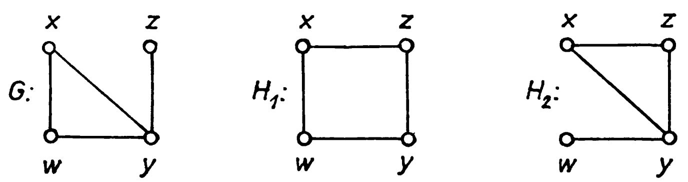

Figure 1

It is immediate that a graph G can be transformed into a graph H by an edge rotation if and only if H can be transformed into G by an edge rotation. More generally, we say simply that G_1 can be *transformed* into G_2, written $G_1 \to G_2$, if either (1) $G_1 \cong G_2$, or (2) there exists a sequence

$$G_1 \cong H_0, H_1, \ldots, H_n \cong G_2 \; (n \geqq 1) \text{ of graphs such that } H_i$$

can be transformed into H_{i+1} by an edge rotation for $i = 0, 1, \ldots, n - 1$. It is obvious that the relation "can be transformed into" is an equivalence relation on the set of all graphs. Moreover, if G_1 and G_2 are graphs for which $G_1 \to G_2$, then clearly G_1 and G_2 have the same order (the same number of vertices) and the same size (the same number of edges). It is perhaps less clear that the converse of the preceding implication is also true.

Proposition 1. *Let G_1 and G_2 be graphs having the same order and the same size. Then $G_1 \to G_2$.*

Proof. If $G_1 \cong G_2$, then $G_1 \to G_2$; so we may assume, without loss of generality, that $G_1 \not\cong G_2$. Suppose that G_1 (and G_2) has order p and size q (necessarily $p \geqq 4$ and $q \geqq 2$). Without loss of generality, we assume that $V(G_1) = V(G_2) = \{v_1, v_2, \ldots \ldots, v_p\}$.

For the complete graph K_p having the vertex set $\{v_1, v_2, \ldots, v_p\}$, we say that an edge $v_a v_b$ $(a < b)$ is less than an edge $v_c v_d$ $(c < d)$, written $v_a v_b < v_c v_d$, if either (i) $a < c$ or (ii) $a = c$ and $b < d$. This produces a linear ordering of the edges e_i, $i = 1, 2, \ldots, \binom{p}{2}$, of K_p, where

$$v_1 v_2 = e_1 < e_2 < e_3 < \ldots < e_{\binom{p}{2}} = v_{p-1} v_p \,. \tag{1}$$

We say that the *weight* of the edge e_i is i. Further, if G is a graph having the vertex set $\{v_1, v_2, \ldots, v_p\}$, then the *weight* of G is defined to be the sum of the weights of its edges, where the weights are determined by (1).

Define the graph H to have the vertex set $\{v_1, v_2, \ldots, v_p\}$ and the q smallest edges of K_p as defined in (1), i.e., $E(H) = \{e_1, e_2, \ldots, e_q\}$. Note that H has weight $\sum_{i=1}^{q} i$. We now show that $G_1 \to H$. Suppose, to the contrary, that G_1 *cannot* be transformed into H. Then let F be a graph with $V(F) = \{v_1, v_2, \ldots, v_p\}$ and minimum weight w such that $G_1 \to F$. Therefore, $w > \sum_{i=1}^{q} i$. This implies that there exist edges $v_a v_b$ and $v_c v_d$ such that $v_a v_b \notin E(F)$, $v_c v_d \in E(F)$ and $v_a v_b < v_c v_d$. Let $F^* = F + v_a v_b - v_c v_d$. We show that $F \to F^*$. Since $G_1 \to F$, this implies that $G_1 \to F^*$. Howeer, since F^* has smaller weight than F, a contradiction is produced, yielding the desired result that $G_1 \to H$. We consider two cases.

Case 1. *Suppose that $a = c$.* Thus $b < d$. By rotating the edge $v_c v_d$ into $v_a v_b$, the graph F is transformed into F^*.

Case 2. *Suppose that $a < c$.* If $b = d$ or $b = c$, then, as in Case 1, we may rotate the edge $v_c v_d$ into $v_a v_b$ so that F is transformed into F^*. Assume, then, that $b \neq d$ and $b \neq c$ so that v_a, v_b, v_c and v_d are four distinct vertices. If $v_b v_d \notin E(F)$, then we may rotate $v_c v_d$ into $v_b v_d$, and then rotate $v_b v_d$ into $v_a v_b$, thereby concluding that F can be transformed into F^*. If $v_b v_d \in E(F)$, then we rotate $v_b v_d$ into $v_a v_b$, after which we rotate $v_c v_d$ into $v_b v_d$, again showing that F can be transformed into F^*.

We now have that $G_1 \to H$. Likewise, $G_2 \to H$. From this, it follows that $G_1 \to G_2$. ■

DISTANCE BETWEEN GRAPHS

Let G_1 and G_2 be two graphs having the same order and the same size. We define the *distance* $d(G_1, G_2)$ between G_1 and G_2 as 0 if $G_1 \cong G_2$ and, otherwise, as the smallest positive integer n for which there exists a sequence $H_0, H_1, \ldots, H_n$ of graphs such that $G_1 \cong H_0$, $G_2 \cong H_n$, and H_i can be transformed into H_{i+1} by an edge rotation for $i = 0, 1, \ldots, n - 1$. By Proposition 1, this "distance" is a well-defined concept. Further, if $\mathcal{G}_{p,q}$ is the set of all graphs having order p and size q, for some fixed integers p and q, then $(\mathcal{G}_{p,q}, d)$ is a metric space.

We make the following observation concerning complements of graphs.

Proposition 2. *Let G_1 and G_2 be two graphs having the same order and the same size. Then*

$$d(G_1, G_2) = d(\bar{G}_1, \bar{G}_2).$$

Proof. If $d(G_1, G_2) = 0$ then $G_1 \cong G_2$, implying that $\bar{G}_1 \cong \bar{G}_2$ and $d(\bar{G}_1, \bar{G}_2) = 0$. Assume then that $d(G_1, G_2) = n \geqq 1$. Hence there exists a sequence

$$G_1 \cong H_0, H_1, \ldots, H_n \cong G_2,$$

where H_i can be transformed into H_{i+1} by an edge rotation for $i = 0, 1, \ldots, n - 1$, where, say, $H_{i+1} = H_i - u_i v_i + u_i w_i$. Observe that $\bar{H}_{i+1} = \bar{H}_i - u_i w_i + u_i v_i$, i.e., $\bar{H}_i$ can be transformed into $\bar{H}_{i+1}$ by an edge rotation. Thus the sequence

$$\bar{G}_1 \cong \bar{H}_0, \bar{H}_1, \ldots, \bar{H}_n \cong \bar{G}_2 \tag{2}$$

implies that $d(\bar{G}_1, \bar{G}_2) \leqq d(G_1, G_2) = n$.

Now by applying the above technique to the sequence (2), we have $d(\bar{\bar{G}}_1, \bar{\bar{G}}_2) \leqq$ $\leqq d(\bar{G}_1, \bar{G}_2)$ or

$$n = d(G_1, G_2) \leqq d(\bar{G}_1, \bar{G}_2) = n,$$

producing the desired result. ■

Next we show that any nonnegative integer is the distance between some pair of graphs.

Proposition 3. *For every nonnegative integer n, there exist graphs* G_1 *and* G_2 *such that* $d(G_1, G_2) = n$.

Proof. If $n = 0$, then for every graph G, $d(G, G) = 0$, so take $G_1 = G_2 = G$. Let $n \geqq 1$ be given. Let $G_1 = (n + 1)K_2$ and $G_2 = K(1, n + 1) \cup nK_1$, so that G_1 and G_2 are graphs of order $2n + 2$ and size $n + 1$. Suppose that $E(G_1) = \{u_0v_0, u_1v_1, \ldots, u_nv_n\}$. Let $H_0 = G_1$ and for $i = 0, 1, \ldots, n - 1$, define

$$H_{i+1} = H_i - u_{i+1}v_{i+1} + u_0v_{i+1} .$$

Note that $H_n \cong G_2$ so that $d(G_1, G_2) \leqq n$. On the other hand, every edge rotation of a graph G produces a graph H such that $|\deg_G v - \deg_H v| \leqq 1$ for every vertex v of G_1. Since G_1 is 1-regular and G_2 contains a vertex of degree $n + 1$, at least n edge rotations are required to transform G_1 into G_2. Thus $d(G_1, G_2) \geqq n$ and the result follows. ∎

In order to present an upper bound for the distance between graphs (having the same order and size), we introduce a new concept. For nonempty graphs G_1 and G_2, we define *a greatest common subgraph* of G_1 and G_2 as any graph G of the maximum size without isolated vertices that is a subgraph of both G_1 and G_2.

While every pair G_1, G_2 of nonempty graphs has a greatest common subgraph, such a subgraph need not be unique. For example, the graphs G_1 and G_2 of Figure 2 (of order 7 and size 6) have three greatest common subgraphs, namely G, G' and G''. Although these subgraphs are all different, they, of course, possess the same maximum size, namely 3, in this case.

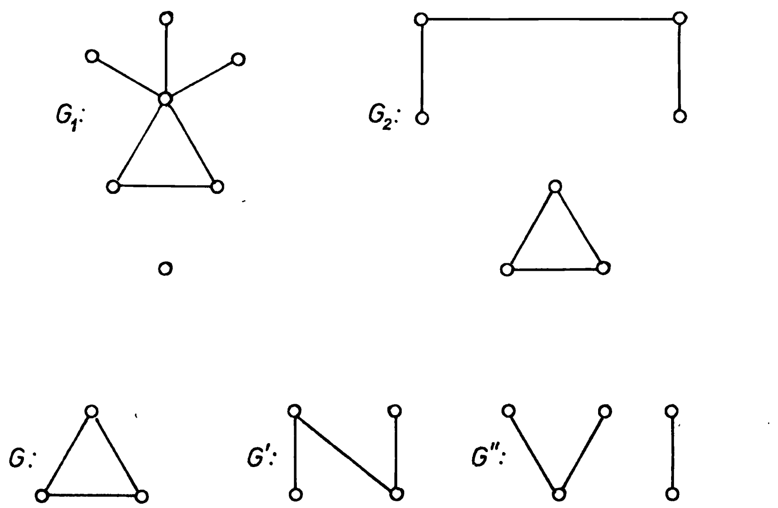

Figure 2

The main reason for introducing greatest common subgraphs lies in the following result.

Proposition 4. *Let G_1 and G_2 be graphs having order p and size $q \geqq 1$, and let G be a greatest common subgraph of G_1 and G_2, where G has size s. Then $d(G_1, G_2) \leqq 2(q - s)$.*

Proof. If $s = q$, then $G_1 \cong G_2$ and $d(G_1, G_2) = 0$. Thus, we assume that $1 \leqq s < q$. Let the vertices of G_1 and G_2 be labeled $v_1, v_2, \ldots, v_p$ so that subgraphs of G_1 and G_2 isomorphic to G are identically labeled. Since $G_1 \not\cong G_2$, the graph G_1 contains an edge $v_i v_j$ that is not in G_2 and G_2 contains an edge $v_k v_l$ that is not in G_1.

Suppose that $v_j = v_k$. Then G_1 can be transformed into $G_1' = G_1 - v_i v_j + v_j v_l$ by an edge rotation and $d(G_1, G_1') = 1$. Hence we may assume that $\{v_i, v_j\} \cap \cap \{v_k, v_l\} = \emptyset$.

Suppose that at least one of v_i and v_j is not adjacent in G_1 to at least one of v_k and v_l; say $v_i v_k \notin E(G_1)$. Then G_1 can be transformed into $G_1^* = G_1 - v_i v_j + v_i v_k$ by rotating $v_i v_j$ into $v_i v_k$, and G_1^* can be transformed into $G_1^{**} = G_1^* - v_i v_k + v_k v_l$ by rotating $v_i v_k$ into $v_k v_l$. Thus $d(G_1, G_1^{**}) \leqq 2$.

Assume then that each of v_i and v_j is adjacent to both v_k and v_l. The graph G_1 can be transformed into $G_1' = G - v_i v_k + v_k v_l$ by rotating $v_i v_k$ into $v_k v_l$, and G_1' can be transformed into $G_1'' = G_1' - v_i v_j + v_i v_k$ by rotating $v_i v_j$ into $v_i v_k$. Therefore, $d(G_1, G_1'') \leqq 2$.

Hence, in any case, G_1 can be transformed into $H_1 = G_1 - v_i v_k + v_k v_l$ and $d(G_1, H_1) \leqq 2$. The graphs H_1 and G_2 have $s + 1$ edges in common. Proceeding as above, we construct a graph H_2 such that $d(G_1, H_2) \leqq 4$, and H_2 and G_2 have $s + 2$ edges in common. Continuing in this manner, we construct a graph $H_{q-s} = G_2$ such that $d(G_1, G_2) \leqq 2(q - s)$. ■

The bound presented in the previous result cannot be improved in general, for if $n \geqq 1$, define

$$G_1 = K_{2n} \cup \bar{K}_{4n^2-4n} \quad \text{and} \quad G_2 = (2n^2 - n) K_2 .$$

Observe that each of G_1 and G_2 has order $4n^2 - 2n$ and size $q = 2n^2 - n$. In this case, G_1 and G_2 have a unique greatest common subgraph $G = nK_2$, which has size $s = n$. Therefore,

$$2(q - s) = 2[(2n^2 - n) - n] = 4n^2 - 4n .$$

The graph G_2 is 1-regular, while G_1 contains $4n^2 - 4n$ isolated vertices. Therefore, $d(G_1, G_2) \geqq 4n^2 - 4n$. By Proposition 4, $d(G_1, G_2) \leqq 2(q - s) = 4n^2 - 4n$, so that $d(G_1, G_2) = 2(q - s)$.

Reference

[1] *B. Zelinka:* On a certain distance between isomorphism classes of graphs. Časopis Pěst. Mat. *100* (1975) 371–373.

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