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ON A CODIMENSION THREE BIFURCATION

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In this paper we study unfoldings of the vector field

$$\dot{x} = X_0(x) = Ax + G(x),$$

where $x = (x_1, x_2, x_3)$, $G \in C^\infty$, $G(x) = o(\|x\|)$, the matrix $A = (a_{ij})$ is equivalent to the nilpotent matrix S with 1's just above the diagonal and 0's elsewhere. Some results contained in this paper have been announced in [15]. The unfoldings of the above vector field, possessing symmetry under the change of sign, $X_0(x) = -X_0(-x)$, are studied in [16].

Under generic hypotheses on the quadratic terms, we derive a normal form for unfoldings of X_0 which enables us to find the bifurcation diagram of the critical points. We show that generically there is a curve $Z_2(Z_{1c})$ in the parameter space, where the linear part of the corresponding vector field, computed at a critical point, has zero as an eigenvalue of multiplicity two (a couple of pure imaginary eigenvalues and one zero eigenvalue). Using Bogdanov's results [3] we describe the bifurcations near the curve Z_2. The case of the codimension two singularity, which occurs on the curve Z_{1c}, is more complicated. It has been partially solved by several authors [5], [7–10]. There are a number of different cases of very complicated bifurcations near the curve Z_{1c}. The problem of global bifurcations of the phase portraits when the parameter goes from a nieghbourhood of Z_2 to a neighbourhood of Z_{1c} remains open.

1. PRELIMINARY LEMMAS

Consider an unfolding of the vector field X_0, represented by the three-parameter family of vector fields

$$\dot{x} = f(x, \varepsilon), \tag{1.1}$$

where $f = (f_1, f_2, f_3) \in C^\infty$, $x = (x_1, x_2, x_3)$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$. We also write $f_\varepsilon(x)$ instead of $f(x, \varepsilon)$.

The vector field $f_0 = X_0$ may be rewritten as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} (Px, x) + h_1(x) \\ (Qx, x) + h_2(x) \\ (Rx, x) + h_3(x) \end{bmatrix}, \tag{1.2}$$

where $P = (p_{ij})$, $Q = (q_{ij})$, $R = (r_{ij})$ are symmetric matrices, $A = (a_{ij})$, $(\cdot, \cdot)$ is the scalar product in R^3, $h_i(x) = o(\|x\|^2)$, $i = 1, 2, 3$.

There exists a linear change of coordinates $y = Nx$ such that (1.2) becomes

$$(1.3) \qquad Y_0: \begin{aligned} \dot{y}_1 &= y_2 + (\tilde{P}y, y) + g_1(y), \\ \dot{y}_2 &= y_3 + (\tilde{Q}y, y) + g_2(y), \\ \dot{y}_3 &= \qquad (\tilde{R}y, y) + g_3(y), \end{aligned}$$

$$\begin{bmatrix} (\tilde{P}y, y) \\ (\tilde{Q}y, y) \\ (\tilde{R}y, y) \end{bmatrix} = N \begin{bmatrix} ((N^{-1})' PN^{-1}y, y) \\ ((N^{-1})' QN^{-1}y, y) \\ ((N^{-1})' RN^{-1}y, y) \end{bmatrix}, \quad g_i(y) = o(\|y\|^2), \quad i = 1, 2, 3,$$

$(N^{-1})'$ is the transpose of N^{-1}.

Lemma 1. *There exists a smooth local diffeomorphism Φ transforming the vector field Y_0 to*

$$(1.4) \qquad \Phi_* Y_0: \dot{x}_1 = x_2, \ \dot{x}_2 = x_3, \ \dot{x}_3 = (Tx, x) + h(x),$$

where $T = \tilde{R} + T_0$, $h(x) = o(\|x\|^2)$, $T_0 = (t_{ij}^0)$ is a symmetric matrix with $t_{11}^0 = 0$, $t_{12}^0 = \tilde{q}_{11}$, $t_{13}^0 = \tilde{p}_{11} + \tilde{q}_{12}$, $\tilde{P} = (\tilde{p}_{ij})$, $\tilde{Q} = (\tilde{q}_{ij})$, $\tilde{R} = (\tilde{r}_{ij})$.

Proof. The diffeomorphism $H: z_1 = y_1$, $z_2 = y_2 + (\tilde{P}y, y) + g_1(y)$, $z_3 = y_3$ transforms (1.3) to the vector field $H_* Y_0(z) = z_2\, \partial/\partial z_1 + (z_3 + (\hat{Q}z, z) + \hat{g}_2(z))\, \partial/\partial z_2 + (\hat{R}z, z) + \hat{g}_3(z))\, \partial/\partial z_3$, where $\hat{g}_i(z) = o(\|z\|^2)$, $i = 2, 3$, $\hat{Q} = \tilde{Q} + \hat{P}$, $\hat{R} = \tilde{R}$, $\hat{P} = (\hat{p}_{ij})$ is a symmetric matrix with $\hat{p}_{11} = 0$, $\hat{p}_{12} = \tilde{p}_{11}$, $\hat{p}_{13} = \tilde{p}_{12}$, $\hat{p}_{22} = 2\tilde{p}_{12}$, $\hat{p}_{23} = \tilde{p}_{13} + \tilde{p}_{22}$, $\hat{p}_{33} = 2\tilde{p}_{23}$. Then $\Phi = K \circ H$, where $K: x_1 = z_1$, $x_2 = z_2$, $x_3 = z_3 + (\hat{Q}z, z) + \hat{g}_2(z)$.

Lemma 2. *Let $T = (t_{ij})$ be the matrix from (1.4). Then the numbers $q = t_{1j}/t_{11}$, $j = 2, 3$, are invariant with respect to regular transformations of coordinates in the phase space that keep the origin fixed.*

Proof. Consider a diffeomorphism of the form $R: y_i = x_i + X_i(x) + o(\|x\|^2)$, $i = 1, 2, 3$, where the functions X_i are homogeneous polynomials of degree 2. We assume that R maps (1.4) to a vector field of the same form. Any diffeomorphism transforming (1.2) to the form (1.4) and keeping the origin fixed, is composed of the mapping $\Psi = \Phi \circ N$, where Φ, N are as above, of a mapping of the form R and of a linear mapping ϱ, which does not change the linear part of the vector field (1.4). The mapping ϱ must be of the form $\varrho(x) = Dx$, where $D = (d_{ij})$, $d_{kk} = \lambda$, $k = 1, 2, 3$, $d_{12} = d_{23} = \varepsilon$, $d_{13} = \delta$, $d_{ij} = 0$ for $i > j$, $\lambda, \varepsilon, \delta$ are real numbers, $\lambda \neq 0$. It suffices to prove the invariance of q with respect to the mappings R and ϱ.

By using the fact that the mapping R preserves the form (1.4) it is easy to check that

$$X_{i+1}(x) = \frac{\partial X_i}{\partial x_1} x_2 + \frac{\partial X_i}{\partial x_2} x_3, \quad i = 1, 2,$$

and therefore

$$X_3(x) = \frac{\partial^2 X_1}{\partial x_1^2} x_2^2 + 2 \frac{\partial^2 X_1}{\partial x_1 \partial x_2} x_2 x_3 + \frac{\partial^2 X_1}{\partial x_2^2} x_3^2 + \frac{\partial X_1}{\partial x_1} x_3 .$$

This implies that the vector field $R_*(\Phi_* Y_0)$ has the form (1.4) with a matrix $T' = (t'_{ij})$ instead of the matrix $T = (t_{ij})$ and $t'_{1j} = t_{1j}$, $j = 1, 2, 3$, i.e. the mapping R does not change the numbers t_{1j}, $j = 1, 2, 3$.

It remains to prove the invariance of q with respect to the mapping ϱ. This mapping transforms the vector field (1.4) to the form (1.3), where $\tilde{P} = (\tilde{p}_{ij}) = \delta \tilde{T}$, $\tilde{Q} = (\tilde{q}_{ij}) = \varepsilon \tilde{T}$, $\tilde{R} = (\tilde{r}_{ij}) = \lambda \tilde{T}$, $\tilde{T} = (\tilde{t}_{ij}) = (D^{-1})' T D^{-1}$, $\tilde{p}_{11} = \delta \lambda^{-2} t_{11}$, $\tilde{q}_{11} = \varepsilon \lambda^{-2} t_{11}$, $\tilde{q}_{12} = -\varepsilon^2 \lambda^{-3} t_{11} + \varepsilon \lambda^{-2} t_{12}$, $\tilde{r}_{12} = -\varepsilon \lambda^{-2} t_{11} + \lambda^{-1} t_{12}$, $r_{11} = \lambda^{-1} t_{11}$, $\tilde{r}_{13} = (\varepsilon^2 \lambda^{-3} - \delta \varepsilon \lambda^{-2})\, t_{11} - \varepsilon \lambda^{-2} t_{12} + \lambda^{-1} t_{13}$. By Lemma 1 there exists a smooth local diffeomorphism transforming the vector field to the form (1.4), with a matrix $\hat{T} = (\hat{t}_{ij}) = \tilde{R} + T_0$ instead of the matrix T, and the first row of the matrix T_0 is $(0, \tilde{q}_{11}, \tilde{q}_{12} + \tilde{p}_{11})$. Therefore $\hat{t}_{11} = \lambda^{-1} t_{11}$, $\hat{t}_{12} = \tilde{r}_{12} + \tilde{q}_{11} = \lambda^{-1} t_{12}$, $\hat{t}_{13} = \tilde{r}_{13} + \tilde{q}_{12} + \tilde{p}_{11} = \lambda^{-1} t_{13}$ and the proof is complete.

2. NORMAL FORM

By Lemma 1 the family (1.1) may be written in the form

$$(2.1) \qquad \begin{aligned} \dot{x}_1 &= x_2 + v_1(x, \varepsilon) , \\ \dot{x}_2 &= x_3 + v_2(x, \varepsilon) . \\ \dot{x}_3 &= t_{11} x_1^2 + t_{12} x_1 x_2 + t_{13} x_1 x_3 + t_{23} x_2 x_3 + t_{22} x_2^2 + t_{33} x_3^2 + v_3(x, \varepsilon) , \end{aligned}$$

where $v_i(x, 0) \equiv 0$, $i = 1, 2$, $v_3(x, 0) = o(\|x\|^2)$.

Assuming $t_{11} \neq 0$, we may introduce new coordinates $y = t_{11} x$ and then (2.1) becomes

$$(2.2) \qquad \begin{aligned} \dot{y}_1 &= y_2 + \tilde{v}_1(y, \varepsilon) , \\ \dot{y}_2 &= y_3 + \tilde{v}_2(y, \varepsilon) , \\ \dot{y}_3 &= y_1^2 + \omega_1 y_1 y_2 + \omega_2 y_1 y_3 + \tilde{t}_{23} y_2 y_3 + \tilde{t}_{22} y_2^2 + \tilde{t}_{33} y_3^3 + \tilde{v}_3(y, \varepsilon) , \end{aligned}$$

where $\tilde{v}_i(y, 0) \equiv 0$, $i = 1, 2$, $\tilde{v}_3(y, 0) = o(\|y\|^2)$, $\omega_j = t_{1j+1}/t_{11}$, $j = 1, 2$, are invariants of the germ, represented by the family (1.1).

Introducing again new coordinates $u_1 = y_1$, $u_2 = y_2 + \tilde{v}_1(y, \varepsilon)$, $u_3 = y_3$, we obtain a family of the form (2.2) with $\tilde{v}_1 \equiv 0$. Transforming the resulting family by the diffeomorphism $z_1 = u_1$, $z_2 = u_2$, $z_3 = u_3 + \tilde{v}_2(y, \varepsilon)$, we get a family of the form (2.2) with $\tilde{v}_1 \equiv 0$, $\tilde{v}_2 \equiv 0$. This family may be written in the form

$$(2.3) \qquad \begin{aligned} \dot{z}_1 &= z_2 , \\ \dot{z}_2 &= z_3 , \\ \dot{y}_3 &= \tilde{F}(z_1, \varepsilon) + z_2 \tilde{Q}_1(z_1, \varepsilon) + z_3 \tilde{Q}_2(z_1, \varepsilon) + z_2 \tilde{Q}_3(z_3, \varepsilon) + \\ &\quad + z_2^2 \tilde{\Psi}_1(z, \varepsilon) + z_3^2 \tilde{\Psi}_2(z, \varepsilon) , \end{aligned}$$

where $\tilde{F}$, $\tilde{Q}_i$, $\tilde{\Psi}_j$, $i = 1, 2, 3$, $j = 1, 2$, are C^∞-functions,

$$\tilde{F}(0,0) = \frac{\partial \tilde{F}(0,0)}{\partial z_1} = 0\,, \quad \frac{\partial^2 \tilde{F}(0,0)}{\partial z_1^2} = 2\,, \quad \frac{\partial \tilde{Q}_i(0,0)}{\partial z_1} = \omega_i\,,$$

$$i = 1, 2\,, \quad \tilde{Q}_k(0,0) = 0\,, \quad k = 1, 2, 3\,.$$

Lemma 3. *If* $\omega_1 \neq 0$, $\omega_2 \neq 0$, *then there exists a smooth regular mapping* $y = y(z, \varepsilon)$, $y(0,0) = 0$, *transforming the family* (2.3) *to the form*

$$\begin{aligned}
\dot{y}_1 &= y_2\,,\\
\dot{y}_2 &= y_3\,,\\
\dot{y}_3 &= F(y_1, \varepsilon) + \beta(\varepsilon)\, y_2 + y_1 y_2 G_1(y_1, \varepsilon) + y_1 y_3 G_2(y_1, \varepsilon) + \\
&\quad + y_2 G_3(y_3, \varepsilon) + y_2^2 \Psi_1(y, \varepsilon) + y_3^2 \Psi_2(y, \varepsilon)\,,
\end{aligned} \tag{2.4}$$

where F, G_1, G_2, G_3, β, Ψ_1, Ψ_2 *are smooth functions,*

$$F(0,0) = \frac{\partial F(0,0)}{\partial y_1} = 0\,, \quad \frac{\partial^2 F(0,0)}{\partial y_1^2} = 2\,,$$

$$G_i(0,0) = \omega_i\,, \quad i = 1, 2\,, \quad \beta(0) = 0\,, \quad G_3(y_3, 0) = 0(|y_3|)\,.$$

Proof. Let $y_1 = z_1 - \alpha(\varepsilon)$, $y_2 = z_2$, $y_3 = z_3$, where α is any smooth function. Then the family (2.3) becomes $\dot{y}_1 = y_2$, $\dot{y}_2 = y_3$, $\dot{y}_3 = \tilde{F}(y_1 + \alpha(\varepsilon), \varepsilon) + y_2 \tilde{Q}_1(y_1 + \alpha(\varepsilon), \varepsilon) + y_3 \tilde{Q}_2(y_1 + \alpha(\varepsilon), \varepsilon) + y_2 \tilde{Q}_3(y_3, \varepsilon) + y_2^2 \hat{\Psi}_1(y, \varepsilon) + y_3^2 \hat{\Psi}_2(y, \varepsilon)$, where $\hat{\Psi}_i(y, \varepsilon) = \tilde{\Psi}_i(y_1 + \alpha(\varepsilon), y_2, y_3, \varepsilon)$, $\tilde{Q}_2(y_1 + \alpha(\varepsilon), \varepsilon) = \tilde{Q}_2(\alpha(\varepsilon), \varepsilon) + y_1 \hat{Q}_2(y_1, \varepsilon)$, $\tilde{Q}_2(0,0) = 0$, $\partial \tilde{Q}_2(0,0)/\partial y_1 = \omega_2$, $\hat{Q}_2(0,0) = \omega_2$. Since $\omega_2 \neq 0$, the implicit function theorem implies that there exists a neighbourhood U of $0 \in R^3$ and a smooth function $\alpha : U \to R^1$ such that $\alpha(\sigma) = 0$, $\tilde{Q}_2(\alpha(\varepsilon), \varepsilon) = 0$ for all $\varepsilon \in U$. From Taylor's expansion of the function $\tilde{Q}_1$ we have $\tilde{Q}_1(y_1 + \alpha(\varepsilon), \varepsilon) = \beta(\varepsilon) + y_1 G_1(y_1, \varepsilon) + o(|y_1|)$, where β, $G_1 \in C^\infty$, $\beta(0) = 0$, $G_1(0,0) = \partial \tilde{Q}_1(0,0)/\partial y_1 = \omega_1$ and so the family obtained has the form (2.4).

If F is the function from Lemma 3, then by the Malgrange-Weierstrass preparation theorem (see [14]) there exist smooth functions $\varphi_i(\varepsilon)$, $\varphi_i(0) = 0$, $i = 1, 2$, $\Theta(y_1, \varepsilon)$, $\Theta(0,0) = 1$, such that $F(y_1, \varepsilon) = (y_1^2 + \varphi_2(\varepsilon)\, y_1 + \varphi_1(\varepsilon))\, \Theta(y_1, \varepsilon)$ and therefore the family (2.4) may be written as

$$\begin{aligned}
\dot{y}_1 &= y_2\,,\\
\dot{y}_2 &= y_3\,,\\
\dot{y}_3 &= (\varphi_1(\varepsilon) + \varphi_2(\varepsilon)\, y_1 + y_1^2 + \varphi_3(\varepsilon)\, y_2 + y_1 y_2 Q_1(y_1, \varepsilon) + \\
&\quad + y_1 y_3 Q_2(y_1, \varepsilon) + y_2 Q_3(y_3, \varepsilon) + y_2^2 \Phi_1(y, \varepsilon) + y_3^2 \Phi_2(y, \varepsilon))\, \Theta(y_1, \varepsilon)\,,
\end{aligned}$$

where Θ, $Q_i \varphi_i$, $\Phi_j \in C^\infty$, $\varphi_i(0) = 0$, $\Phi_j(0,0) = 0$, $i = 1, 2, 3$, $j = 1, 2$, $Q_k(0,0) = \omega_k$, $k = 1, 2$, $\Theta(0,0) = 1$.

We have assumed $\omega_1 \neq 0, \omega_2 \neq 0$ in the previous lemmas. Now we show that these conditions are generically satisfied in the space of three-parameter families of vector fields of the form (1.1). To this aim, we define some algebraic manifolds.

Let S be the nilpotent matrix with 1's just above the diagonal and 0's elsewhere. For the matrix $A = (a_{ij})$ of the linear part of the vector field X_0 there exists a regular matrix $N = (c_{ij})$ such that $NAN^{-1} = S$. Since rank $A = 2$, there exists at least one nonzero minor of order 2. There is no loss of generality to assume

$$A_3 = \det \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} \neq 0 .$$

This condition corresponds to a stratum of some algebraic manifold, which will be specified later. Under the assumption that some other minor of order 2 is nonzero, we have to do with another stratum of this algebraic manifold and all computations for this case are similar to those for $A_3 \neq 0$.

First let us express the elements of the matrix N as functions of elements of the matrix A. If $A_3 \neq 0$, then

$$c_{11} = \frac{a_{23}}{A_3} \det \begin{bmatrix} a_{12} & A_2 \\ a_{13} & A_3 \end{bmatrix}, \quad c_{12} = \frac{a_{13}}{A_3} \det \begin{bmatrix} a_{12} & A_2 \\ a_{13} & A_3 \end{bmatrix}, \quad c_{13} = 0 ,$$

$$c_{21} = \frac{1}{A_3} \det \begin{bmatrix} A_2 & a_{23} \\ A_3 & a_{33} \end{bmatrix}, \quad c_{22} = \frac{1}{A_3} \det \begin{bmatrix} a_{12} & A_2 \\ a_{13} & A_3 \end{bmatrix}, \quad c_{23} = 0 ,$$

$c_{31} = A_1$, $c_{32} = A_2$, $c_{33} = A_3$, where

$$A_1 = \det \begin{bmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{bmatrix}, \quad A_2 = \det \begin{bmatrix} a_{32} & a_{12} \\ a_{33} & a_{13} \end{bmatrix}.$$

The characteristic equation of the matrix $A = (a_{ij})$ is

$$-\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0 ,$$

where $c_0 = \det A$, $c_1 = a_{23}a_{32} + a_{11}a_{33} + a_{12}a_{21} - a_{22}a_{33} - a_{13}a_{31} - a_{11}a_{22}$ $c_2 = \operatorname{Sp} A = a_{11} + a_{22} + a_{33}$. Therefore A has zero as an eigenvalue of multiplicity 3 if and only if $\det A = 0$, $\operatorname{Sp} A = 0$, $c_1 = 0$.

Denote by Γ_3^∞ the set of all smooth vector fields on R^3. Let $j^k v(x)$ be the k-jet of $v \in \Gamma_3^\infty$ at a point x and let $J_3^k(x)$ be the set of all such k-jets. For $v \in \Gamma_3^\infty$, $j^2 v(x) = (v(x), D\, v(x), D^2 v(x))$, we may identify $D\, v(x)$ with

$$\left(\frac{\partial v_1(x)}{\partial x_1}, \frac{\partial v_1(x)}{\partial x_2}, \frac{\partial v_1(x)}{\partial x_3}, \ldots, \frac{\partial v_3(x)}{\partial x_3}\right) \in R^9$$

and because of the symmetry of the matrices $D^2 v_k(x)$ we may identify $D^2 v(x)$ with

$$\left(\frac{\partial^2 v_1(x)}{\partial x_1^2}, \frac{\partial^2 v_1(x)}{\partial x_2^2}, \frac{\partial^2 v_1(x)}{\partial x_3^2}, \frac{\partial^2 v_1(x)}{\partial x_1\,\partial x_2}, \frac{\partial^2 v_1(x)}{\partial x_1\,\partial x_3}, \frac{\partial^2 v_1(x)}{\partial x_2\,\partial x_3}, \ldots, \frac{\partial^2 v_3(x)}{\partial x_2\,\partial x_3}\right) \in R^{18} .$$

This means that the 2-jet $j^2\, v(x)$ may be identified with

$$\left(v(x), \frac{\partial v_1(x)}{\partial x_1}, \ldots, \frac{\partial v_3(x)}{\partial x_3}, \frac{\partial^2 v_1(x)}{\partial x_1^2}, \ldots, \frac{\partial^2 v_3(x)}{\partial x_2\, \partial x_3}\right) \in R^{30}.$$

Let us define the following sets:
$T_k = \{(a, A, B) \in J_3^2 \colon F_i(a, A, B) = 0,\ \ i = 1, 2, 3,\ \ F_4(a, A, B) = t_{1k} = 0,\ \ a = 0,$ rank $A = 2\}$, $k = 1, 2, 3$, where $F_1 = \operatorname{Sp} A$, $F_2 = \det A$, $F_3 = a_{23}a_{32} + a_{11}a_{33} + a_{12}a_{21} - a_{22}a_{33} - a_{13}a_{31} - a_{11}a_{22}$, $A = (a_{ij})$ and t_{11}, t_{12}, t_{13} are the elements of the matrix T from (1.4), which are functions of the elements of the matrices A, B (the elements of B are in fact the elements of the matrices P, Q, R from (1.2)). Direct computations show that

$$\begin{aligned} t_{11} = {} & c_{31}(\alpha_{11}c'_{11} + \alpha_{12}c'_{21} + \alpha_{13}c'_{31}) + c_{32}(\beta_{11}c'_{11} + \beta_{12}c'_{21} + \\ & + \beta_{13}c'_{31}) + c_{33}(\gamma_{11}c'_{11} + \gamma_{12}c'_{21} + \gamma_{13}c'_{31}), \\ t_{12} = {} & c_{31}(\alpha_{11}c'_{12} + \alpha_{12}c'_{22} + \alpha_{13}c'_{32}) + c_{32}(\beta_{11}c'_{12} + \beta_{12}c'_{22} + \\ & + \beta_{13}c'_{32}) + c_{33}(\gamma_{11}c'_{12} + \gamma_{12}c'_{22} + \gamma_{13}c'_{32}) + c_{21}(\alpha_{11}c'_{11} + \\ & + \alpha_{12}c'_{21} + \alpha_{13}c'_{31}) + c_{22}(\beta_{11}c'_{11} + \beta_{12}c'_{21} + \beta_{13}c'_{31}), \\ t_{13} = {} & (c_{31}\alpha_{13} + c_{32}\beta_{13} + c_{33}\gamma_{13})\, c'_{33} + c_{21}(\alpha_{11}c'_{12} + \alpha_{12}c'_{22} + \\ & + \alpha_{13}c'_{32}) + c_{22}(\beta_{11}c'_{12} + \beta_{12}c'_{22} + \beta_{13}c'_{32}) + c_{11}(\alpha_{11}c'_{11} + \\ & + \alpha_{12}c'_{21} + \alpha_{13}c'_{31}) + c_{12}(\beta_{11}c'_{11} + \beta_{12}c'_{21} + \beta_{13}c'_{31}), \end{aligned}$$

where $\alpha_{11} = c'_{11}p_{11} + c'_{21}p_{12} + c'_{31}p_{13}$, $\alpha_{12} = c'_{11}p_{12} + c'_{21}p_{22} + c'_{31}p_{23}$, $\alpha_{13} = c'_{11}p_{13} + c'_{21}p_{23} + c'_{31}p_{33}$ (the same for β_{1k} and γ_{1k}, $k = 1, 2, 3$, where we have q_{ij} and r_{ij}, respectively, instead of p_{ij}), $NAN^{-1} = S$, S is as above, $N = (c_{ij})$, $N^{-1} = (c_{ij})$. The elements c_{ij} are functions of the elements of the matrix A expressed as above (we assume $A_3 \neq 0$).

Lemma 4. *The sets T_1, T_2, T_3 are smooth submanifolds of J_3^2 of codimension 7.*

Proof. (1) for T_2: Let $F = (F_1, F_2, F_3, F_4)\colon R^{30} \to R^4$, where F_i, $i = 1, 2, 3, 4$, are the functions from the definition of the sets T_k. It suffices to show that rank $DF = 4$.

$$F_{ij} = \det \begin{bmatrix} \dfrac{\partial F_1}{\partial a_{11}} & \dfrac{\partial F_1}{\partial a_{31}} & \dfrac{\partial F_1}{\partial a_{21}} & \dfrac{\partial F_1}{\partial r_{ij}} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial F_4}{\partial a_{11}} & \dfrac{\partial F_4}{\partial a_{31}} & \dfrac{\partial F_4}{\partial a_{21}} & \dfrac{\partial F_4}{\partial r_{ij}} \end{bmatrix} = \frac{\partial F_4}{\partial r_{ij}} A_{23},$$

where $A_{23} = a_{12}A_3 - a_{13}A_2$. It suffices to show that $F_{11}^2 + F_{12}^2 \neq 0$. By the above formulae for c_{ij} we have that $c_{11} = k_1 A_{23}$, $c_{12} = k_2 A_{23}$, $c_{13} = 0$ and hence $A_{23} \neq 0$.

Therefore it suffices to show that

$$\Delta = \left(\frac{\partial F_4}{\partial r_{11}}\right)^2 + \left(\frac{\partial F_4}{\partial r_{12}}\right)^2 \neq 0 .$$

If we express c'_{ij} as functions of c_{ij}, then the formula for t_{12} yields $t_{12} = - d^2 c_{33}^3 c_{12} c_{22} r_{11} + d^2 c_{33}^3 (c_{12} c_{21} + c_{22} c_{11}) r_{12} +$ (terms independent of r_{11} and r_{12}), $d^{-1} = \det N$ and therefore $\Delta = d^4 c_{33}^6 (c_{12}^2 c_{22}^2 + (c_{12} c_{21} + c_{22} c_{11})^2)$. Since $c_{33} = A_3 \neq 0$, $c_{22} = kA_{23} \neq 0$, we obtain that $\Delta = 0$ if and only if $c_{11} = c_{12} = 0$. However, $c_{13} = 0$, N is regular and therefore $\Delta \neq 0$.

(2) for T_1: $F_4 = t_{11} = d\, c_{33}^3 c_{22}^2 r_{11} +$ (terms independent of r_{11}), $c_{22} = kA_{23} \neq 0$, $c_{33} = A_3 \neq 0$ and therefore $\partial F_4/\partial r_{11} \neq 0$, i.e. rank $DF = 4$.

(3) for T_3: $F_4 = t_{13} = d^2 c_{32}^2 c_{22} (c_{11} c_{22} - c_{12} c_{21}) q_{12} +$ (terms independent of q_{12}). This implies that $\partial F_4/\partial q_{12} \neq 0$, i.e. rank $DF = 4$ and the proof of the lemma is complete.

Denote by D^∞ the set of all smooth mapping from $R^3 \times R^3$ into R^3 and for any $f \in D^\infty$ define the mapping $\Psi_f: R^3 \times R^3 \to J_3^2$, $\Psi_f(x, \varepsilon) = j^2 f_\varepsilon(x)$, $(x, \varepsilon) \in R^3 \times R^3$ ($f_\varepsilon(x) = f(x, \varepsilon)$). As a consequence of Lemma 4 and Thom's transversality theorem (see e.g. [13, Theorem 3.1]) we obtain

Lemma 5. (1) *There exists a residual subset D_1^∞ of D^∞ such that if $f \in D_1^\infty$, then $\Psi_f(R^3 \times R^3) \cap (T_1 \cup T_2 \cup T_3) = \emptyset$.*

(2) *If $X \subset R^3 \times R^3$ is a compact set, then there exists an open dense subset D_X of D^∞ such that if $f \in D_X$, then $\Psi_f(X) \cap (T_1 \cup T_2 \cup T_3) = \emptyset$.*

Let $\Sigma = \{(a, A) \in J_3^1: a = 0, \det A = 0, \operatorname{Sp} A = 0, c_1 = 0, A$ is a nonzero matrix$\}$, $A = (a_{ij})$, c_1 are as above. From the above computations it follows that Σ is a smooth submanifold of J_3^1 of codimension 6.

Definition. *The family (1.1) is called nondegenrate, if $t_{11} \cdot t_{12} \cdot t_{13} \neq 0$ and*

$$\Phi_f \,\overline{\pitchfork}_{(0,0)}\, \Sigma \tag{2.6}$$

(Φ_f transversally intersects Σ at $(0, 0)$), where $\Phi_f(x, \varepsilon) = j^1 f_\varepsilon(x)$.

Denote by H^∞ the set of all families of vector fields of the form (1.1). As a consequence of Lemma 5 and Thom's transversality theorem we obtain the following lemma.

Lemma 6. *The set of all nondegenerate families of vector fields $H_1^\infty \subset H^\infty$ is open dense in H^∞.*

Let $f \in H_1^\infty$ and suppose that it is already in the form (2.5). Define the mapping $\sigma_f: R^6 \to R^6$, $\sigma_f(y, \varepsilon) = (f(y, \varepsilon), \operatorname{Sp} D_y f_\varepsilon(y), \det D_y f_\varepsilon(y), H_\varepsilon(y))$, where $D_y f_\varepsilon(y) = (a_{ij}(y, \varepsilon))$ is the differential of the mapping f at y, $\operatorname{Sp} D_y f_\varepsilon(y) = a_{11} + a_{22} + a_{33}$, $H_\varepsilon(y) = -a_{22} a_{33} + a_{32} a_{23} + a_{11} a_{33} - a_{13} a_{31} - a_{11} a_{22} + a_{12} a_{21}$. The form of the mapping σ_f and the forms of the functions defining the set imply that the transversality condition (2.6) is equivalent to the regularity of the mapping σ_f at the origin, i.e. to the condition $\det D\sigma_f(0, 0) \neq 0$.

If $f = (f_1, f_2, f_3)$, then $\operatorname{Sp} D_y f_\varepsilon(y) = \partial f_3/\partial y_3$, $\det D_y f_\varepsilon(y) = \partial f_3/\partial y_1$, $H_\varepsilon(y) = \partial f_3/\partial y_2$. Using the form of the family (2.5) one can show that $\det D\sigma_f(0, 0) = -\omega_2 \det D\varphi(0)$, $\varphi = (\varphi_1, \varphi_2, \varphi_3)$. Since $\omega_2 \neq 0$ and $\det \sigma_f(0, 0) \neq 0$ for $f \in H_1^\infty$, we obtain that $\det D\,\varphi(0) \neq 0$. This enables us to introduce new coordinates in the parameter space: $\mu_i = \varphi_i(\varepsilon)$, $i = 1, 2, 3$, and we obtain a family of the form (2.5) with μ_i, $Q_i(y_1, \varphi^{-1}(\mu))$ $(i = 1, 2)$, $Q_3(y_3, \varphi^{-1}(\mu))$, $\Phi_j(y, \varphi^{-1}(\mu))$ $\Theta(y_1, \varphi^{-1}(\mu))$ instead of $\varphi_i(\varepsilon)$, $Q_i(y_1, \varepsilon)$, $Q_3(y_3, \varepsilon)$, $\Phi_j(y, \varepsilon)$, $\Theta(y_1, \varepsilon)$, respectively, where $\mu = (\mu_1, \mu_2, \mu_3)$. Dividing the right-hand side of the resulting family by the function $\Theta(y_1, \varphi^{-1}(\mu))$ the family becomes

$$\dot{z}_1 = z_2\tilde{\Theta}(z_1, \mu)\,, \quad \dot{z}_2 = z_3\tilde{\Theta}(z_1, \mu)\,, \quad \dot{z}_3 = R(z, \mu)\,, \tag{2.7}$$

where $\tilde{\Theta}$, $R \in C^\infty$, $\tilde{\Theta}(0, 0) = 1$ and R has the same form as the right-hand side of (2.5) with $\Theta \equiv 1$. This family is C^∞-equivalent to (2.5). Now, if we put $u_1 = z_1$, $u_2 = z_2\tilde{\Theta}(z_1, \mu)$, $u_3 = z_3$, the family becomes

$$\dot{u}_1 = u_2\,, \quad \dot{u}_2 = u_3\hat{\Theta}(u_1, \mu)\,, \quad u_3 = \hat{R}(u, \mu)\,,$$

where $\hat{\Theta}$, $\hat{R} \in C^\infty$, $\hat{\Theta}(0, 0) = 1$, $\hat{R}(u, \mu)$ has the same form as R. Finally, introducing new coordinates $y_1 = u_1$, $y_2 = u_2$, $y_3 = u_3\hat{\Theta}(u_1, \mu)$, one obtains a family of the form

$$\begin{aligned}
\dot{y}_1 &= y_2\,, \\
\dot{y}_2 &= y_3\,, \\
\dot{y}_3 &= \mu_1 + \mu_2 y_1 + y_1^2 + \mu_3 y_2 + y_1 y_2 Q_1(y_1, \mu) + \\
&\quad + y_1 y_3 Q_2(y_1, \mu) + y_2 Q_3(y_3, \mu) + y_2^2 \Phi_1(y, \mu) + y_3^2 \Phi_2(y, \mu)\,,
\end{aligned} \tag{2.8}$$

where Q_1, Q_2, Q_3, Φ_1, Φ_2 are smooth functions, $Q_1(0, 0) = \omega_1$, $Q_2(0, 0) = \omega_2$.

We have proved the following theorem.

Theorem. *There exists an open dense subset H_1^∞ of the set H^∞ of all three-parameter families of vector fields of the form (1.1) such that if $f \in H_1^\infty$, then f is non-degenerate, and it is possible to transform this family by a smooth regular transformation of coordinates in a sufficiently small neighbourhood of the origin in $R^3 \times R^3$ to the form (2.8), where ω_1, ω_2 are invariants of the germ, represented by the family f.*

3. BIFURCATION DIAGRAM

Let $f \in H_1^\infty$ be a family of the form (2.8). All critical points of this family have the form $(y_1, 0, 0)$, where y_1 is a real root of the equation

$$y^2 + \mu_2 y + \mu_1 = 0\,.$$

Let U be a neighbourhood of the origin in the parameter space and let S_k ($k = 0, 1, 2$) be the set of all $\mu \in U$ for which (2.8) has k critical points.

Lemma 7. *There exists a smooth function* $\mu_1 = S(\mu_2)$ *such that* $S_1 = \{\mu = (\mu_1, \mu_2, \mu_3) \in U: \mu_1 = S(\mu_2),\ S(0) = S'(0) = 0,\ S''(0) > 0\}$, *i.e.* S_1 *is a fold dividing U into two components, one of which is* S_0 *and the other is* S_2.

If $\mu \in S_2$, then the vector field (2.8) has two critical points $F = (\xi_1, 0, 0)$, $G = (\xi_2, 0, 0)$, where $\xi_1 = -\frac{1}{2}(\mu_2 - v)$, $\xi_2 = \frac{1}{2}(-\mu_2 - v)$, $v = (\mu_2^2 - 4\mu_1)^{1/2}$.

Let $S_{12} = S_1 \cup S_2$ and let $K = (\xi, 0, 0)$ be a critical point of (2.8). Denote by $L(K)$ the matrix of the linear part of (2.8), computed at K. The characteristic equation of $L(K)$ is

$$\lambda^3 - a_2\lambda^2 - a_1\lambda - a_0 = 0\,, \tag{3.1}$$

where $|a_0| = |v|$, $a_1 = (\mu_3 + \xi)\,Q_1(\xi, \mu)$, $a_2 = \xi Q_2(\xi, \mu)$. If $\mu \in S_1$, then the vector field (2.8) has one critical point $K = (\xi, 0, 0)$, where $\xi = -\frac{1}{2}\mu_2$. If $\mu \in S_2$, then the matrix $L(F)$ ($L(G)$) has the characteristic equation of the form (3.1), where $\xi = \xi_1$ ($\xi = \xi_2$) and $a_0 = v > 0$ ($a_0 = -v < 0$).

First assume $\mu \in S_1$. Then the matrix $L(K)$ has zero as an eigenvalue. Obviously, it is of multiplicity 2 if and only if, in addition to the identity $\mu_2^2 - 4\mu_1 = 0$ defining the set S_1, the following holds:

$$H(\mu_2, \mu_3) = (\mu_3 - \tfrac{1}{2}\mu_2)\,Q_1(-\tfrac{1}{2}\mu_2, \mu) = 0\,, \quad \mu_2 \neq 0\,.$$

Since $Q_1(0, 0) = \omega_1 \neq 0$, the last identity is satisfied in a sufficiently small neighbourhood of the origin only if $\mu_3 = \eta(\mu_2) = \frac{1}{2}\mu_2$. If $\chi(t) = (\frac{1}{4}t^2, t, \eta(t))$ and W is a neighbourhood of the origin in R^1, then $\chi(W)$ is a one-dimensional smooth submanifold of S_1. For $\mu \in Z_2(K) = \chi(W) \setminus \{0\}$, the matrix $L(K)$ has zero as an eigenvalue of multiplicity 2 and the third eigenvalue is $\lambda_3 = -\frac{1}{2}\mu_2 Q_1(-\frac{1}{2}\mu_2, \mu)$. The matrix $L(K)$ has zero as an eigenvalue of multiplicity 1 and a couple of pure imaginary eigenvalues if and only if $a_0 = 0$, $a_2 = 0$, $a_1 < 0$, i.e. $\mu_1 = 0$, $\mu_2 = 0$, $\mu_3\omega_1 < 0$. Denote $Z_{1c} = \{\mu: \mu_1 = 0, \mu_2 = 0, \mu_3\omega_1 < 0\}$.

We have proved the following lemma.

Lemma 8. *There exist one-dimensional smooth submanifolds* Z_2 *and* Z_{1c} *of* S_1 *such that the following holds:*

(1) Z_2 *is the set of all* $\mu \in U$ *(U is a neighbourhood of the origin) for which the matrix* $L(K)$ *has eigenvalues:* $\lambda_1 = \lambda_2 = 0, \lambda_3 \neq 0$, *where* $\operatorname{sign} \lambda_3 = -\operatorname{sign} \mu_2\omega_1$;

(2) Z_{1c} *is the set of all* $\mu \in U$ *for which the matrix has one zero eigenvalue and a couple of pure imaginary eigenvalues*

$$\bar{Z}_2 \setminus Z_2 = \{0\}\,, \quad \bar{Z}_{1c} \setminus Z_{1c} = \{0\}\,. \tag{3}$$

Now assume $\mu \in S_2$. By means of the substitution $z + \frac{1}{3}a_2$ for λ in the characteristic equation (3.1) of the matrix $L(K)$ we obtain

$$z^3 + 3pz + 2q = 0\,, \tag{3.2}$$

where

(3.3) $$p = -\tfrac{1}{3}(a_1 + \tfrac{1}{3}a_2^2), \quad q = -\tfrac{1}{2}(a_0 + \tfrac{1}{3}a_1a_2 + \tfrac{2}{27}a_2^3).$$

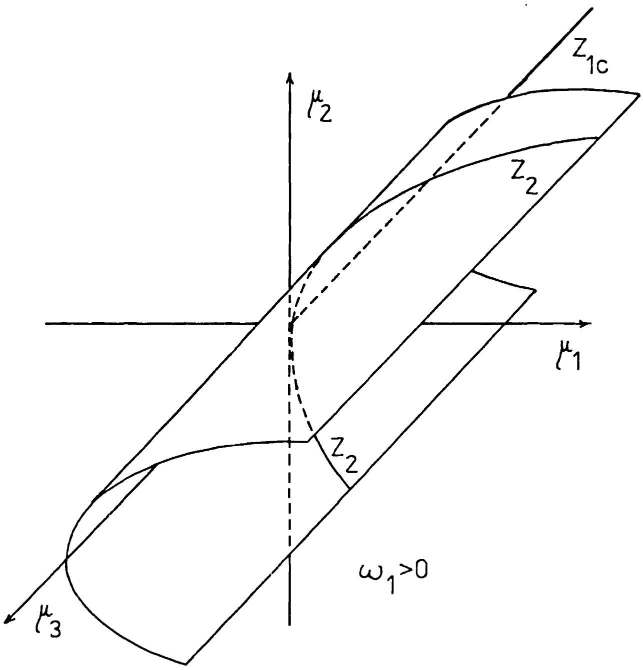

Fig. 1.

The discriminant of the equation (3.2) is $D = D(\mu) = q^2 + p^3$. Let us introduce new coordinates on S_{12} via the mapping

(3.4)
$$\begin{aligned} v_1 &= a_0 = (\mu_2^2 - 4\mu_1)^{1/2}, \\ \varrho_F\colon\ v_2 &= a_2 = -\tfrac{1}{2}(\mu_2 - (\mu_2^2 - 4\mu_1)^{1/2})\, Q_2(\xi_1, \mu), \\ v_3 &= a_1 = [\mu_3 - \tfrac{1}{2}(\mu_2 - (\mu_2^2 - 4\mu_1)^{1/2})]\, Q_1(\xi_1, \mu). \end{aligned}$$

Obviously, ϱ_F is a smooth diffeomorphism on S_2, but it is not C^1 on S_1 and

(3.5) $$\tilde{S}_1 = \varrho_F(S_1) = \partial\varrho_F(S_{12}) = \{v = (v_1, v_2, v_3)\colon v_1 = 0\}.$$

In these coordinates the characteristic equation of $L(F)$ is

(3.6) $$\lambda^3 - v_2\lambda^2 - v_3\lambda - v_1 = 0.$$

The discriminant of this equation is $D_F = D_F(v) = p^3 + q^2$, where

(3.7) $$p = -\tfrac{1}{3}(v_3 + \tfrac{1}{3}v_2^2), \quad q = -\tfrac{1}{2}(v_1 + \tfrac{1}{3}v_2v_3 + \tfrac{2}{27}v_2^3).$$

Denote $\mathscr{D}_F = \{v\colon D_F(v) = 0\}$, $\mathscr{D}_F^+ = \{v\colon D_F(v) > 0\}$, $\mathscr{D}_F^- = \{v\colon D_F(v) < 0\}$.

Lemma 9. *If* $v \in \mathscr{D}_F^-(\mathscr{D}_F; \mathscr{D}_F^+)$, *then the equation* (3.1) *has three distinct real roots (two distinct real roots; one real and a couple of complex roots).*

$\mathscr{D}_F = \mathscr{F}^+ \cup \mathscr{F}^-$, where $\mathscr{F}^\pm = \{v: v_1 = F^\pm(v_2, v_3),\ v_3 + \frac{1}{3}v_2^2 \geqq 0\}$,

$$F^\pm(v_2, v_3) = -\tfrac{1}{3}(v_2 v_3 + \tfrac{2}{9}v_2^3) \pm \frac{2}{\sqrt{27}}(v_3 + \tfrac{1}{3}v_2^2)^{3/2}. \tag{3.8}$$

The functions F^+, F^- are smooth on $P_F^+ = \{v: v_3 + \frac{1}{3}v_2^2 > 0\}$, but only C^1 on $P_F^0 = \{v: v_3 + \frac{1}{3}v_2^2 = 0\}$. $F^\pm(v_2, 0) = \frac{2}{27}(-v_2^3 \pm |v_2|^2)$ and therefore $F^+(v_2, 0) = 0$ for $v_2 \geqq 0$, $F^+(v_2, 0) = -\frac{4}{27}v_2^3 > 0$ for $v_2 < 0$, $F^-(v_2, 0) = -\frac{4}{27}v_2^3 < 0$ for $v_2 > 0$. $F^-(v_2, 0) = 0$ for $v_2 \leqq 0$. Since

$$\frac{\partial F^\pm(v_2, v_3)}{\partial v_3} = -\tfrac{1}{3}v_2 \pm \frac{1}{\sqrt{3}}(v_3 + \tfrac{1}{3}v_2^2)^{1/2},$$

we have

$$\frac{\partial F^\pm(v_2, 0)}{\partial v_3} = \tfrac{1}{3}(-v_2 \pm |v_2|)$$

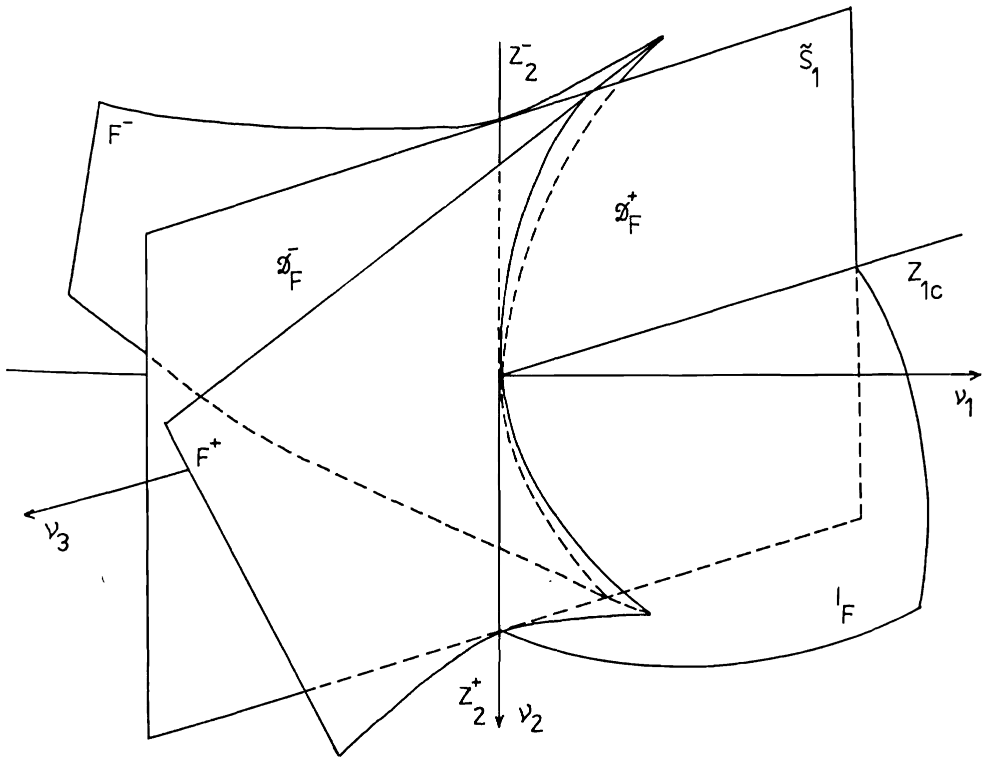

Fig. 2.

and therefore

$$\frac{\partial F^+(v_2, 0)}{\partial v_3} = 0 \quad \text{for} \quad v_2 \geqq 0\,, \quad \frac{\partial F^-(v_2, 0)}{\partial v_3} = 0 \quad \text{for} \quad v_2 \leqq 0\,,$$

$$\frac{\partial^2 F^\pm(v_2, 0)}{\partial v_3^2} = \pm \frac{1}{2|v_2|}\,.$$

Moreover, it is obvious that $F^+(v_2, v_3) = F^-(v_2, v_3)$ if and only if $v_3 = -\frac{1}{3}v_2^2$. The above properties of the function F^+, F^- enable us to sketch the picture of the set $\mathscr{D}_F$ (Fig. 2). Since $\varrho_F(S_2) = \tilde{S}_2 = \{v\colon v_1 > 0\}$, we are interested in the restriction of $\mathscr{D}_F$, $\mathscr{D}_F^-$, $\mathscr{D}_F^+$ to this set only.

Obviously $\tilde{Z}_2 = \varrho_F(Z_2) = \{v\colon v_1 = 0,\ v_3 = 0\}$ and $\tilde{Z}_{1c} = \varrho_F(Z_{1c}) = \{v\colon v_1 = 0,\ v_3 < 0\}$.

Now we are interested in such $v \in \mathscr{D}_F^+$ for which the equation (3.6) has a couple of pure imaginary roots. For $v \in \mathscr{D}_F^+$ there is one real root $\lambda_1 = u + v + \frac{1}{3}v_2$ and a couple of complex ones $\lambda_{2,3} = \frac{1}{3}v_2 - \frac{1}{2}(u + v) \pm \mathrm{i}(\sqrt{3}/2)(u - v)$, where $u = (-q + (D_F)^{1/2})^{1/3}$, $v = (-q - (D_F)^{1/2})^{1/3}$, $q, \mathscr{D}_F$ are as above. This implies that $\operatorname{Re} \lambda_{2,3} = 0$ if and only if $v \in I_F = \{v \in \mathscr{D}_F^+ : H_F(v_1, v_2, v_3) = 0\}$, where $H_F(v_1, v_2, v_3) = 2v_2 - 3((-q + (D_F)^{1/2})^{1/3} + (-q - (D_F)^{1/2})^{1/3})$. For any $v_3^0 < 0$ we have $H_F(0, 0, v_3^0) = 0$. The function H_F is C^1 in a neighbourhood of the point $(0, 0, v_3^0)$ and $\partial H_F(0, 0, v_3^0)/\partial v_1 = 3/v_3^0$. Therefore there is a C^1-function $v_1 = h(v_2, v_3)$ defined in a neighbourhood V of $(0, v_3^0)$ such that $h(0, v_3^0) = 0$ and $H_F(h(v_2, v_3), v_2, v_3) = 0$ in V. Moreover, $\partial h(0, v_3^0)/\partial v_2 = -v_3^0 > 0$ and hence the function $h(v_2, v_3^0)$ increases near the point $v_2 = 0$. We have

$$\frac{\partial H_F}{\partial v_1} = -(D_F)^{-1/2}\left(-\frac{\partial q}{\partial v_1}(D_F)^{1/2} + \frac{1}{2}\frac{\partial D_F}{\partial v_1}\right)((D_F)^{1/2} - q)^{-3/2} -$$

$$- \left(\frac{\partial q}{\partial v_1}(D_F)^{1/2} + \frac{1}{2}\frac{\partial D_F}{\partial v_1}\right)((D_F)^{1/2} + q)^{-3/2} =$$

$$= -\tfrac{1}{2}(D_F)^{-1/2}(((D_F)^{1/2} - q)^{1/3} + ((D_F)^{1/2} + q)^{1/3}) \neq 0 \quad \text{for} \quad v \in \mathscr{D}_F^+\,.$$

Therefore the set I_F is a two-dimensional C^1-manifold defined not only locally near the set $\tilde{Z}_{1c}$. We can express the set $I_F \setminus \{v\colon v_1 = v_3 = 0,\ v_2 > 0\}$ as the graph of a C^1-function $v_1 = h(v_2, v_3)$, $v_3 < 0$, $v_2 \geqq 0$. Since $H_F(0, v_2, 0) = 0$, $\partial H_F(0, v_2, 0)/\partial v_1 \neq 0$ for any $v_2 > 0$, the uniqueness of the implicit function implies that $\lim_{v_3 \to 0} h(v_2, v_3) = 0$. Defining $h(v_2, 0) = 0$, we obtain that I_F is the graph of a function $v_1 = h(v_2, v_3)$ defined for all $v_2 \geqq 0$, $v_3 \leqq 0$, which is C^1 on $\{v\colon v_2 > 0,\ v_3 \leqq 0\}$. The boundary of the set I_F is $\{v\colon v_1 = 0,\ v_2 = 0,\ v_3 \leqq 0\} \cup \{v\colon v_1 = 0,\ v_3 = 0,\ v_2 \geqq 0\}$.

Since for $v \in \mathscr{D}_F$ the equation (3.6) has one root of multiplicity two, it has no complex root and therefore the surface I_F does not intersect the surface $\mathscr{D}_F$.

We prove that

$$\alpha = \lim_{v_3 \to 0} \frac{\partial h(v_2, v_3)}{\partial v_3} < 0 \quad \text{for any} \quad v_2 > 0$$

sufficiently small. We have

$$\frac{\partial h(v_2, v_3)}{\partial v_3} = -\left(\frac{\partial H_F(h')}{\partial v_3}\right)\left(\frac{\partial H_F(h')}{\partial v_1}\right)^{-1},$$

where $h' = (h(v_2, v_3), v_2, v_3))$ and $\partial H_F/\partial v_3 = -(D_F)^{-1/2}(-\partial q/\partial v_3\,(((D_F)^{1/2} - q)^{1/3} + ((D_F)^{1/2} + q)^{1/3}) - \frac{1}{2}(((D_F)^{1/2} + q)^{2/3} - ((D_F)^{1/2} - q)^{2/3}))$. Using the above formulae for $\partial H_F/\partial v_1$ and $\partial H_F/\partial v_3$ we obtain

$$\alpha = -2\lim_{v_3 \to 0}\left(-\frac{\partial q(h(v_2, v_3), v_2, v_3)}{\partial v_3} - \tfrac{1}{2}(((D_F)^{1/2} + q)^{1/3} - ((D_F)^{1/2} - q)^{1/3})\right).$$

Since $h(v_2, 0) = 0,\ D_F(0, v_2, 0) = 0$,

$$q = \left(-\frac{v_2}{3^3}\right)^3, \quad \frac{\partial q(0, v_2, 0)}{\partial v_3} = -\tfrac{1}{6}v_2,$$

we obtain that $\alpha = -\frac{1}{3}v_2 < 0$ for $v_2 > 0$. This together with the fact that the set $I_F \cap \mathcal{D}_F$ is empty implies that I_F looks like in Fig. 2.

New let us consider the critical point $G = (\xi_2, 0, 0)$. Similarly to the case of the critical point F, we introduce new coordinates via the mapping

$$\text{(3.9)} \qquad \begin{aligned} \varkappa_1 &= -(\mu_2^2 - 4\mu_1)^{1/2}, \\ \varrho_G\colon \varkappa_2 &= -\tfrac{1}{2}(\mu_2 + (\mu_2^2 - 4\mu_1)^{1/2}\, Q_2(\xi_2, \mu), \\ \varkappa_3 &= \mu_3 - \tfrac{1}{2}(\mu_2 + (\mu_2^2 - 4\mu_1)^{1/2})\, Q_1(\xi_2, \mu). \end{aligned}$$

The mapping ϱ_G is a smooth diffeomorphism on S_2 and

$$\text{(3.10)} \qquad \begin{aligned} \hat{S}_1 &= \varrho_G(S_1) = \partial\varrho_G(S_{12}) = \{\varkappa = (\varkappa_1, \varkappa_2, \varkappa_3) : \varkappa_1 = 0\}, \\ \mu_1 &= \tfrac{1}{4}(\varphi_2^2(\varkappa) - \varkappa_1^2) = \varphi_1(\varkappa), \\ \varrho_G^{-1}\colon \mu_2 &= \varphi_2(\varkappa), \\ \mu_3 &= \varphi_3(\varkappa), \end{aligned}$$

where the functions φ_2, φ_3 satisfy the identities $\varkappa_2 = -\frac{1}{2}(\varphi_2(\varkappa) + \varkappa_1)\, Q_2(-\frac{1}{2}(\varphi_2(\varkappa) + \varkappa_1), \varphi(\varkappa)), \varkappa_3 = [\varphi_3(\varkappa) - \frac{1}{2}(\varphi_2(\varkappa) + \varkappa_1)]\, Q_1(-\frac{1}{2}(\varphi_2(\varkappa) + \varkappa_1), \varphi(\varkappa)), \varphi = (\varphi_1, \varphi_2, \varphi_3)$. Since $Q_1(0) = \omega_1 \neq 0$, $Q_2(0) = \omega_2 \neq 0$, the existence of the functions $\varphi_2, \varphi_3 \in C^\infty$ follows from the implicit function theorem. From these identities we obtain

$$\frac{\partial\varphi_2(0)}{\partial\varkappa_1} = -1, \quad \frac{\partial\varphi_2(0)}{\partial\varkappa_2} = -\frac{2}{\omega_2}, \quad \frac{\partial\varphi_2(0)}{\partial\varkappa_3} = 0,$$

$$\frac{\partial\varphi_3(0)}{\partial\varkappa_1} = 0, \quad \frac{\partial\varphi_3(0)}{\partial\varkappa_2} = -\frac{1}{\omega_2}, \quad \frac{\partial\varphi_3(0)}{\partial\varkappa_3} = \frac{1}{\omega_1}$$

and therefore

$$(3.11)\qquad \mu_1 = \frac{1}{4}\left(-\varkappa_1 - \frac{2}{\omega_2}\varkappa_2 - \varkappa_1^2 + h_2(\varkappa)\right),$$

$$\varrho_G^{-1}:\ \mu_2 = -\varkappa_1 - \frac{2}{\omega_2}\varkappa_2 + h_2(\varkappa),$$

$$\mu_3 = -\frac{1}{\omega_2}\varkappa_2 + \frac{1}{\omega_1}\varkappa_3 + h_3(\varkappa),$$

where $h_2(\varkappa)$, $h_3(\varkappa) = o(\|\varkappa\|)$.

Hence we obtain

$$\begin{aligned} & v_1 = -\varkappa_1, \\ H = \varrho_F \circ \varrho_G^{-1}:\ & v_2 = \varkappa_2 + \tilde{h}_2(\varkappa), \\ & v_3 = \varkappa_3 + \tilde{h}_3(\varkappa), \end{aligned}$$

where $\tilde{h}_2, \tilde{h}_3 = o(\|\varkappa\|)$. Since $H(0, \varkappa_2, \varkappa_3) = (0, \varkappa_2, \varkappa_3)$, we have $\tilde{h}_i(\varkappa) = \varkappa_1 \tilde{H}_i(\varkappa)$, $i = 1, 2$. The inverse mapping H^{-1} has the same form as H, i.e.

$$\begin{aligned} & \varkappa_1 = -v_1, \\ H^{-1}:\ & \varkappa_2 = v_2 + v_1 H_2(v), \\ & \varkappa_3 = v_3 + v_1 H_3(v), \end{aligned}$$

where $H_i(v) = O(\|v\|)$, $i = 1, 2$. Therefore the characteristic equation of the matrix $L(G)$ has the form

$$(3.12)\qquad \lambda^3 - (v_2 + v_1 H_2(v))\lambda^2 - (v_3 + v_1 H_3(v))\lambda + v_1 = 0.$$

The discriminant of this equation is $D_G = D_G(v) = \tilde{p}^3 + \tilde{q}^2$, where $\tilde{p} = \tilde{p}(v) = \hat{p}(H^{-1}(v))$, $\tilde{q} = \tilde{q}(v) = \hat{g}(H^{-1}(v))$, $\hat{p} = -\frac{1}{3}(\varkappa_3 + \frac{1}{3}\varkappa_2^2)$, $\hat{g} = -\frac{1}{2}(\varkappa_1 + \frac{1}{3}\varkappa_2\varkappa_3 + \frac{2}{27}\varkappa_2^3)$. Let $\mathscr{D}_G = \{v\colon D_G(v) = 0\}$, $\mathscr{D}_G^+ = \{v\colon D_G(v) > 0\}$, $\mathscr{D}_G^- = \{v\colon D_G(v) < 0\}$.

In the $\varkappa$-coordinates we have the same bifurcation diagram as we have obtained for the critical point F in the v-coordinates. In order to obtain the bifurcation diagram not only for F and G separately, but also for F and G as a couple, we need to sketch the bifurcation diagram for G also in the v-coordinates.

From the form of the mapping H it follows that H maps the $\varkappa_3$-axis onto the v_3-axis onto the v_3-axis, the $\varkappa_2$-axis onto the v_2-axis and the $\varkappa_1$-axis is mapped by H onto a curve, which has its tangent at the origin close to the v_1-axis.

The discriminant surface $\mathscr{D}_G$ has the form $\mathscr{D}_G = H^+ \cup H^-$, with $H^\pm = \{v\colon v_1 = \tilde{F}^\pm(v_2, v_3),\ v_1 \geqq 0\}$, where $\tilde{F}^\pm(v_2, v_3)$ is the solution of the implicit equation

$$v_1 + F^\pm(v_2 + v_1 H_2(v_2, v_3),\ v_3 + v_1 H_3(v_2, v_3)) = 0.$$

From this equation, the uniqueness of its solutions and from the properties of the functions F^+, F^- mentioned above it follows that the functions $\tilde{F}^+, \tilde{F}^-$ have the following properties:

$$\tilde{F}^+(v_2, 0) = 0 \text{ for } v_2 \geqq 0\,, \qquad \tilde{F}^+(v_2, 0) < 0 \text{ for } v_2 < 0\,,$$
$$\tilde{F}^-(v_2, 0) > 0 \text{ for } v_2 > 0\,, \qquad \tilde{F}^-(v_2, 0) = 0 \text{ for } v_2 \leqq 0\,,$$
$$\frac{\partial \tilde{F}^+(v_2, 0)}{\partial v_3} = 0 \quad \text{for} \quad v_2 \geqq 0\,, \qquad \frac{\partial \tilde{F}^+(v_2, 0)}{\partial v_3} < 0 \quad \text{for} \quad v_2 < 0\,,$$
$$\frac{\partial \tilde{F}^-(v_2, 0)}{\partial v_3} > 0 \quad \text{for} \quad v_2 > 0\,, \qquad \frac{\partial \tilde{F}^-(v_2, 0)}{\partial v_3} = 0 \quad \text{for} \quad v_2 \leqq 0\,,$$
$$\frac{\partial^2 \tilde{F}^+(v_2, 0)}{\partial v_3^2} < 0 \quad \text{for} \quad v_2 > 0 \quad \text{and} \quad \frac{\partial^2 \tilde{F}^-(v_2, 0)}{\partial v_3^2} > 0$$

for $v_2 < 0$. The properties of the functions $\tilde{F}^+, \tilde{F}^-$ are the same as for the functions $-F^+$ and $-F^-$, respectively. From these properties we obtain that the surface $\mathscr{D}_G$ looks like in Fig. 3.

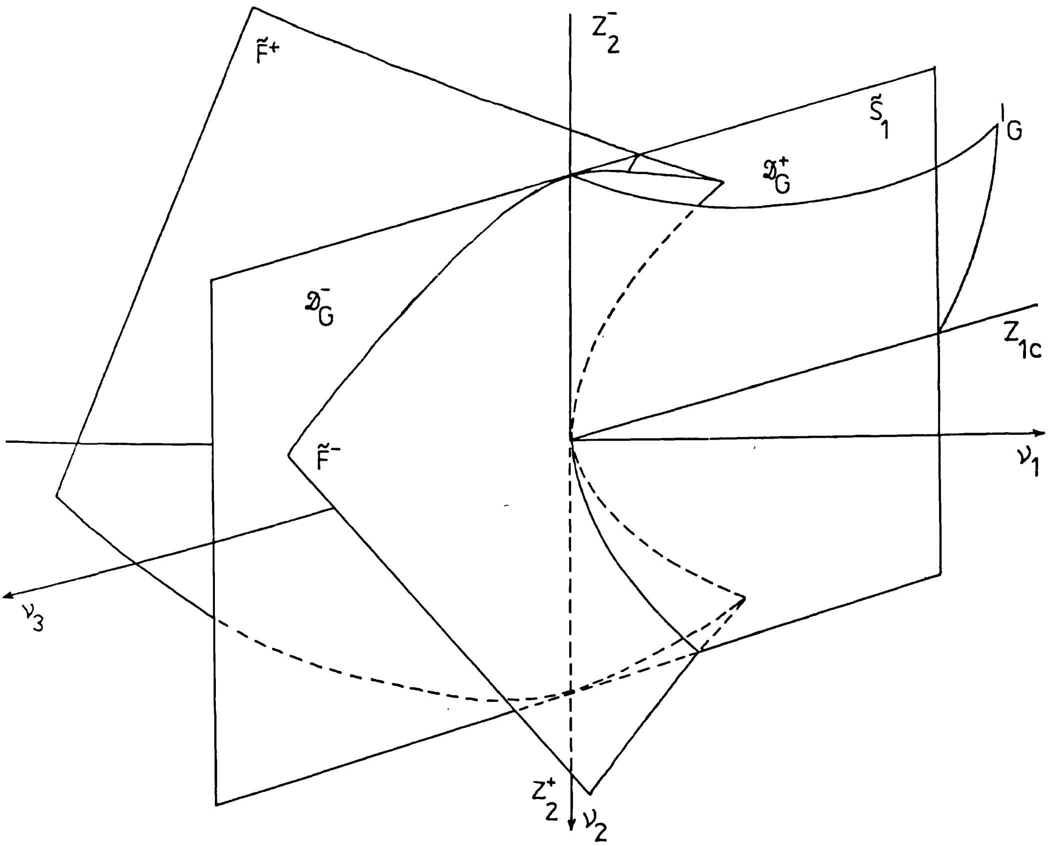

Fig. 3.

Now we are interested in such $v \in \mathscr{D}_G^+$ for which the characteristic equation of the matrix $L(G)$ has a couple of pure imaginary eigenvalues. For $v \in \mathscr{D}_G^+$ the equation (3.12) has one real root $\beta_1 = U + V + \frac{1}{3}\varkappa_2$ and a couple of complex ones $\beta_{2,3} = \frac{1}{3}\varkappa_2 - \frac{1}{2}(U + V) \pm i(\sqrt{3}/2)(U - V)$, where $\varkappa_2 = v_2 + v_1 H_2(v)$, $U = (-\tilde{q} + (D_G)^{1/2})^{1/3}$, $V = (\tilde{q} - (D_G)^{1/2})^{1/3}$. This implies that $\operatorname{Re} \beta_{2,3} = 0$ if and only if $v \in I_G = \{v \in \mathscr{D}_G^+ : H_G(v_1, v_2, v_3) = 0\}$, where $H_G(v_1, v_2, v_3) = 2\varkappa_2 - 3((-\tilde{q} + (D_G)^{1/2})^{1/3} + (-\tilde{q} + (D_G)^{1/2})^{1/3})$. For any $v_3^0 < 0$ we have $H_G(0, 0, v_3^0) = 0$. The function H_G is C^1 in a neighbourhood of the point $(0, 0, v_3^0)$, and $\partial H_G(0, 0, v_3^0)/\partial v_1 = -v_3^0 \neq 0$. Therefore there is a C^1-function $v_1 = k(v_2, v_3)$ defined in a neigbourhood of the point $(0, v_3^0)$ such that $k(0, v_3^0) = 0$ and $H_G(k(v_2, v_3), v_2, v_3) = 0$ in this neighbourhood. Moreover, from the implicit equation we have $\partial k(0, v_3^0)/\partial v_2 = v_3^0 < 0$ for $v_3^0 < 0$. Similarly to the case of the set I_F, it is possible to extend the function $v_1 = k(v_2, v_3)$ to a function $\tilde{k}$ defined on the set $\{v : v_2 \leqq 0, v_3 \leqq 0\}$ so that $\tilde{k} \in C^1$ on $\{v : v_2 \leqq 0, v_3 < 0\}$, $\tilde{k}(v_2, 0) = 0$ for $v_2 \leqq 0$, $\tilde{k}(0, v_3) = 0$ for $v_3 \leqq 0$ and $I_G = \operatorname{graph} k$. Moreover,

$$\lim_{v_3 \to 0} \frac{\partial k(v_2, v_3)}{\partial v_3} < 0 \quad \text{for any} \quad v_2 < 0 \, .$$

Similarly to the case of the set I_F, it is possible to show that the surface I_G does not intersect the surface $\mathscr{D}_G$. We have shown that I_G looks like in Fig. 3.

For $v_1 \in \tilde{S}_1$ there is only one critical point K, for which the matrix $L(K)$ has the eigenvalues $\lambda_1 = 0, \lambda_{2,3} = \frac{1}{2}(v_2 \pm (v_2^2 + 4v_3)^{1/2}$. The sets $\tilde{Z}_2, \tilde{Z}_{1c} \subset \tilde{S}_1$ (see Lemma 8)

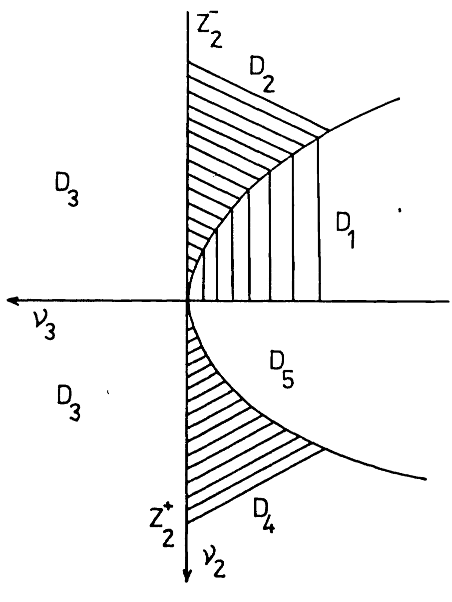

Fig. 4.

and $R = \{v \in \tilde{S}_1 : v_2^2 + 4v_3 = 0\}$ divide the set $\tilde{S}_1$ into the following components:

$$D_1 = \{v \in \tilde{S}_1 : \Psi(v_2, v_3) = v_2^2 + 4v_3 < 0,\ v_2 < 0\},$$
$$D_2 = \{v \in \tilde{S}_1 : \Psi > 0,\ v_2 < 0,\ v_3 < 0\}, \quad D_3 = \{v \in \tilde{S} : v_3 > 0\},$$
$$D_4 = \{v \in \tilde{S}_1 : \Psi > 0,\ v_2 > 0,\ v_3 < 0\}, \quad D_5 = \{v \in \tilde{S}_1 : \Psi < 0,\ v_2 > 0\}$$

(see Fig. 4).

We have the following list of signs of eigenvalues of the matrix $L(K)$:

$\lambda_1 = 0$ for all $v \in \tilde{S}_1$ and

$$D_1: \operatorname{Re} \lambda_{2,3} < 0, \quad D_2: \lambda_2 < 0, \quad \lambda_3 < 0, \quad D_3: \lambda_2 > 0, \quad \lambda_3 < 0,$$
$$D_4: \lambda_2 > 0, \quad \lambda_3 > 0, \quad D_5: \operatorname{Re} \lambda_{2,3} > 0,$$
$$Z_2^-: \lambda_2 = 0, \quad \lambda_3 < 0, \quad Z_2^+: \lambda_3 = 0, \quad \lambda_2 > 0, \quad \tilde{Z}_{1c}: \lambda_{2,3} = \pm i\omega,$$

$\omega \neq 0$, where $\tilde{Z}_2 = Z_2^+ \cup Z_2^-$, $Z_2^+ = \{v \in \tilde{Z}_2 : v_2 > 0\}$, $Z_2^- = \{v \in \tilde{Z}_2 : v_2 < 0\}$.

Let us introduce the following notations: $\mathscr{D}_1 = \mathscr{D}_F^- \cap \mathscr{D}_G^-$, $\mathscr{D}_2 = \mathscr{D}_F^+ \cap \mathscr{D}_G^-$, $\mathscr{D}_3 = \mathscr{D}_F^- \cap \mathscr{D}_G^+$; $I_F^+(I_G^+)(I_F^-(I_G^-))$ is the set of all $v \in \mathscr{D}_F^+(\mathscr{D}_G^+)$ for which the matrix

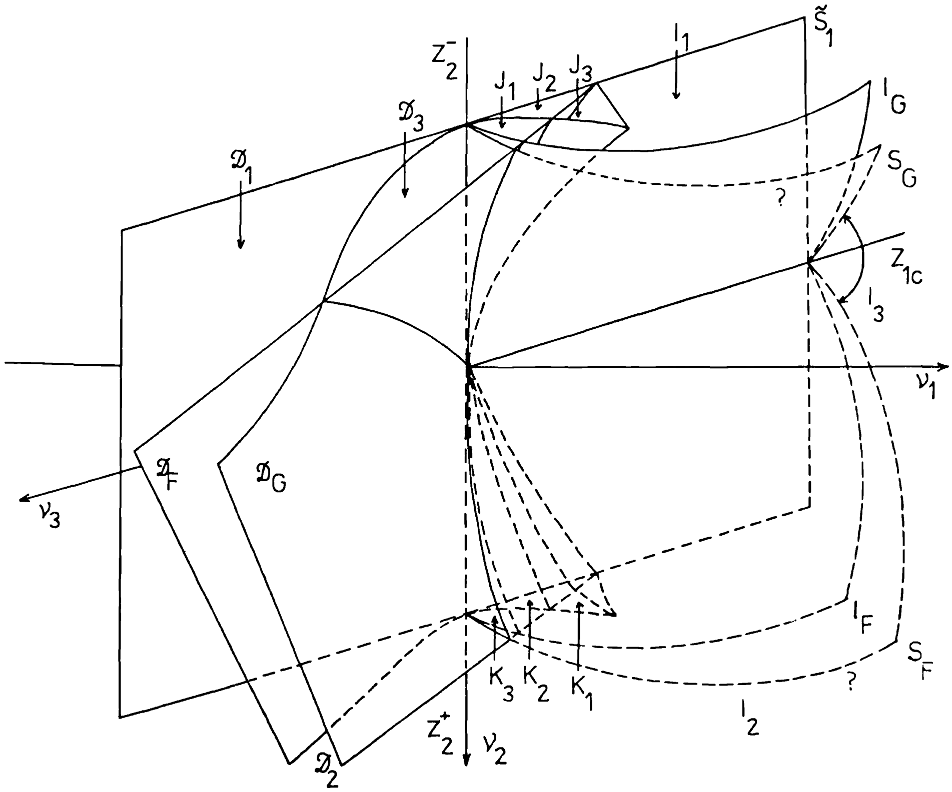

Fig. 5.

$L(F)\,(L(G))$ has a couple of complex eigenvalues with positive (negative) real parts,

$$I_1 = I_F^- \cap I_G^-\,,\quad I_2 = I_F^+ \cap I_G^+\,,\quad I_3 = I_F^- \cap I_G^+\,,\quad J_1 = I_1 \cap \mathscr{D}_2\,,$$
$$J_2 = I_1 \cap \mathscr{D}_1\,,\quad J_3 = I_1 \cap \mathscr{D}_3\,,\quad K_1 = I_2 \cap \mathscr{D}_3\,,\quad K_2 = I_2 \cap \mathscr{D}_1\,,$$
$$K_3 = I_1 \cap \mathscr{D}_2\,,$$
$$A_1 = \mathscr{D}_F \cap \mathscr{D}_G \cap \{v\colon v_2 < 0,\ v_3 < 0\}\,,$$
$$A_2 = \mathscr{D}_F \cap \mathscr{D}_G \cap \{v\colon v_2 < 0,\ v_3 > 0\}\,,$$
$$A_3 = \mathscr{D}_F \cap \mathscr{D}_G \cap \{v\colon v_2 < 0,\ v_3 > 0\}\,,$$
$$B_1 = \mathscr{D}_F \cap I_G\,,\quad B_2 = \mathscr{D}_G \cap I_F \quad \text{(see Fig. 5)}\,.$$

If the matrix $L(F)\,(L(G))$ has only real eigenvalues, then we denote them by $\lambda_1, \lambda_2, \lambda_3$ $(\beta_1, \beta_2, \beta_3)$. If $v \in \mathscr{D}_F^+(\mathscr{D}_G^+)$, then the matrix $L(F)\,(L(G))$ has one real and a couple of complex eigenvalues. Let us denote the real eigenvalue by $\lambda_2(\beta_3)$ and the complex one by $\lambda(\beta)$. Then $\det L(F) = \lambda_2|\lambda|^2 = v_1 > 0$, $\det L(G) = \beta_3|\beta|^2 = -v_1 < 0$ and therefore $\lambda_2 > 0$, $\beta_3 < 0$. Since $\lambda_2 + 2\,\mathrm{Re}\,\lambda = v_2$ and $\beta_3 + 2\,\mathrm{Re}\,\beta = v_2 + v_1 H_2(v)$ we obtain that $\mathrm{Re}\,\lambda < 0$ for $v_2 < 0$ and $\mathrm{Re}\,\beta > 0$ for $v_2 > 0$, v_1 sufficiently small. These properties of the eigenvalues together with the list of signs of eigenvalues of the matrix $L(K)$ for $v \in \tilde{S}_1$ enable us to deduce the following list of signs of eigenvalues for $v_1 > 0$:

I_1: $\mathrm{Re}\,\lambda < 0$, $\lambda_2 > 0$, $\mathrm{Re}\,\beta < 0$, $\beta_3 < 0$,

I_2: $\mathrm{Re}\,\lambda > 0$, $\lambda_2 > 0$, $\mathrm{Re}\,\beta > 0$, $\beta_3 < 0$,

I_3: $\mathrm{Re}\,\lambda < 0$, $\lambda_2 > 0$, $\mathrm{Re}\,\beta > 0$, $\beta_3 < 0$,

$\mathscr{D}_1$: $\lambda_1 < 0$, $\lambda_2 > 0$, $\lambda_3 < 0$, $\beta_1 > 0$, $\beta_2 > 0$, $\beta_3 < 0$,

$\mathscr{D}_2$: $\mathrm{Re}\,\lambda < 0$, $\lambda_2 > 0$, $\beta_1 > 0$, $\beta_2 > 0$, $\beta_3 < 0$,

$\mathscr{D}_3$: $\lambda_1 < 0$, $\lambda_2 > 0$, $\lambda_3 < 0$, $\mathrm{Re}\,\beta > 0$, $\beta_3 < 0$,

J_1: $\lambda_1 < 0$, $\lambda_2 > 0$, $\lambda_3 < 0$, $\mathrm{Rr}\,\beta < 0$, $\beta_3 < 0$,

J_2: $\lambda_1 < 0$, $\lambda_2 > 0$, $\lambda_3 < 0$, $\beta_1 < 0$, $\beta_2 < 0$, $\beta_3 < 0$,

J_3: $\mathrm{Re}\,\lambda < 0$, $\lambda_2 > 0$, $\beta_1 < 0$, $\beta_2 < 0$, $\beta_3 < 0$,

K_1: $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, $\mathrm{Re}\,\beta > 0$, $\beta_3 < 0$,

K_2: $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, $\beta_1 > 0$, $\beta_2 > 0$, $\beta_3 < 0$,

K_3: $\mathrm{Re}\,\lambda > 0$, $\lambda_2 > 0$, $\beta_1 > 0$, $\beta_2 > 0$, $\beta_3 < 0$,

A_1: $\lambda_1 = \lambda_3 < 0$, $\lambda_2 > 0$, $\beta_1 = \beta_2 < 0$, $\beta_3 < 0$,

A_2: $\lambda_1 = \lambda_3 < 0$, $\lambda_2 > 0$, $\beta_1 = \beta_2 < 0$, $\beta_3 < 0$,

A_3: $\lambda_1 = \lambda_3 > 0$, $\lambda_2 > 0$, $\beta_1 = \beta_2 > 0$, $\beta_3 < 0$,

B_1: $\lambda_1 = \lambda_3 < 0$, $\lambda_2 > 0$, $\beta_{2,3} = \pm i\omega$, $\omega \neq 0$, $\beta_3 < 0$,

B_2: $\lambda_{2,3} = \pm i\gamma$, $\gamma \neq 0$, $\lambda_2 > 0$, $\beta_1 = \beta_2 > 0$, $\beta_3 < 0$.

In this section we study the bifurcations of the family (2.8). Although we have obtained a relatively simple bifurcation diagram for the critical points, the bifurcation diagram for the corresponding eigenvalues indicates that the bifurcations of the phase portraits are complicated.

For $\mu^0 \in Z_2$ we have $(\mu_2^0)^2 - 4\mu_1^0 = 0$, $2\mu_3^0 - \mu_2^0 = 0$. The point $K = (-\frac{1}{2}\mu_2^0, 0, 0)$ is the unique critical point of the vector field v_{μ^0} (we denote by v_μ the vector field corresponding to the parameter μ). Let $\xi_i = \xi_i(\mu)$, $i = 1, 2$, be the roots of the equation $y^2 + \mu_2 y + \mu_1 = 0$ such that $\xi_i(\mu^0) = -\frac{1}{2}\mu_2^0$ (we assume $\mu \in S_1 \cup S_2$). If $y_1 - \xi_1 = x_1$, $y_2 = x_2$, $y_3 = x_3$, then the family (2.8) becomes

$$
\begin{aligned}
(4.1) \qquad \dot{x}_1 &= x_2\,, \\
\dot{x}_2 &= x_3\,, \\
\dot{x}_3 &= x_1(x_1 + \xi_1 - \xi_2) + \mu_3 x_2 \tilde{Q}_1(x_1, \mu) + \xi_1 x_2 \tilde{Q}_1(x_1, \mu) + \\
&\quad + x_1 x_2 \tilde{Q}_1(x_1, \mu) + \xi_1 x_3 \tilde{Q}_2(x_1, \mu) + x_1 x_3 \tilde{Q}_2(x_1, \mu) + \\
&\quad + x_2 x_3 \tilde{Q}_3(x_3, \mu) + x_2^2 \tilde{\Phi}_1(x, \mu) + x_3^2 \tilde{\Phi}_2(x, \mu)\,,
\end{aligned}
$$

where the functions $\tilde{Q}_i$, $\tilde{\Phi}_j$ have the same properties as the functions Q_i, Φ_j from (2.8). The family (4.1) has two critical points $K_1 = (0, 0, 0)$ and $K_2 = (\xi, 0, 0)$, where $\xi = \xi_2 - \xi_1$. The matrix of the linearization at K_1 is $L(K_1) = A(\mu) = (a_{ij})$, where $a_{12} = a_{23} = 1$, $a_{31} = \xi_1 - \xi_2$, $a_{32} = (\mu_3 + \xi_1)\,\tilde{Q}_1(0, \mu)$, $a_{33} = \xi_1 \tilde{Q}_2(0, \mu)$ and the other entries are equal to zero. For $\mu^0 \in Z_2$ also $a_{31} = a_{32} = 0$ and $a_{33} = \gamma = -\frac{1}{2}\mu_2^0 \tilde{Q}_2(0, 0, \mu_2^0, 0)$. If

$$
C = \begin{pmatrix} -\gamma & 1-\gamma & 1 \\ 0 & -\gamma & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{then} \quad C^{-1} = \begin{pmatrix} -\gamma^{-1} & \gamma^{-1} - \gamma^{-2} & \gamma^{-2} \\ 0 & -\gamma^{-1} & \gamma^{-1} \\ 0 & 0 & 1 \end{pmatrix}
$$

and using the change of coordinates $u = Cx$ we obtain

$$
(4.2) \qquad \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + B_0(\mu) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} F_0(u, \mu) \\ F_0(u, \mu) \\ F_0(u, \mu) \end{pmatrix},
$$

where $F_0(u, \mu) = f(C^{-1}u, \mu)$, f is the nonlinear part of the right-hand side of the third equation of (4.1), $B_0(\mu^0) = 0$, $F_0(u, \mu^0) = A_{200}u_1^2 + A_{020}u_2^2 + A_{002}u_3^2 + A_{110}u_1u_2 + A_{101}u_1u_3 + A_{011}u_2u_3 + o(\|u\|^2)$. By [6, Theorem 2.2] (see also [4], [11] the parametrized central manifold can be expressed as the graph of a function $u_3 = h(u_1, u_2, \mu)$ defined locally, in a neighbourhood of the point $(0, 0, \mu^0)$ for which

$$
h(0, 0, \mu^0) = \frac{\partial h(0, 0, \mu^0)}{\partial u_1} = \frac{\partial h(0, 0, \mu^0)}{\partial u_2} = 0\,.
$$

Therefore the reduction of the family (4.2) to the central manifold has the form

$$(4.3)\qquad \begin{pmatrix}\dot u_1\\ \dot u_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} + B(\mu)\begin{pmatrix}u_1\\ u_2\end{pmatrix} + \begin{pmatrix}F(u,\mu)\\ F(u,\mu)\end{pmatrix},$$

where $B(\mu^0) = 0$, $F(u, \mu^0) = A_{200}u_1^2 + A_{110}u_1u_2 + A_{020}u_2^2 + o(\|u\|^2)$, $A_{200} = \gamma^{-2}$, $A_{110} = -2\gamma^{-1}(\gamma^{-1} - \gamma^{-2}) + \gamma^{-2}Q_1(0, \mu^0)$.

Let us restrict the set of parameters to a neighbourhood $U(\mu^0) \subset P(\mu^0)$ of the point μ^0, where $P(\mu^0)$ is a two-dimensional surface crossing the set Z_2 transversally at μ^0. Using Bogdanov's method (see [3]) it is possible to rewrite the family (4.3) in suitable coordinates on $U(\mu^0)$, $\varepsilon = \varrho(\mu)$, $v = \delta(u)$, $\varrho(\mu^0) = 0$, $\delta(0) = 0$ to the form

$$(4.4)\qquad \begin{aligned}\dot v_1 &= v_2\,,\\ \dot v_2 &= \varepsilon_1 + \varepsilon_2 v_1 + g(v, \varepsilon)\,,\end{aligned}$$

where $g(v, 0) = (Qv, v) + o(\|v\|^2)$, $Q = (q_{ij})$ is a symmetric matrix with $q_{11} \neq 0$. By [3, Lemma 2] $q_{12} = \tilde q_{12} \cdot \tilde g_{11}^{-1}$, where $\tilde q_{11} = \gamma^{-2}$, $\tilde q_{12} = -2\gamma^{-1}(\gamma^{-1} - \gamma^{-2}) + \gamma^{-2}Q_1(0, \mu^0)$. Therefore sign $q_{12} = -$sign $\mu_2^0\omega_2$ for μ_2^0 sufficiently small.

Denote by $v_\varepsilon^+(v_\varepsilon^-)$ the family (4.4) with $q_{12} > 0$ $(q_{12} < 0)$. We remark that it is possible to transform the family v_ε^- to the same form with $q_{12} > 0$ by using the change of coordinates $x_2 \to -x_2$, $t \to -t$. The complete bifurcation diagram for the family v_ε^+ is described in [1, 3].

Now it is convenient to use the ν-coordinales (see (3.4)). Since $\nu_2 = -\frac{1}{2}(\mu_2 - \nu_1) \cdot Q_2(\xi_1, \mu)$, we have that $q_{12} > 0$ $(q_{12} < 0)$ for $\nu^0 = (0, \nu_2^0, 0) \in Z_2^+(Z_2^-)$. This means that the bifurcations near $\nu^0 \in Z_2^+(Z_2^-)$ correspond to the bifurcations of the family $v_\varepsilon^+(v_\varepsilon^-)$.

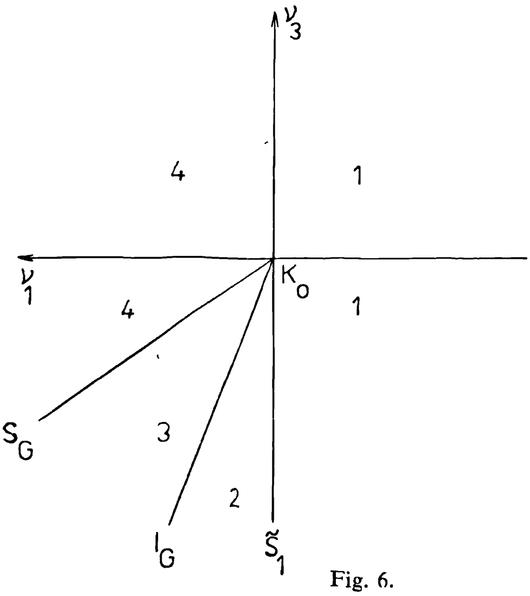

Fig. 6.

Assume $\nu^0 \in Z_2^-$. For the family v_ε^- there exists a curve R (see [3]), on which a stable focus bifurcates into a stable closed orbit and the focus becomes unstable. By the bifurcation diagram shown in Fig. 5, this Hopf bifurcation may occur only near

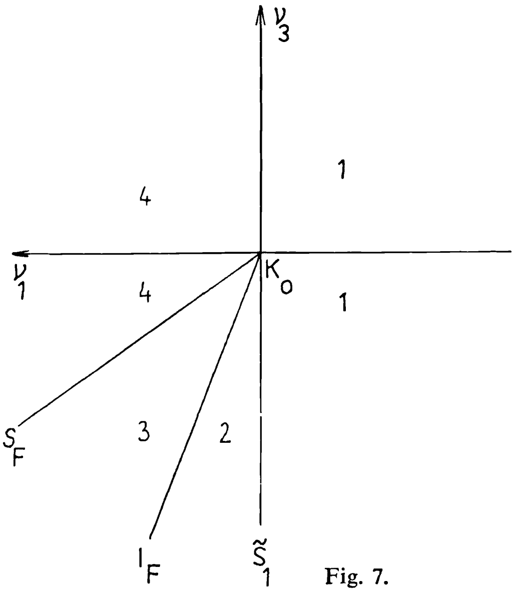

Fig. 7.

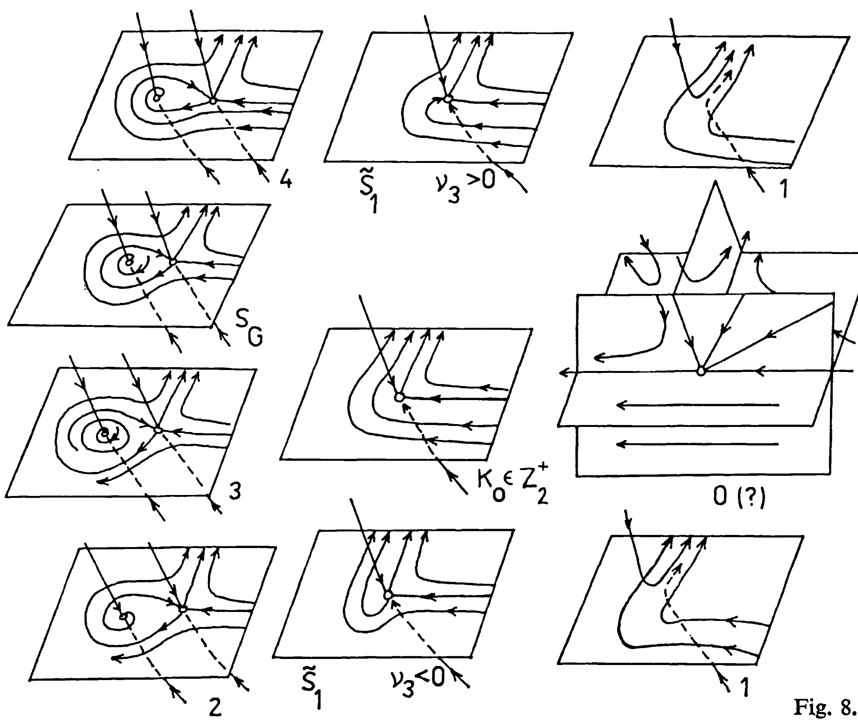

Fig. 8.

the point G. For $\nu \in I_1(I_3)$ the matrix $L(G)$ has one real eigenvalue $\beta_3 < 0$ and a couple of complex eigenvalues β, $\bar{\beta}$ with $\operatorname{Re} \beta < 0$ ($\operatorname{Re} \beta > 0$). This means that if the parameter goes in the direction $I_1 \to I_3$, crossing the surface I_G transversally, then the stable focus G bifurcates into a stable closed orbit and the focus becomes unstable. This determines the orientation of Bogdanov's bifurcation cycle. By [3] there must be a curve P in $U(\mu^0) \cap I_3$ with the end-point at ν^0 such that if the parameter ν approaches this curve, the period of the closed orbit tends to infinity, i.e. the closed orbit bifurcates into a homoclinic orbit. This implies that for the family (2.8) (in the ν-coordinates) there is a surface $S_G \subset I_3 \cap \mathscr{D}_3$ such that if the parameter ν approaches this surface, the period of the closed orbit, arising on I_G, tends to infinity. Since for a parameter from the set Z_{1c} the corresponding central manifold is three-dimensional, the two-dimensional central manifold corresponding to a parameter from the set Z_2^- is destroyed if the parameter passes out of a neighbourhood of the set Z_2^-. Therefore the global properties of the surface S_G cannot be found by the methods of plane vector fields and so it is difficult to find them. We know the form of S_G near Z_2^-.

If $\nu^0 \in Z_2^+$, then by the bifurcation diagram, the Hopf bifurcation may occur near the point F only. For the family v_ε^+ there exists a curve, on which an unstable focus

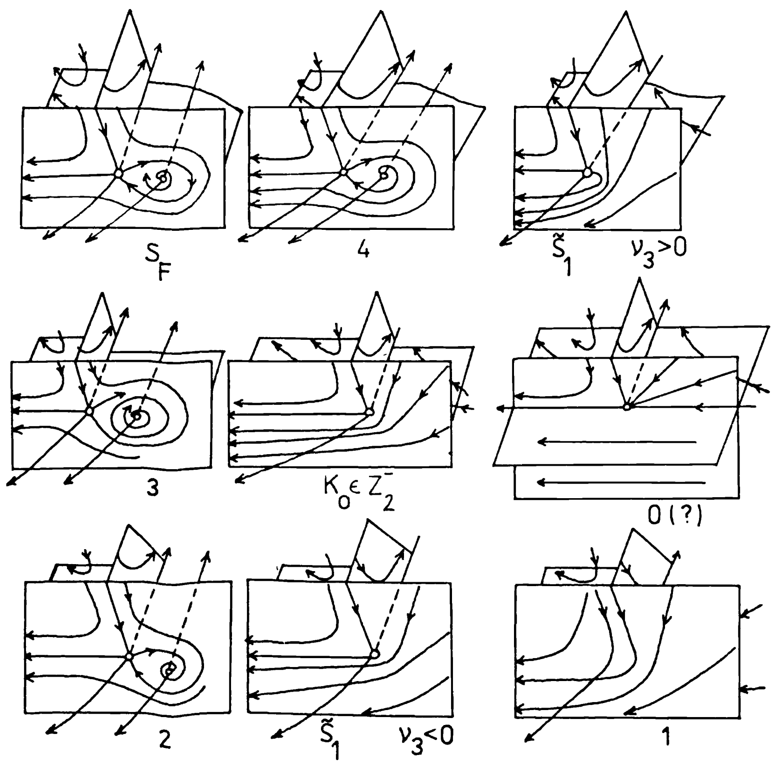

Fig. 9.

bifurcates into an unstable closed orbit. For $v \in K_3(I_3)$, the matrix $L(F)$ has one real eigenvalue $\lambda_2 > 0$ and a couple of complex eigenvalues λ, $\bar{\lambda}$ with Re $\lambda > 0$ (Re $\lambda < 0$). This means that if the parameter v goes in the direction $K_3 \to I_3$, crossing the surface I_F transversally, an unstable focus bifurcates into an unstable closed orbit and this determines the orientation of Bogdanov's cycle. Similarly as above, Bogdanov's results imply that there must be a surface $S_F \subset I_3 \cap \mathscr{D}_2$ on which the closed orbit arising on I_F bifurcates into a homoclinic orbit. The problem of global properties of S_F remains open.

From the above considerations we conclude that in a neighbourhood of $v^0 \in \tilde{Z}_2$ the bifurcation diagram and the bifurcations look like in Figures 6–9.

We have described the bifurcations near the set $\tilde{Z}_2$. For the results bifurcations near the set $\tilde{Z}_{1c}$ we refer to the papers [5], [7–10]. The problem how the phase portraits appearing for the parameter from a neighbourhood of $\tilde{Z}_2$ may bifurcate into different phase portraits corresponding to the values of the parameters from a neighbourhood of the set $\tilde{Z}_{1c}$ remains open.

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