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THE DIRICHLET PROBLEM AND WEIGHTED SPACES I

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0. INTRODUCTION

0.1. Let us consider the differential operator of order $2k$,

$$(Au)(x) = \sum_{|\alpha|,|\beta| \leq k} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x) D^\beta u(x)), \tag{0.1}$$

defined on a domain $\Omega \subset \mathbb{R}^N$, and the associated bilinear form

$$a(u, v) = \sum_{|\alpha|,|\beta| \leq k} \int_\Omega a_{\alpha\beta}(x) D^\beta u \, D^\alpha v \, dx . \tag{0.2}$$

Further, let us consider the Sobolev space

$$W^{k,2}(\Omega)$$

and its subspace

$$W_0^{k,2}(\Omega) = \overline{C_0^\infty(\Omega)} .$$

Throughout this paper, only *real* functions will be considered.

0.2. Let us recall that to find a weak solution of the Dirichlet problem for the operator A means to find, for a given function $u_0 \in W^{k,2}(\Omega)$ and for a given continuous linear functional $f \in (W_0^{k,2}(\Omega))^*$, a function

$$u \in W_0^{k,2}(\Omega)$$

such that

(i) $u - u_0 \in W_0^{k,2}(\Omega)$;

(ii) $a(u, v) = \langle f, v \rangle$ for every $v \in W_0^{k,2}(\Omega)$.

Here the function u_0 represents the right-hand side in the boundary condition and condition (i) can be interpreted in the following sense:

$$\frac{\partial^i u}{\partial n^i} = \frac{\partial^i u_0}{\partial n^i} \quad \text{on} \quad \partial\Omega , \quad i = 0, 1, \ldots, k - 1 ,$$

where $\partial\Omega$ is the boundary of the domain Ω and n the outer normal to $\partial\Omega$. The functional f represents the right-hand side in the (formal) differential equation

$$Au = f;$$

the symbol $\langle\cdot,\cdot\rangle$ expresses the duality pairing between $(W_0^{k,p}(\Omega))^*$ and $W_0^{k,p}(\Omega)$.

0.3. The Lax-Milgram Lemma. *Let V be a Hilbert space and $b(u, v)$ a bilinear form defined on $V \times V$. Let this form be continuous — i.e. let there exist a constant $c_1 > 0$ such that*

$$|b(u, v)| \leqq c_1 \|u\|_V \|v\|_V \tag{0.3}$$

holds for all $u, v \in V$, and V-elliptic — i.e. let there exist a constant $c_2 > 0$ such that

$$b(u, u) \geqq c_2 \|u\|_V^2 \tag{0.4}$$

holds for all $u \in V$. Further, let f be a functional from V^*. Then there exists one and only one element $u \in V$ such that

$$b(u, v) = \langle f, v\rangle \quad \textit{for every} \quad v \in V \tag{0.5}$$

and

$$\|u\|_V \leqq c_3 \|f\|_{V^*} \tag{0.6}$$

(with constant c_3 independent of u_0 and f).

0.4. As is well known — see e.g. [7], [11], with help of the Lax-Milgram Lemma it is possible to prove the *existence* and *uniqueness of the weak solution* of the Dirichlet problem from Section 0.2, provided the differential operator A from (0.1) satisfies certain conditions: It suffices that $a_{\alpha\beta}$ be bounded measurable functions,

$$a_{\alpha\beta} \in L^\infty(\Omega) \tag{0.7}$$

and that the ellipticity condition

$$\sum_{|\alpha|,|\beta| \leqq k} a_{\alpha\beta}(x)\, \xi_\alpha \xi_\beta \geqq c_4 \sum_{|\alpha| \leqq k} |\xi_\alpha|^2 \tag{0.8}$$

hold for a.e. $x \in \Omega$ and for all real vectors $\xi = \{\xi_\alpha, |\alpha| \leqq k\}$ with a constant $c_4 > 0$ independent of ξ. Under certain additional assumptions on the domain Ω, condition (0.8) can be weakened e.g. in the sense that the summation on the right-hand side ranges only over the multiindices of the length $k : |\alpha| = k$.

The aim of this paper is to show that the method of the proof of existence of a weak solution *can be extended to a broader class of differential operators*, viz. to equations, for which the classical Sobolev spaces cannot be used. We shall show how it is possible to construct, for a given operator A, a suitable *weighted* Sobolev space $W^{k,2}(\Omega, S)$, in which the existence and uniqueness of the weak solution of the Dirichlet problem is already guaranteed.

Now, we shall define the space just mentioned and the corresponding Dirichlet problem.

0.5. The weighted Sobolev space. Let Ω be a domain in $\mathbb{R}^N$; the symbol

$$\mathscr{W}(\Omega) \tag{0.9}$$

denotes the set of all measurable, a.e. in Ω positive functions. Such functions will be called *weight functions.*

Let us denote by $\boldsymbol{M}(N, k)$ the set of all N-dimensional multiindices of length at most k:

$$\boldsymbol{M}(N, k) = \{\alpha \in \mathbb{N}_0^N,\ |\alpha| \leqq k\}\,.$$

Let $\boldsymbol{M}$ be a fixed subset of $\boldsymbol{M}(N, k)$ and let $\boldsymbol{M}$ contain at least one multiindex of length k. Further, let a collection of weight functions

$$S = \{w_\alpha = w_\alpha(x),\ w_\alpha \in \mathscr{W}(\Omega),\ \alpha \in \boldsymbol{M}\} \tag{0.10}$$

be given; the collection S will be called shortly a *weight.*

For $p > 1$ we introduce the linear space

$$W^{k,p}(\Omega, S) \tag{0.11}$$

as the set of all measurable functions $u = u(x)$ defined on Ω which have on Ω (generalized) derivatives $D^\alpha u$ for $\alpha \in \boldsymbol{M}$ such that

$$\|D^\alpha u\|_{p,w_\alpha}^p = \int_\Omega |D^\alpha u(x)|^p\, w_\alpha(x)\, dx < \infty\,. \tag{0.12}$$

Let us suppose that $W^{k,p}(\Omega, S)$ is a Banach space with respect to the norm

$$\|u\|_{k,p,S} = \Big(\sum_{\alpha \in \boldsymbol{M}} \|D^\alpha u\|_{p,w_\alpha}^p\Big)^{1/p} \tag{0.13}$$

and let us introduce the so called "nulled-space"

$$W_0^{k,p}(\Omega, S) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{k,p,S}} \tag{0.14}$$

0.6. Remarks. (i) If we take $\boldsymbol{M} = \boldsymbol{M}(N, k)$ and $w_\alpha(x) \equiv 1$ for $\alpha \in \boldsymbol{M}$, we obtain the classical Sobolev space $W^{k,p}(\Omega)$.

(ii) The assumption that (0.13) is a norm and that $W^{k,p}(\Omega, S)$ is a Banach space with respect to this norm imposes some conditions on the set $\boldsymbol{M}$ as well as the collection S. The reader can easily verify that the space $W^{k,p}(\Omega, S)$ will be complete, if e.g. $\theta = (0, 0, \ldots, 0)$ belongs to $\boldsymbol{M}$ and if

(0.15) $$w_\alpha^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\Omega)\,, \quad \alpha \in \boldsymbol{M}\,.$$

Thus restricts to some extent the choice of the weight functions. In a forthcoming paper [5] we shall show how conditions (0.15) can be avoided.

(iii) In order to be able to introduce the "nulled-space" $W_0^{k,p}(\Omega, S)$, the set $C_0^\infty(\Omega)$ has to be a subset of the space $W^{k,p}(\Omega, S)$. This will be fulfilled iff

(0.16) $$w_\alpha \in L^1_{\mathrm{loc}}(\Omega)\,, \quad \alpha \in \boldsymbol{M}\,.$$

However, even this condition can be avoided – see [5].

(iv) The norm in the "nulled-space" $W_0^{k,p}(\Omega, S)$ is again given by (0.13). In the case of the classical Sobolev space it is known that the expression

$$|||u|||_{k,p} = \Big(\sum_{|\alpha|=k} \|D^\alpha u\|_p^p\Big)^{1/p}\,,$$

where the summation ranges only over the derivatives of the highest, k-th, order, gives an equivalent norm in $W_0^{k,p}(\Omega)$. An analogous, but more complicated situation occurs for weighted spaces; we shall use estimates of the type

(0.17) $$\|u\|_{p,w_0} \leqq \mathrm{const}\left(\sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{p,w_i}^p\right)^{1/p},$$

which hold for functions $u \in C_0^\infty(\Omega)$ with a constant independent of u, under certain conditions on the weight functions $w_0, w_1, \ldots, w_N$. More details about equivalent norms can be found e.g. in [4], [10]; now we shall state Hardy's inequality, which is an important tool for deriving estimates of the type (0.17) and which will be used in the sequel in some examples.

0.7. Hardy's inequality (see [4]). Let $u = u(t)$ be a function from $C_0^\infty(0, \infty)$. Then for $\varepsilon \neq p - 1$ and $p > 1$ the following inequality holds:

(0.18) $$\int_0^\infty |u(t)|^p\, t^{\varepsilon - p}\, \mathrm{d}t \leqq \left(\frac{p}{|\varepsilon - p + 1|}\right)^p \int_0^\infty |u'(t)|^p\, t^\varepsilon\, \mathrm{d}t\,.$$

0.8. The Dirichlet problem. Let A be the differential operator (0.1) and $a(u, v)$ the corresponding bilinear form (0.2). Let $W^{k,2}(\Omega, S)$ be the weighted Sobolev space from Section 0.5 (with $p = 2$) and let us suppose that the coefficients $a_{\alpha\beta}$ of the operator A are such that the expression $a(u, v)$ is meaningful for every $u, v \in W^{k,2}(\Omega, S)$. Let u_0 be a given function from $W^{k,2}(\Omega, S)$ and f a continuous linear functional from $(W_0^{k,2}(\Omega, S))^*$. A function $u \in W^{k,2}(\Omega, S)$ is called a *weak solution of the Dirichlet problem in the space* $W^{k,2}(\Omega, S)$, if it satisfies

(i) $u - u_0 \in W_0^{k,2}(\Omega, S)$;

(ii) $a(u, v) = \langle f, v\rangle$ for every $v \in W_0^{k,2}(\Omega, S)$.

1.1. Example. Let us consider a special differential operator of the second order

$$(1.1)\qquad (Au)(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)\frac{\partial u}{\partial x_i}\right) + a_0(x)\,u\,, \quad x \in \Omega\,.$$

In this case the bilinear form (0.2) has the form

$$(1.2)\qquad a(u, v) = \sum_{i=1}^{N} \int_\Omega a_i(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x + \int_\Omega a_0(x)\,uv\,\mathrm{d}x\,,$$

and, in particular,

$$(1.3)\qquad a(u, u) = \int_\Omega |u|^2\,a_0(x)\,\mathrm{d}x + \sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^2 a_i(x)\,\mathrm{d}x\,.$$

If all the coefficients $a_i(x)$ are weight functions, i.e. if

$$(1.4)\qquad a_i \in \mathscr{W}(\Omega) \quad \text{for} \quad i = 0, 1, \ldots, N\,,$$

then the expression (1.3) is nothing else than the square of the norm in the weighted space

$$(1.5)\qquad W^{1,2}(\Omega, S)$$

with the collection

$$(1.6)\qquad S = \{a_0, a_1, \ldots, a_N\}\,,$$

i.e., we have

$$(1.7)\qquad a(u, u) = \|u\|_{1,2,S}^2\,.$$

At the same time it follows from the Hölder inequality that

$$\left|\int_\Omega a_i(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x\right| \leqq \int_\Omega \sqrt{[a_i(x)]}\left|\frac{\partial u}{\partial x_i}\right| \sqrt{[a_i(x)]}\left|\frac{\partial v}{\partial x_i}\right|\mathrm{d}x \leqq$$

$$\leqq \left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^2 a_i(x)\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega \left|\frac{\partial v}{\partial x_i}\right|^2 a_i(x)\,\mathrm{d}x\right)^{1/2} \leqq \|u\|_{1,2,S}\,\|v\|_{1,2,S}$$

and since the other terms in (1.2) can be estimated analogously, we eventually have

$$(1.8)\qquad |a(u, v)| \leqq (N + 1)\,\|u\|_{1,2,S}\,\|v\|_{1,2,S}\,.$$

Inequality (1.8) and identity (1.7) show that the form $a(u, v)$ fulfils conditions (0.3) and (0.4) of the Lax-Milgram Lemma. Therefore, one can expect that the existence and uniqueness of a weak solution of the Dirichlet problem in the weighted space $W^{1,2}(\Omega, S)$ can be easily proved by applying this Lemma. Let us note that the weight S is determined directly by the coefficients of the differential operator A.

(i) Let the coefficients a_i fulfil the following conditions: There exist constants c_1, c_2 such that

$$a_i(x) \leqq c_1\,, \tag{1.9}$$

$$a_i(x) \geqq c_2 > 0 \tag{1.10}$$

for a.e. $x \in \Omega$ and for $i = 0, 1, \ldots, N$. Then the above mentioned approach brings nothing new, since – as can be easily shown – the weighted space $W^{1,2}(\Omega, S)$ coincides with the *classical* Sobolev space $W^{1,2}(\Omega)$: the expression (1.7) defines a norm which is, in view of conditions (1.9) and (1.10), equivalent to the usual norm in $W^{1,2}(\Omega)$. The application of a weighted space is *purely formal* in this case.

(ii) Let us assume that one or both the conditions (1.9), (1.10) are violated for *some* of the functions $a_i(x)$. [Coefficients of the form

$$a_i(x) = [\operatorname{dist}(x, \Gamma)]^\varepsilon\,, \tag{1.11}$$

with Γ a part of the closure $\bar{\Omega}$ of the domain Ω, meas $\Gamma = 0$, and with ε a (non-zero) real number, may serve as a typical example of such functions: condition (1.9) is violated for $\varepsilon < 0$ and condition (1.10) for $\varepsilon > 0$.] In this case, the classical Sobolev space $W^{1,2}(\Omega)$ *cannot be used* in general; on the other hand, one can use the weighted space $W^{1,2}(\Omega, S)$ and so enlarge the class of operators A for which the Dirichlet problem is uniquely solvable. Let us note that if the conditions

$$\frac{1}{a_i} \in L^1_{\text{loc}}(\Omega)\,, \quad a_i \in L^1_{\text{loc}}(\Omega)\,, \quad i = 0, 1, \ldots, N \tag{1.12}$$

are fulfilled in addition to conditions (1.4), then $W^{1,2}(\Omega, S)$ is a Hilbert space and the definition of the space $W_0^{1,2}(\Omega, S)$ makes sense – see Remark 0.6 (ii), (iii).

(iii) Conditions (1.4) can be *weakened* as well: Let us assume that (1.4) takes place only for $i = 1, \ldots, N$, while for $i = 0$ we have

$$a_0(x) = -\lambda\, b_0(x) \quad \text{with} \quad \lambda \geqq 0\,, \quad b_0 \in \mathscr{W}(\Omega) \cap L^1_{\text{loc}}(\Omega)\,, \quad b_0^{-1} \in L^1_{\text{loc}}(\Omega)\,. \tag{1.13}$$

Then the expression $[a(u, u)]^{1/2}$ in general fails to have all the properties of a norm and consequently, the weighted space $W^{1,2}(\Omega, S)$ can no more be introduced in a natural way. Therefore we choose

$$S = \{b_0, a_1, \ldots, a_N\} \tag{1.14}$$

and consider the spaces $W^{1,2}(\Omega, S)$ and $W_0^{1,2}(\Omega, S)$ with this *new* collection of weight functions. In the same way as in part (ii) we prove that

$$|a(u, v)| \leqq (N + \lambda)\, \|u\|_{1,2,S}\, \|v\|_{1,2,S} \tag{1.15}$$

holds, so that condition (0.3) of the Lax-Milgramm Lemma is fulfilled. It remains to find out when condition (0.4) is fulfilled. To this end let us assume that the estimate

$$\|u\|^2_{2,b_0} \leqq c \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|^2_{2,a_i} \tag{1.16}$$

holds for all functions $u \in W_0^{1,2}(\Omega, S)$ with a constant $c > 0$ independent of u. Then we have

$$\|u\|^2_{1,2,S} = \|u\|^2_{2,b_0} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|^2_{2,a_i} \leqq (c+1) \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|^2_{2,a_i}, \tag{1.17}$$

i.e., the expressions

$$\|u\|_{1,2,S} = \left(\|u\|^2_{2,b_0} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|^2_{2,a_i} \right)^{1/2} \tag{1.18}$$

and

$$|||u|||_{1,2,S} = \left(\sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|^2_{2,a_i} \right)^{1/2} \tag{1.19}$$

are equivalent norms on $W_0^{1,2}(\Omega, S)$. Making use of inequalities (1.16) and (1.17) we obtain

$$a(u, u) = \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|^2_{2,a_i} - \lambda \|u\|^2_{2,b_0} \geqq (1 - \lambda c) \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|^2_{2,a_i} \geqq \frac{1 - \lambda c}{c + 1} \|u\|^2_{1,2,S}.$$

Hence it follows that condition (0.4) is fulfilled provided

$$0 \leqq \lambda < \frac{1}{c}. \tag{1.20}$$

In other words, for these values of λ it is again possible to prove existence and uniqueness of a weak solution of the Dirichlet problem in the space $W_0^{1,2}(\Omega, S)$ by virtue of the Lax-Milgram Lemma.

[If $a_0 \equiv 0$, i.e. $\lambda = 0$, then b_0 may be chosen in various ways, nevertheless, always so that the estimate (1.16) holds, which guarantees the equivalence of the norms (1.18) and (1.19).]

The following example demonstrates that weighted spaces are useful for solving the Dirichlet problem on *unbounded* domains.

1.2. Example. Consider the Dirichlet problem for the Laplace operator $-\Delta$. This is the operator from (1.1) with the coefficients $a_0 = 0$, $a_1 = a_2 = \ldots = a_N = 1$. Apparently we should do with the classical Sobolev spaces $W^{1,2}(\Omega)$ and $W_0^{1,2}(\Omega)$. However, if the domain Ω is unbounded, then the corresponding bilinear form

$$a(u, v) = \sum_{i=1}^{N} \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx$$

in general *fails to be* $W_0^{1,2}(\Omega)$*-elliptic*, i.e., there need not exist a constant $c > 0$ such that

$$a(u, u) \geqq c \|u\|^2_{1,2,S}$$

holds for all functions $u \in W_0^{1,2}(\Omega)$. This follows from the fact that for certain types of the domains Ω, the expression $[a(u, u)]^{1/2}$ is not an equivalent norm on the space $W_0^{1,2}(\Omega)$ – cf. [1]. Hence condition (0.4) is not fulfilled and the classical Sobolev spaces cannot be used in connection with the Lax-Milgram Lemma.

Nevertheless, it can be shown that for a suitably chosen weight function b_0 – e.g. $b_0(x) = (1 + |x|)^\varepsilon$, see [3] – the expression $[a(u, u)]^{1/2}$ will be an equivalent norm on the *weighted* space $W_0^{1,2}(\Omega, S)$ provided we choose $S = \{b_0, 1, 2, \ldots, 1\}$.*) Consequently, if we choose $V = W_0^{1,2}(\Omega, S)$ then condition (0.4) is fulfilled and, since the validity of condition (0.3) is obvious, we can prove the existence of a weak solution of the Dirichlet problem in the space $W_0^{1,2}(\Omega, S)$.

1.3. Example. Consider a plane domain $(N = 2)$ and choose the square $(0, 1) \times \times (0, 1)$ for the domain Ω. Further, consider the fourth-order differential operator

$$\frac{\partial^2}{\partial x_1\, \partial x_2}\left(\frac{\partial^2 u}{\partial x_1\, \partial x_2}\right). \tag{1.21}$$

In this case the corresponding bilinear form (0.2) is of the form

$$a(u, v) = \int_\Omega \frac{\partial^2 u}{\partial x_1\, \partial x_2} \frac{\partial^2 v}{\partial x_1\, \partial x_2}\, dx_1\, dx_2\,. \tag{1.22}$$

If we worked with the space $V = W_0^{2,2}(\Omega)$ we should easily find out that the form $a(u, v)$ satisfies condition (0.3) but *not* condition (0.4). So the classical Sobolev space cannot be used. Therefore, let us choose the weighted space

$$W^{2,2}(\Omega, S)\,, \quad S = \{1, 0, 0, 0, 1, 0\}\,, \text{ **)}$$

i.e. the space with a norm given by the formula

$$\|u\|_{2,2,S}^2 = \int_\Omega |u|^2\, dx_1\, dx_2 + \int_\Omega \left|\frac{\partial^2 u}{\partial x_1\, \partial x_2}\right|^2 dx_1\, dx_2\,. \tag{1.23}$$

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*) In the paper [3] the inequality $\varepsilon < -N$ is considered. However, it can be shown that we can choose $\varepsilon \leqq -2$.

**) This notation does not comply with that introduced in Sec. 0.5. It means that $\mathbf{M} = \{(0, 0), (1, 1)\}$ and $S = \{w_\alpha, \alpha \in \mathbf{M}\}$, where $w_{(0,0)} = w_{(1,1)} = 1$. Analogous notation will be also used in the forthcoming examples: the notation $S = \{a, b, c, d, e, f\}$ for $N = 2$ and $k = 2$ will mean that the weight functions $a, b, \ldots, f$ are arranged in the same order as they successively appear at the functions

$$u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1\, \partial x_2}, \frac{\partial^2 u}{\partial x_2^2};$$

the "zero" weight functions, which *do not appear* in the corresponding norm at all, are written down as well.

It is immediately seen that condition (0.3) is fulfilled provided we choose $V = W_0^{2,2}(\Omega, S)$. Thus it remains to prove that condition (0.4) is fulfilled as well, i.e. the form $a(u, v)$ is V-elliptic. To this end it suffices to prove that the expressions

$$[a(u, u)]^{1/2} \quad \text{and} \quad \|u\|_{2,2,S} \tag{1.24}$$

are equivalent norms on $W_0^{2,2}(\Omega, S)$. Let us prove it: Since

$$\int_\Omega |u|^2 \, dx_1 \, dx_2 = \int_0^1 \left(\int_0^1 |u(x_1, x_2)|^2 \, dx_1 \right) dx_2$$

and since

$$\int_0^1 |u(x_1, x_2)|^2 \, dx_1 \leqq \int_0^1 |u(x_1, x_2)|^2 \frac{1}{x_1^2} \, dx_1 \leqq 4 \int_0^1 \left| \frac{\partial u}{\partial x_1} (x_1, x_2) \right|^2 dx_1$$

for each function $u \in C_0^\infty(\Omega)$ [the former inequality follows from the fact that $x \in \Omega$ and hence $x_1 \leqq 1$, the latter is a consequence of Hardy's inequality – cf. (0.18) – for $p = 2$ and $\varepsilon = 0$, since for every x_2 we have $u(x_1, x_2) = C_0^\infty(0, 1)$ and after extending it by zero it belongs even to $C_0^\infty(0, \infty)$], we obtain by integrating by x_2 from 0 to 1:

$$\int_\Omega |u|^2 \, dx_1 \, dx_2 \leqq 4 \int_\Omega \left| \frac{\partial u}{\partial x_1} \right|^2 dx_1 \, dx_2 \, . \tag{1.25}$$

Applying the same procedure to the function $\partial u / \partial x_1$ and the variable x_2, we obtain

$$\int_0^1 \left| \frac{\partial u}{\partial x_1} (x_1, x_2) \right|^2 dx_2 \leqq \int_0^1 \left| \frac{\partial u}{\partial x_1} (x_1, x_2) \right|^2 \frac{1}{x_2^2} \, dx_2 \leqq 4 \int_0^1 \left| \frac{\partial^2 u}{\partial x_1 \, \partial x_2} (x_1, x_2) \right|^2 dx_2$$

that is

$$\int_\Omega \left| \frac{\partial u}{\partial x_1} \right|^2 dx_1 \, dx_2 \leqq 4 \int_\Omega \left| \frac{\partial^2 u}{\partial x_1 \, \partial x_2} \right|^2 dx_1 \, dx_2 \, .$$

This together with (1.25) implies

$$\int_\Omega |u|^2 \, dx_1 \, dx_2 \leqq 16 \int_\Omega \left| \frac{\partial^2 u}{\partial x_1 \, \partial x_2} \right|^2 dx_1 \, dx_2 = 16 a(u, u) \tag{1.26}$$

for every function $u \in C_0^\infty(\Omega)$ and hence also for every function $u \in W_0^{2,2}(\Omega, S)$. Now, it follows from (1.22), (1.23) and (1.26) that

$$a(u, u) \leqq \int_\Omega |u|^2 \, dx_1 \, dx_2 + \int_\Omega \left| \frac{\partial^2 u}{\partial x_1 \, \partial x_2} \right|^2 dx_1 \, dx_2 = \|u\|_{2,2,S}^2 \leqq 17 \, a(u, u)$$

for $u \in W_0^{2,2}(\Omega, S)$. The inequality yields that the norms in (1.24) are equivalent, as well as that the form $a(u, v)$ is $W_0^{2,2}(\Omega, S)$-elliptic (with the constant $\frac{1}{17}$).

Consequently, there exists a unique weak solution of the Dirichlet problem for the operator (1.21) in the space $W^{2,2}(\Omega, S)$.

1.4. Remarks. (i) The weighted space $W^{2,2}(\Omega, S)$ from the previous example is actually not a *weighted* space but rather an *anisotropic* one (with the *dominating mixed derivative* $\partial^2 u/\partial x_1\, \partial x_2$). We have chosen this example in view of its simplicity; nonetheless, we shall present another similar example involving more complicated weights (see Example 1.5).

(ii) In Section (0.8) we have introduced the Dirichlet problem in a completely abstract way, by means of the spaces $W^{k,2}(\Omega, S)$ and $W_0^{k,2}(\Omega, S)$ without specifying in more detail in which way we are to understand the *boundary value conditions to be satisfied.* The boundary condition is expressed by condition (i) in 0.8: $u - u_0 \in$ $\in W_0^{k,2}(\Omega, S)$ means that "u (i.e., the solution) behaves on $\partial\Omega$ as u_0 (i.e., the given function)". Evidently, when the weighted spaces are involved we cannot in general characterize the "boundary behavior of the function u on $\partial\Omega$" so satisfactorily as in the case of the classical Sobolev spaces, when $u \in W_0^{k,p}(\Omega)$ means that "$D^\gamma u = 0$ on $\partial\Omega$ in the sense of traces for $|\gamma| \leqq k - 1$". This is connected with the fact that a full description of the properties of the *traces* of functions from weighted spaces has not yet been available. For this reason we stop at the formulation of the Dirichlet problem in the form from Definition 0.8, not going into its *interpretation* for the time being.

1.5. Example. Consider again a plane domain ($N = 2$), choosing the first quadrant $(0, \infty) \times (0, \infty)$ for the domain Ω. Further, consider the fourth-order differential operator

$$(1.27)\quad Au = \frac{\partial^2}{\partial x_1\, \partial x_2}\left(x_1^{\delta_1} x_2^{\delta_2} \frac{\partial^2 u}{\partial x_1\, \partial x_2}\right) - \frac{\partial}{\partial x_1}\left(x_1^{\gamma_1} x_2^{\gamma_2} \frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(x_1^{\beta_1} x_2^{\beta_2} \frac{\partial u}{\partial x_2}\right).$$

The corresponding bilinear form is

(1.28)
$$a(u, v) = \int_\Omega \frac{\partial^2 u}{\partial x_1\, \partial x_2} \frac{\partial^2 v}{\partial x_1\, \partial x_2} x_1^{\delta_1} x_2^{\delta_2}\, dx_1\, dx_2 + \int_\Omega \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_1} x_1^{\gamma_1} x_2^{\gamma_2}\, dx_1\, dx_2 + \int_\Omega \frac{\partial u}{\partial x_2} \frac{\partial v}{\partial x_2} x_1^{\beta_1} x_2^{\beta_2}\, dx_1\, dx_2 .$$

(i) Consider the weighted space

$$(1.29)\qquad W^{2,2}(\Omega, S)$$

where the collection S is chosen in the following way:

$$(1.30)\qquad S = \{x_1^{\delta_1 - 2} x_2^{\delta_2 - 2}, x_1^{\gamma_1} x_2^{\gamma_2}, x_1^{\beta_1} x_2^{\beta_2}, 0, x_1^{\delta_1} x_2^{\delta_2}, 0\} .$$

Thus the weight functions at $\partial u/\partial x_1$, $\partial u/\partial x_2$ and $\partial^2 u/\partial x_1\,\partial x_2$ are determined directly by the corresponding coefficients in the operator (1.27); the choice of the weight function at the function u in the form $x_1^{\delta_1-2}x_2^{\delta_2-2}$ is made on purpose: namely, if $\delta_1 \neq 1$, $\delta_2 \neq 1$, we obtain by using twice Hardy's inequality (0.18) (for $p = 2$ and $\varepsilon = \delta_1$, $\varepsilon = \delta_2$, respectively) the relation

(1.31)

$$\int_\Omega |u(x_1, x_2)|^2\, x_1^{\delta_1-2}x_2^{\delta_2-2}\,\mathrm{d}x_1\,\mathrm{d}x_2 = \int_0^\infty \left(\int_0^\infty |u(x_1x_2)|^2\, x_1^{\delta_1-2}\,\mathrm{d}x_1\right) x_2^{\delta_2-2}\,\mathrm{d}x_2 \leqq$$

$$\leqq \frac{4}{|\delta_1 - 1|^2}\int_0^\infty\left(\int_0^\infty \left|\frac{\partial u}{\partial x_1}(x_1, x_2)\right|^2 x_1^{\delta_1}\,\mathrm{d}x_1\right) x_2^{\delta_2-2}\,\mathrm{d}x_2 =$$

$$= \frac{4}{|\delta_1 - 1|^2}\int_0^\infty\left(\int_0^\infty \left|\frac{\partial u}{\partial x_1}(x_1, x_2)\right|^2 x_2^{\delta_2-2}\,\mathrm{d}x_2\right) x_1^{\delta_1}\,\mathrm{d}x_1 \leqq$$

$$= \frac{16}{|\delta_1 - 1|^2\,|\delta_2 - 1|^2}\int_0^\infty\left(\int_0^\infty \left|\frac{\partial^2 u}{\partial x_1\,\partial x_2}(x_1, x_2)\right|^2 x_2^{\delta_2}\right)\mathrm{d}x_2\, x_1^{\delta_1}\,\mathrm{d}x_1 =$$

$$= \frac{16}{(|\delta_1 - 1|\,.\,|\delta_2 - 1|)^2}\int_\Omega \left|\frac{\partial^2 u}{\partial x_1\,\partial x_2}\right|^2 x_1^{\delta_1}x_2^{\delta_2}\,\mathrm{d}x_1\,\mathrm{d}x_2\,,$$

which holds for functions from $C_0^\infty(\Omega)$ and hence also for functions $u \in W_0^{2,2}(\Omega, S)$. Since the norm on $W^{2,2}(\Omega, S)$ is given by the formula

$$\|u\|_{2,2,S}^2 = a(u, u) + \int_\Omega |u|^2\, x_1^{\delta_1-2}x_2^{\delta_2-2}\,\mathrm{d}x_1\,\mathrm{d}x_2$$

we obtain by virtue of (1.31) the estimate

(1.32) $\quad \|u\|_{2,2,S}^2 \leqq c\, a(u, u)$ with $c = [1 + 16(|\delta_1 - 1|\,.\,|\delta_2 - 1|)^{-2}]\,.$

However, this means that the form $a(u, v)$ is $W_0^{2,2}(\Omega, S)$-elliptic, i.e., it satisfies condition (0.4). Since it can be easily shown by means of the Hölder inequality that the form $a(u, v)$ satisfies condition (0.3) as well, we actually arrived at the following assertion: If $\delta_1 \neq 1$, $\delta_2 \neq 1$ and if the collection S is chosen according to (1.30), then there is a unique weak solution of the Dirichlet problem for the operator A from (1.27) in the weighted space $W^{2,2}(\Omega, S)$.

At the same time, inequality (1.32) together with the obvious inequality $\|u\|_{2,2,S}^2 \geqq$ $\geqq a(u, u)$ asserts that for $\delta_1 \neq 1$, $\delta_2 \neq 1$ the expressions $\|u\|_{2,2,S}$ and $[a(u, u)]^{1/2}$ are *equivalent norms* on $W_0^{2,2}(\Omega, S)$:

(ii) In the preceding considerations we have not at all used those parts of the norm in $W^{2,2}(\Omega, S)$ that involve the first derivatives $\partial u/\partial x_1$, $\partial u/\partial x_2$. Therefore, let us consider the problem whether the Dirichlet problem for the operator (1.27) is solvable in the space $W^{2,2}(\Omega, \tilde{S})$, where we put

(1.33) $$\tilde{S} = \{x_1^{\delta_1-2}x_2^{\delta_2-2}, 0, 0, 0, x_1^{\delta_1}x_2^{\delta_2}, 0\}\,.$$

In the same way as in part (i) we can show that the form $a(u, v)$ is $W_0^{2,2}(\Omega, \tilde{S})$-elliptic, i.e., it fulfils condition (0.4). Let us have a look at condition (0.3): If the inequality

$$|a(u, v)| \leqq c_1 \|u\|_{2,2,\tilde{S}} \|v\|_{2,2,\tilde{S}}$$

were satisfied, then

$$|a(u, u)| \leqq c_1 \|u\|_{2,2,\tilde{S}}^2$$

would hold as well. This in particular would mean that for $u \in C_0^\infty(\Omega)$ we should have the estimate

$$\int_\Omega \left|\frac{\partial u}{\partial x_1}\right|^2 x_1^{\gamma_1}x_2^{\gamma_2}\,dx_1\,dx_2 \leqq c_1\|u\|_{2,2,\tilde{S}}^2 =$$

$$= c_1\left[\int_\Omega |u|^2 x_1^{\delta_1-2}x_2^{\delta_2-2}\,dx_1\,dx_2 + \int_\Omega \left|\frac{\partial^2 u}{\partial x_1\,\partial x_2}\right|^2 x_1^{\delta_1}x_2^{\delta_2}\,dx_1\,dx_2\right]$$

as well as an analogous estimate with

$$\int_\Omega \left|\frac{\partial u}{\partial x_2}\right|^2 x_1^{\beta_1}x_2^{\beta_2}\,dx_1\,dx_2$$

on the left-hand side. However, there are counterexamples available demonstrating that these estimates can hold only if

(1.34) $$\gamma_1 = \delta_1\,, \quad \gamma_2 = \delta_2 - 2\,,$$
$$\beta_2 = \delta_2\,, \quad \beta_1 = \delta_1 - 2\,.$$

If (1.34) fails to hold, condition (0.3) is not fulfilled, either. Conversely, by Hardy's inequality we can show that condition (0.3) really holds provided conditions (1.34) (and the conditions $\delta_1 \neq 1$, $\delta_2 \neq 1$) are fulfilled.

We conclude: A weak solution of the Dirichlet problem for the operator from (1.27) with the collection $\tilde{S}$ from (1.33) exists if and only if $\delta_1 \neq 1$, $\delta_2 \neq 1$ and conditions (1.34) are fulfilled.

2. DIRICHLET PROBLEM: A SIMPLE CASE

2.1. Differential operator. Let us assume that the coefficients $a_{\alpha\beta}$ of the differential operator A from (0.1), i.e. the operator

(2.1) $$Au = \sum_{|\alpha|,|\beta|\leqq k} (-1)^{|\alpha|}\,D^\alpha(a_{\alpha\beta}\,D^\beta u)\,,$$

satisfy the following conditions:

A.1 $$a_{\alpha\alpha} \in \mathscr{W}(\Omega) \quad \text{for} \quad |\alpha| \leqq k\,;$$

A.2 $$a_{\alpha\alpha} \in L^1_{\mathrm{loc}}(\Omega), \quad \frac{1}{a_{\alpha\alpha}} \in L^1_{\mathrm{loc}}(\Omega) \quad \text{for} \quad |\alpha| \leqq k;$$

A.3 there is a constant $c_1 > 0$ such that

$$|a_{\alpha\beta}(x)| \leqq c_1 \sqrt{[a_{\alpha\alpha}(x)\, a_{\beta\beta}(x)]} \quad \text{for a.e. } x \in \Omega \quad (|\alpha|, |\beta| \leqq k); \tag{2.2}$$

A.4 there is a constant $c_2 > 0$ such that for an arbitrary vector $\xi = \{\xi_\alpha, |\alpha| \leqq k\}$ and for a.e. $x \in \Omega$ we have

$$\sum_{|\alpha|,|\beta| \leqq k} a_{\alpha\beta}(x)\, \xi_\alpha \xi_\beta \geqq c_2 \sum_{|\alpha| \leqq k} a_{\alpha\alpha}(x)\, \xi_\alpha^2 . \tag{2.3}$$

Let us recall that $\mathscr{W}(\Omega)$ is the set of weight functions — cf. Sec. 0.5.

2.2. Weighted space. We will consider the space $W^{k,2}(\Omega, S)$ with

$$S = \{w_\alpha(x) = a_{\alpha\alpha}(x), |\alpha| \leqq k\}; \tag{2.4}$$

thus we choose here $\boldsymbol{M} = \boldsymbol{M}(N, k)$. In view of condition **A.2**, $W^{k,2}(\Omega, S)$ is a complete (Hilbert) space and it also makes sense to define the space $W_0^{k,2}(\Omega, S)$.

2.3. Theorem. *Let the coefficients $a_{\alpha\beta}$ of the operator A from (2.1) be defined on a domain $\Omega \subset \mathbb{R}^N$ and satisfy conditions* **A.1**–**A.4.** *Then there is a unique weak solution u of the Dirichlet problem in the space $W^{k,2}(\Omega, S)$. Further, there is a positive constant c independent of the function u_0 and the functional f, such that*

$$\|u\|_{k,2,S} \leqq c(\|u_0\|_{k,2,S} + \|f\|_{(W_0^{k,2}(\Omega,S))^*}) .$$

Proof. 1° By using property **A.3**, the Hölder inequality and the definition of the norm in the space $W^{k,2}(\Omega, S)$ we arrive at the estimate

$$\begin{aligned} a(u, v) &\leqq \sum_{|\alpha|,|\beta| \leqq k} \int_\Omega |a_{\alpha\beta}(x)|\, |D^\beta u(x)|\, |D^\alpha v(x)| \,dx \leqq \\ &\leqq c_1 \sum_{|\alpha|,|\beta| \leqq k} \int_\Omega |D^\beta u(x)| \sqrt{[a_{\beta\beta}(x)]}\, |D^\alpha v(x)| \sqrt{[a_{\alpha\alpha}(x)]} \,dx \leqq \\ &\leqq c_1 \sum_{|\alpha|,|\beta| \leqq k} \left(\int_\Omega |D^\beta u(x)|^2 w_\beta(x) \,dx\right)^{1/2} \left(\int_\Omega |D^\alpha v(x)|^2 w_\alpha(x) \,dx\right)^{1/2} \leqq \\ &\leqq c_3 \|u\|_{k,2,S} \|v\|_{k,2,S} , \end{aligned} \tag{2.5}$$

which holds for arbitrary functions $u, v \in W^{k,2}(\Omega, S)$.

2° At the same time, inequalities (2.5) imply that the expression $a(u_0, v)$ for a fixed $u_0 \in W_0^{k,2}(\Omega, S)$ is a continuous linear functional over the space $W^{k,2}(\Omega, S)$ as well as over its arbitrary subspace, thus, in particular, over the subspace $V = W_0^{k,2}(\Omega, S)$.

3° Setting $\xi_\alpha = D^\alpha u(x)$ in (2.3) and integrating the resulting inequality over Ω, we immediately obtain the estimate

$$a(u, u) \geqq c_2 \|u\|_{k,2,S}^2 , \tag{2.6}$$

which holds for each function $u \in W^{k,2}(\Omega, S)$ and, a *fortiori*, for each $u \in W_0^{k,2}(\Omega, S)$.

4° In view of inequalities (2.5) and (2.6) we may apply the Lax-Milgram Lemma 0.3 with $b(u, v) = a(u, v)$, $V = W_0^{k,2}(\Omega, S)$ and choosing the functional $g \in V^*$ as follows:

$$\langle g, v\rangle = \langle f, v\rangle - a(u_0, v) ;$$

here $f \in V^*$ and $u_0 \in W^{k,2}(\Omega, S)$ are respectively the functional and the function from Definition 0.8 of a weak solution of the Dirichlet problem. By point 2° of our proof, g indeed belongs to V^* and by Theorem 0.3 there is a unique function $w \in V$ such that

$$a(w, v) = \langle g, v\rangle \quad \text{for every} \quad v \in V . \tag{2.7}$$

If we now put

$$u = w + u_0 \tag{2.8}$$

then $u - u_0 = w \in V = W_0^{k,2}(\Omega, S)$, i.e., condition (i) from Definition 0.8 is fulfilled. At the same time condition (ii) from (0.8) is fulfilled, since (2.7) yields

$$a(u, v) = a(w + u_0, v) = a(w, v) + a(u_0, v) =$$
$$= \langle g, v\rangle + a(u_0, v) = \langle f, v\rangle .$$

Consequently, the function u from (2.8) is the required, uniquely determined weak solution of the Dirichlet problem in $W^{k,2}(\Omega, S)$. Moreover, in virtue of (0.6) we have

$$\|u\|_{k,2,S} \leqq c(\|u_0\|_{k,2,S} + \|f\|_{V^*}) .$$

2.4. Remark. It is immediately verified that condition **A.3** could be written also in the following form:

A.3* There is a constant $\tilde{c}_1 > 0$ such that

$$|a_{\alpha\beta}(x)| \leqq \tilde{c}_1 \sqrt{[a_{\alpha\alpha}(x)\, a_{\beta\beta}(x)]} \quad \text{for a.e.} \quad x \in \Omega \tag{2.2*}$$

and for $\alpha \neq \beta$, $|\alpha|, |\beta| \leqq k$.

It can be shown that condition **A.4** guaranteeing the V-ellipticity of the form $a(u, v)$ already follows from condition **A.3*** provided the *constant* $\tilde{c}_1$ *is sufficiently small.* Indeed, we have

$$\sum_{|\alpha|,|\beta| \leqq k} a_{\alpha\beta}(x)\, \xi_\alpha \xi_\beta = \sum_{\alpha=\beta} + \sum_{\alpha\neq\beta} \geqq$$
$$\geqq \sum_{\alpha=\beta} - \Big|\sum_{\alpha\neq\beta}\Big| \geqq \sum_{|\alpha| \leqq k} a_{\alpha\alpha}(x)\, \xi_\alpha^2 - \sum_{\alpha\neq\beta} |a_{\alpha\beta}(x)\, \xi_\alpha \xi_\beta| .$$

The last summand is estimated by means of (2.2*) and the inequality $ab \leqq \frac{1}{2}(a^2 + b^2)$:

$$\sum_{\alpha \neq \beta} |a_{\alpha\beta}(x)\, \xi_\alpha \xi_\beta| \leqq \tilde{c}_1 \sum_{\alpha \neq \beta} \sqrt{[a_{\alpha\alpha}(x)]}\, |\xi_\alpha|\, \sqrt{[a_{\beta\beta}(x)]}\, |\xi_\beta| \leqq$$

$$\leqq \tfrac{1}{2}\tilde{c}_1 \sum_{\alpha \neq \beta} (a_{\alpha\alpha}(x)\, \xi_\alpha^2 + a_{\beta\beta}(x)\, \xi_\beta^2) \leqq \tfrac{1}{2}\tilde{c}_1\, 2(\varkappa - 1) \sum_{|\alpha| \leqq k} a_{\alpha\alpha}(x)\, \xi_\alpha^2 ,$$

where $\varkappa$ is the number of multiindices of length at most k. Hence

$$\sum_{|\alpha|,|\beta| \leqq k} a_{\alpha\beta}(x)\, \xi_\alpha \xi_\beta \geqq (1 - \tilde{c}_1(\varkappa - 1)) \sum_{|\alpha \leqq k|} a_{\alpha\alpha}(x)\, \xi_\alpha^2 ,$$

which means that for

(2.8) $$\tilde{c}_1 < \frac{1}{\varkappa - 1}$$

condition **A.4** is fulfilled, i.e. inequality (2.3) with the constant $c_2 = 1 - \tilde{c}_1(\varkappa - 1)$ holds.

2.5. Weakening conditions A.1—A.4. Conditions **A.1—A.4** represent the simplest assumptions that essentially immediately yield the existence theorem. Since they are simple, they are very rough, too. Let us therefore present some (again rather obvious) generalizations of these conditions.

(i) It is evident that all the preceding arguments remain valid if some of the coefficients $a_{\alpha\alpha}$ vanish. Thus condition **A.1** can be replaced by the following one:

A.1* Denote by $\boldsymbol{M}$ the set of those multiindices $\alpha \in \boldsymbol{M}(N, k)$ for which $a_{\alpha\alpha} \in \mathscr{W}(\Omega)$, and let $a_{\beta\beta}(x) \equiv 0$ for $\beta \notin \boldsymbol{M}$. Let the set $\boldsymbol{M}$ contain at least one multiindex of length k and let the expression

(2.9) $$\|u\|_{k,2,S} = \left(\sum_{\alpha \in \boldsymbol{M}} \int_\Omega |D^\alpha u|^2\, a_{\alpha\alpha}(x)\, dx \right)^{1/2}$$

be a norm on the space $W^{k,2}(\Omega, S)$ with the collection

(2.10) $$S = \{w_\alpha(x) = a_{\alpha\alpha}(x),\ \alpha \in \boldsymbol{M}\} .$$

Conditions **A.2—A.4** do not change, except that condition **A.2** is relevant only for $\alpha \in \boldsymbol{M}$ and that it suffices to perform the summation on the right-hand side of (2.3) only over $\alpha \in \boldsymbol{M}$. Remark 2.4 remains valid, too, with the only change consisting in replacing the number $\varkappa$ in (2.8) by the number of multiindices in the set $\boldsymbol{M}$.

(ii) Let us pass back to the space $W^{k,2}(\Omega, S)$ from Sec. 2.2 and let us assume that the expression

(2.11) $$|||u|||_{k,2,S} = \left(\sum_{\alpha \in \boldsymbol{M}_1} \int_\Omega |D^\alpha u(x)|^2\, w_\alpha(x)\, dx \right)^{1/2}$$

where the sum is taken over a *certain subset* $\mathbf{M}_1 \subset \mathbf{M}(N, k)$, is again a norm on the space $W_0^{k,2}(\Omega, S)$, equivalent to the original norm $\|u\|_{k,2,S}$. Then condition **A.4** may be replaced by a weaker condition

A.4* There exists a constant $c_2^* > 0$ such that for any vector $\xi = \{\xi_\alpha, |\alpha| \leq k\}$, the inequality

(2.12) $$\sum_{|\alpha|,|\beta| \leq k} a_{\alpha\beta}(x)\, \xi_\alpha \xi_\beta \geqq c_2^* \sum_{\alpha \in \mathbf{M}_1} a_{\alpha\alpha}(x)\, \xi_\alpha^2$$

holds.

(iii) The collection S of weight functions from (2.4) is determined directly by the "diagonal" coefficients $a_{\alpha\alpha}$ of the operator A. Consider now another collection

(2.13) $$\tilde{S} = \{\tilde{w}_\alpha(x), |\alpha| \leq k,\ \tilde{w}_\alpha \in \mathscr{W}(\Omega)\},$$

which possesses the following property: the spaces

$$W^{k,2}(\Omega, S) \quad \text{and} \quad W^{k,2}(\Omega, \tilde{S})$$

are *different* (in the sense that the norms $\|u\|_{k,2,S}$ and $\|u\|_{k,2,\tilde{S}}$ are not equivalent), but the identity

(2.14) $$W_0^{k,2}(\Omega, S) = W_0^{k,2}(\Omega, \tilde{S})$$

holds (in the sense that the *both norms* just mentioned *are equivalent* on the set $C_0^\infty(\Omega)$). Since we take the space $W_0^{k,2}(\Omega, S)$ for the space V to which we apply Theorem 0.3, we may, in view of (2.14), use the collection $\tilde{S}$ as well.

The advantage of this approach consists in the fact that it allows of greater variability in the choice of the function u_0, which represents the boundary conditions. In fact, it may happen that the given function u_0 does not belong to the space $W^{k,2}(\Omega, S)$ but it does belong to $W^{k,2}(\Omega, \tilde{S})$. Naturally, we have to solve the Dirichlet problem in that space to which u_0 belongs; this means, in this case, in the space $W^{k,2}(\Omega, \tilde{S})$.

Let us notice that, if the function u_0 simultaneously belongs to two *different* spaces $W^{k,2}(\Omega, S)$ and $W^{k,2}(\Omega, \tilde{S})$ (the corresponding norms being equivalent on $C_0^\infty(\Omega)$, so that the spaces $W_0^{k,2}(\Omega, S)$ and $W_0^{k,2}(\Omega, \tilde{S})$ *coincide*), we can solve the Dirichlet problem in the former as well as in the latter space. In both cases we obtain the *same solution*, for we construct them by means of a uniquely determined function $w \in W_0^{k,2}(\Omega, S) = W_0^{k,2}(\Omega, \tilde{S})$ (cf. identity (2.8)).

Concerning the choice of the function u_0, see Example 2.6.

(iv) The choice of two different collections S and $\tilde{S}$ that give the same "nulled" space, described in part (iii), allows of greater variability also in the coefficients of the operator A: in virtue of the eqvivalence of the norms on the "nulled" spaces we can replace some of the inequalities (2.2) and (2.3) by the corresponding inequalities of the type

(2.15) $$|a_{\alpha\beta}(x)| \leq c_1 \sqrt{[\tilde{w}_\alpha(x)\, \tilde{w}_\beta(x)]},$$

$$(2.16) \qquad \sum_{|\alpha|,|\beta|\leqq k} a_{\alpha\beta}(x)\,\xi_\alpha\xi_\beta \geqq c_2 \sum_{|\alpha|\leqq k} \tilde{w}_\alpha(x)\,\xi_\alpha^2 .$$

2.6. Example. Consider a plane domain ($N = 2$) and take the halfplane $\{(x_1, x_2), x_2 > 1\}$ for Ω. Further, consider the operator

$$Au = -\frac{\partial}{\partial x_1}\left(x_2^{-2}\frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(x_2^2\frac{\partial u}{\partial x_2}\right) + u .$$

By Sec. 1.1, the weighted space corresponding to this operator is the space $W^{1,2}(\Omega, S)$ with the collection

$$S = \{1, x_2^{-2}, x_2^2\} .$$

If we consider still another system

$$\tilde{S} = \{x_2^{-3/2}, x_2^{-2}, x_2^2\} ,$$

we have $W^{1,2}(\Omega, S) \neq W^{1,2}(\Omega, \tilde{S})$, since e.g. the function $u_0(x_1, x_2) = (x_2^2 + 1)^{-1}$ belongs to $W^{1,2}(\Omega, \tilde{S})$ but not to $W^{1,2}(\Omega, S)$. On the other hand, $W_0^{1,2}(\Omega, S) = W_0^{1,2}(\Omega, \tilde{S})$ since the respective norms are equivalent as a consequence of the inequalities

$$\iint_\Omega |u|^2\, x_2^{-3/2}\, dx_1\, dx_2 \leqq \iint_\Omega |u|^2\, dx_1\, dx_2 \leqq 4 \iint_\Omega \left|\frac{\partial u}{\partial x_2}\right|^2 x_2^2\, dx_1\, dx_2 ,$$

which are valid for every $u \in C_0^\infty(\Omega)$ (the first inequality follows from the fact that $x_2 > 1$ for $(x_1, x_2) \in \Omega$, the other is a consequence of Hardy's inequality (0.18) with respect to x_2 for $p = \varepsilon = 2$).

Let us now solve the Dirichlet problem with the boundary condition given by the function $u_0(x_1, x_2) = (x_2^2 + 1)^{-1}$. As $u_0 \notin W^{1,2}(\Omega, S)$, we cannot look for a solution in this space, but we can look for it in the space $W^{1,2}(\Omega, \tilde{S})$.

3. DIRICHLET PROBLEM: MORE COMPLICATED CASE

3.1. In Chap. 2 we worked with a collection S whose elements were determined by the "diagonal" coefficients of the operator A. In Sec. 2.1 we assumed that *all* the coefficients $a_{\alpha\alpha}$ belong to $\mathscr{W}(\Omega)$ (condition **A.1**), and we weakened this condition by admitting that some of the coefficients $a_{\alpha\alpha}$ vanish (condition **A.1***). Nevertheless, as is seen e.g. from Example 1.1 (iii), coefficients $a_{\alpha\alpha}$ which are negative or change they sign are admissible, too, though they neither belong to $\mathscr{W}(\Omega)$ nor vanish identically. However, it is necessary for us to be able to estimate the terms that correspond to these coefficients in the norm of the space $W^{k,2}(\Omega, S)$ by the other terms, that is, we have to be able to determine certain coefficients that are *decisive*. The situation is similar to that occuring in the case of classical elliptic equations: in this latter case we use the classical Sobolev space $W^{k,2}(\Omega)$ and the decisive coefficients in the Dirichlet problem are the coefficients $a_{\alpha\alpha}$ with $|\alpha| = k$, i.e. the *coefficients of the highest order.*

In our general case it is not so easy to determine the decisive coefficients; there has to be at least one coefficient of the highest order among then. However, as Example 1.5 demonstrates, in some cases we cannot do without terms with coefficients $a_{\alpha\alpha}$ of "lower orders", i.e. with $|\alpha| < k$. Let us formulate these considerations a little more precisely.

3.2. Coefficients of the differential operator and the weight function. Let us again consider the differential operator A from (2.1) and let us denote by $\boldsymbol{M}_0$ the set of such multiindices $\alpha \in \boldsymbol{M}(N, k)$ for which $a_{\alpha\alpha} \in \mathscr{W}(\Omega)$. Let the following condition be fulfilled:

B.1 The set $\boldsymbol{M}_0$ contains at least one multiindex of length k; moreover, $a_{\alpha\alpha} \in L^1_{\text{loc}}(\Omega)$ for $\alpha \in \boldsymbol{M}_0$.

Let us now choose a set $\boldsymbol{M}_1 \subset \boldsymbol{M}_0$ so that

B.2 for $\alpha \in \boldsymbol{M}_0$ there is a constant $\tilde{c}_\alpha > 0$ such that

$$\|D^\alpha u\|^2_{2,a_{\alpha\alpha}} \leqq \tilde{c}_\alpha \sum_{\gamma \in \boldsymbol{M}_1} \|D^\gamma u\|^2_{2,a_{\gamma\gamma}} \tag{3.1}$$

holds for all $u \in C_0^\infty(\Omega)$.

Denote the sum on the right-hand side by $|||u|||^2_{\boldsymbol{M}_1}$, i.e.

$$|||u|||_{\boldsymbol{M}_1} = \Big(\sum_{\gamma \in \boldsymbol{M}_1} \|D^\gamma u\|^2_{2,a_{\gamma\gamma}}\Big)^{1/2} . \tag{3.2}$$

The expression $|||\cdot|||_{\boldsymbol{M}_1}$ possesses all the properties of a *seminorm.*

The set $\boldsymbol{M}_1$ can be chosen in various ways; condition (3.1) will be certainly fulfilled if we set $\boldsymbol{M}_1 = \boldsymbol{M}_0$. However, we shall naturally strive for choosing the set $\boldsymbol{M}_1$ *as small as possible*; practically this means that we try to estimate *the greatest possible* number of terms of the form $\|D^\alpha u\|^2_{2,a_{\alpha\alpha}}$ by combinations of the *smallest possible* number of analogous terms with other multiindices. When doing this we will assume that the set $\boldsymbol{M}_1$ chosen satisfies the following conditions:

B.3
$$\frac{1}{a_{\alpha\alpha}} \in L^1_{\text{loc}}(\Omega) \quad \text{for} \quad \alpha \in \boldsymbol{M}_1 ;$$

B.4 there exists a constant $c_1 > 0$ such that

$$|a_{\alpha\beta}(x)| \leqq c_1 \sqrt{[a_{\alpha\alpha}(x)\, a_{\beta\beta}(x)]} \text{ for a.e. } x \in \Omega \text{ and for } \alpha, \beta \in \boldsymbol{M}_1; \tag{3.3}$$

B.5 there exists a constant $c_2 > 0$ such that

$$a(u, u) \geqq c_2 |||u|||^2_{\boldsymbol{M}_1} \tag{3.4}$$

holds for all functions $u \in C_0^\infty(\Omega)$, where $a(u, v)$ is the bilinear form corresponding to the operator A – cf. (0.2).

Now let us choose a set $\boldsymbol{M}_2 \subset \boldsymbol{M}(N, k) - \boldsymbol{M}_1$ and weight functions $w_\alpha \in \mathscr{W}(\Omega)$ or $\alpha \in \boldsymbol{M}_2$ so that they satisfy the following conditions:

B.6 $$w_\alpha \in L^1_{\mathrm{loc}}(\Omega) \quad \text{and} \quad \frac{1}{w_\alpha} \in L^1_{\mathrm{loc}}(\Omega) \quad \text{for} \quad \alpha \in \boldsymbol{M}_2\,;$$

further, there is a constant $c_\alpha > 0$ such that for all $u \in C_0^\infty(\Omega)$ the inequality

$$\|\mathrm{D}^\alpha u\|^2_{2,w_\alpha} \leqq c_\alpha |||u|||^2_{\boldsymbol{M}_1} \tag{3.5}$$

holds.

B.7 For every pair $\alpha, \beta \in \boldsymbol{M}_1 \cup \boldsymbol{M}_2$ there is a constant $c_{\alpha\beta} > 0$ such that

$$|a_{\alpha\beta}(x)| \leqq c_{\alpha\beta} \sqrt{[w_\alpha(x)\, w_\beta(x)]} \quad \text{for a.e.} \quad x \in \Omega\,; \tag{3.6}$$

here we put

$$w_\gamma(x) = a_{\gamma\gamma}(x) \quad \text{for} \quad \gamma \in \boldsymbol{M}_1\,. \tag{3.7}$$

If $\alpha \notin \boldsymbol{M}_1 \cup \boldsymbol{M}_2$ or $\beta \notin \boldsymbol{M}_1 \cup \boldsymbol{M}_2$, then $a_{\alpha\beta}(x) \equiv 0$.

3.3. Remark. The first part of condition **B.7** already includes condition **B.4**. Nonetheless, we have presented it explicitly, as Sec. 3.2 actually provides *instructions* how to proceed in a particular case when constructing the sets $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ and choosing the functions w_α for $\alpha \in \boldsymbol{M}_2$. If we do not succeed in fulfilling condition **B.4**, our theory is inpracticable and it is of no use constructing the set $\boldsymbol{M}_2$ and verifying conditions **B.5**, **B.6** and **B.7**.

3.4. Weighted space. Let us now choose the set $\boldsymbol{M}$ so that

$$\boldsymbol{M}_1 \subset \boldsymbol{M} \subset \boldsymbol{M}_1 \cup \boldsymbol{M}_2\,, \tag{3.8}$$

holds and denote $S = \{w_\alpha, \alpha \in \boldsymbol{M}\}$ (for $\alpha \in \boldsymbol{M}_1$ we choose w_α accordingly to (3.7)). Moreover, we choose the set $\boldsymbol{M}$ so as to satisfy the condition

B.8 the space $W^{k,2}(\Omega, S)$ is a (complete) Hilbert space with the norm

$$\|u\|_{k,2,S} = \Big(\sum_{\alpha\in\boldsymbol{M}} \|\mathrm{D}^\alpha u\|^2_{2,\omega_\alpha}\Big)^{1/2}\,. \tag{3.9}$$

3.5. Remarks. (i) As $\boldsymbol{M} \subset \boldsymbol{M}_1 \cup \boldsymbol{M}_2$, it follows from (3.5) that for all $u \in C_0^\infty(\Omega)$ the inequality

$$\|u\|^2_{k,2,S} \leqq c_3 |||u|||^2_{\boldsymbol{M}_1} \tag{3.10}$$

is satisfied with a constant $c_3 = 1 + \sum_{\alpha \in \mathbf{M}_2} c_\alpha > 0$ independent of u. As $\mathbf{M}_1 \subset \mathbf{M}$, the converse inequality (with $c_3 = 1$) holds as well, which means that the seminorm $|||\cdot|||_{\mathbf{M}_1}$ is *a norm on the space* $W_0^{k,2}(\Omega, S)$, equivalent to the norm (3.9).

(ii) It immediately follows from (3.4) and (3.10) that the form $a(u, v)$ is $W_0^{k,2}(\Omega, S)$-elliptic; indeed,

$$a(u, u) \geqq \frac{c_2}{c_3} \|u\|_{k,2,S}^2 \tag{3.11}$$

for all $u \in C_0^\infty(\Omega)$. Condition **B.6** and, above all, inequality (3.5) thus play a crucial role in our considerations.

Now we can state the main result:

3.6. Theorem. *Let A be the operator from* (2.1) *and $W^{k,2}(\Omega, S)$ the weighted space from* Sec. 3.4. *Let the coefficients $a_{\alpha\beta}$ of the operator A and the weight functions w_γ satisfy conditions* **B.1—B.8**. *Then there is a unique weak solution of the Dirichlet problem in the space $W^{k,2}(\Omega, S)$.*

Proof is left to the reader since it is analogous to that of Theorem 2.3. The $W_0^{k,2}(\Omega, S)$-ellipticity of the form $a(u, v)$ follows from (3.11); conditions **B.7**, **B.6** and **B.2** then guarantee the validity of condition (0.3) of the Lax-Milgram Lemma.

3.7. Remark. When choosing the set $\mathbf{M}$ we have, with regard to (3.8), a certain liberty, so that generally we can construct *various* spaces $W^{k,2}(\Omega, S)$ (more precisely, various collections S depending on the choice of the set $\mathbf{M}$). Naturally, Remark 3.5 (i) implies that the corresponding "nulled" spaces $W_0^{k,2}(\Omega, S)$ *coincide* for all choices of the set $\mathbf{M}$. This fact can be made use of when choosing the function u_0 that represents the "boundary conditions"; the situation is the same as in Sec. 2.5 (iii).

3.8. The algebraic condition of ellipticity. In Chap. 2 the $W_0^{k,2}$-ellipticity of the form $a(u, v)$ was guaranteed by the algebraic condition **A.4** – see (2.3), while here it was formulated in the "integral form" – see (3.4).

Because of the fact that the verification of algebraic conditions is sometimes easier than of integral ones, let us introduce a certain analogue of condition **A.4**. Let the following condition hold:

B.5* There is a constant $c_2 > 0$ such that any vector $\xi = \{\xi_\alpha, |\alpha| \leqq k\}$ satisfies

$$\sum_{|\alpha|,|\beta| \leqq k} a_{\alpha\beta}(x)\, \xi_\alpha \xi_\beta \geqq c_2 \sum_{\gamma \in \mathbf{M}_1} a_{\gamma\gamma}(x)\, \xi_\gamma^2 \tag{3.12}$$

for a.e. $x \in \Omega$.

First of all, it is evident that (3.12) immediately implies, by choosing $\xi_\alpha = D^\alpha u(x)$ and integrating over the set Ω, the inequality (3.4), so that condition **B.5** is fulfilled provided condition **B.5*** is.

Of course, the algebraic condition **B.5*** is substantially *more restrictive*: it immediately follows from (3.12) that

$$a_{\alpha\alpha}(x) \geqq 0 \text{ a.e. in } \Omega ,$$

for *all* $\alpha \in \boldsymbol{M}(N, k)$ (not only for $\alpha \in \boldsymbol{M}_0$, where the property $a_{\alpha\alpha} \in \mathscr{W}(\Omega)$ was a condition for constructing the set $\boldsymbol{M}_0$). Indeed, if for some $\beta \notin \boldsymbol{M}_0$ the coefficient $a_{\beta\beta}$ was negative on a set of positive measure, then inequality (3.12) with the choice $\{\xi_\alpha = 0 \text{ for } \alpha \neq \beta,\ \xi_\beta = 1\}$ would yield a contradiction.

On the other hand, condition **B.5** admits also diagonal coefficients that are negative or change signs – cf. Example 1.1 (iii) or the forthcoming Example 3.9.

3.9. Example. Let Ω be a plane domain and let us choose

$$Au = \frac{\partial^4 u}{\partial x_1^4} - 2\lambda \frac{\partial^4 u}{\partial x_1^2\, \partial x_2^2} + \frac{\partial^4 u}{\partial x_2^4}, \quad \lambda > 0 . \tag{3.13}$$

Thus, we now have $a_{(2,0)(2,0)}(x) = a_{(0,2)(0,2)}(x) \equiv 1$, $a_{(1,1)(1,1)} = -2\lambda$, $a_{\alpha\beta}(x) \equiv 0$ for the other α, β such that $|\alpha| \leqq 2$, $|\beta| \leqq 2$. The "diagonal" coefficient $a_{(1,1)(1,1)}$ is negative and hence it does not belong to $\mathscr{W}(\Omega)$; this is why $\boldsymbol{M}_0 = \{(2,0), (0,2)\}$. Since condition **B.2** should hold, we cannot reduce the set $\boldsymbol{M}_0$, hence $\boldsymbol{M}_1 = \boldsymbol{M}_0$. Condition **B.5*** naturally fails to hold, since inequality (3.12) has the form

$$\xi_{(2,0)}^2 - 2\lambda \xi_{(1,1)}^2 + \xi_{(0,2)}^2 \geqq c_2(\xi_{(2,0)}^2 + \xi_{(0,2)}^2)$$

and this condition evidently is violated for $\xi_{(2,0)} = \xi_{(0,2)} = 0$, $\xi_{(1,1)} = 1$. Nonetheless, condition **B.5**, i.e. inequality (3.4), is satisfied for small λ's. Indeed,

$$|||u|||_{\boldsymbol{M}_1}^2 = \iint_\Omega \left[\left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2\right] \mathrm{d}x_1\, \mathrm{d}x_2$$

and

$$a(u, u) = |||u|||_{\boldsymbol{M}_1}^2 - 2\lambda \iint_\Omega \left(\frac{\partial^2 u}{\partial x_1\, \partial x_2}\right)^2 \mathrm{d}x_1\, \mathrm{d}x_2 . \tag{3.14}$$

Since we consider functions $u \in C_0^\infty(\Omega)$, we can extend the function u by zero onto the whole plane $\mathbb{R}^2$ and integrate always over $\mathbb{R}^2$. Passing to the Fourier Transform we easily prove

(3.15)
$$\iint_\Omega \left(\frac{\partial^2 u}{\partial x_1\, \partial x_2}\right)^2 \mathrm{d}x_1\, \mathrm{d}x_2 \leqq \frac{1}{2} \iint_\Omega \left[\left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2\right] \mathrm{d}x_1\, \mathrm{d}x_2 = \tfrac{1}{2}|||u|||_{\boldsymbol{M}_1}^2 .$$

Hence and from (3.14) we obtain

$$a(u, u) \geqq (1 - \lambda) |||u|||^2_{M_1} \quad \text{for} \quad u \in C_0^\infty(\Omega);$$

thus for $\lambda < 1$ condition (3.4) is fulfilled.

Let us see now how we can choose, in our case, the set M_2 and the weight functions w_α for $\alpha \in M_2$. Since $a_{(1,1)(1,1)}(x) = -2\lambda \neq 0$, the multiindex (1,1) does not belong to $M_0(= M_1)$. Condition **B.7** implies that necessarily $(1,1) \in M_2$, and the weight function $w_{(1,1)}$ must satisfy (3.6), i.e.,

$$(3.16) \qquad |-2\lambda| \leqq \tilde{c} \sqrt{[w_{(1,1)}(x)\, w_{(1,1)}(x)]} = \tilde{c}\, w_{(1,1)}(x) \quad \text{for a.e.} \quad x \in \Omega .$$

Simultaneously (3.5) must hold, that is,

$$\iint_\Omega \left| \frac{\partial^2 u}{\partial x_1\, \partial x_2} \right|^2 w_{(1,1)}(x)\, dx_1\, dx_2 \leqq \tilde{\tilde{c}} |||u|||^2_{M_1} .$$

Taking into account (3.15) we see that this inequality is fulfilled by the weight function $w_{(1,1)}(x) \equiv 1$, which at the same time satisfies condition (3.16), too. Thus, if we choose

$$M_2 = \{(0, 0), (1, 1)\}, \quad w_{(1,1)}(x) \equiv 1, \quad w_{(0,0)}(x) = (1 + |x|)^\varepsilon$$

($\varepsilon \leqq -2$; $\varepsilon = 0$ if the domain Ω is bounded), condition **B.8** will be fulfilled as well. In view of Remark 3.7, the existence of a weak solution of the Dirichlet problem is guaranteed in *two* spaces $W^{k,2}(\Omega, S)$: either with the choice $S = \{(1 + |x|)^\varepsilon, 0, 0, 1, 1, 1\}$ $(M = M_1 \cup M_2)$ or $S = \{(1 + |x|)^\varepsilon, 0, 0, 1, 0, 1\}$ $(M = M_1 \cup \{(0, 0)\})$. The choice $M = M_1$ is *inadmissible* since then condition **B.8** would not hold in general.

If the domain Ω is bounded, then M_2 may be chosen so that $M_1 \cup M_2 = \{(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2)\} = M(2, 2)$; here we take $w_\alpha(x) \equiv 1$ for $\alpha \in M_2$. In this case we in fact solve the Dirichlet problem for the operator (3.13) in the classical Sobolev space $W^{2,2}(\Omega)$.

Here it has always been essential that $0 < \lambda < 1$. Let us note that for these values the operator A from (3.13) is *not elliptic* in the classical sense, since there is a *nonzero* vector ξ such that

$$\sum_{|\alpha|,|\beta| \leqq 2} a_{\alpha\beta}(x)\, \xi_\alpha \xi_\beta = \xi^2_{(2,0)} - 2\lambda \xi^2_{(1,1)} + \xi^2_{(0,2)} = 0 .$$

3.10. Example. Let us consider the domain as well as the operator from Example 1.3. Here $M(N, k) = M(2, 2)$ and $M_0 = \{(1, 1)\}$, since $a_{(1,1)(1,1)}(x) \equiv 1$ is the only nonzero coefficient. We again have choose $M_1 = M_0$. The reader can easily verify (using the arguments of Sec. 1.3) that the choice

$$M_2 = \{(0, 0, (1, 0), (0, 1)\} \quad \text{and} \quad w_\alpha(x) \equiv 1 \quad \text{for} \quad \alpha \in M_2$$

leads to the desired result, i.e., solvability of the Dirichlet problem in the corresponding space $W^{2,2}(\Omega, S)$ with $S = \{w_\alpha, \alpha \in \boldsymbol{M}\}$ and $\boldsymbol{M}$ determined by the inclusions

$$\boldsymbol{M}_1 \cup \{(0,0)\} \subset \boldsymbol{M} \subset \boldsymbol{M}_1 \cup \boldsymbol{M}_2 .$$

Let us recall that the space $W^{2,2}(\Omega, S)$ from Example 1.3 corresponds to the choice $\boldsymbol{M} = \boldsymbol{M}_1 \cup \{(0,0)\}$.

3.11. Example. Consider the domain as well as the operator from Example 1.5. There we had $a_{(1,1)(1,1)}(x) = x_1^{\delta_1} x_2^{\delta_2}$, $a_{(1,0)(1,0)}(x) = x_1^{\gamma_1} x_2^{\gamma_2}$, $a_{(0,1)(0,1)}(x) = x_1^{\beta_1} x_2^{\beta_2}$, and the other coefficients $a_{\alpha\beta}$ vanished. Hence $\boldsymbol{M}_0 = \{(1,0), (0,1), (1,1)\}$. We have shown that the choice $\boldsymbol{M}_1 = \boldsymbol{M}_0$ and $\boldsymbol{M}_2 = \{(0,0)\}$ with $w_{(0,0)}(x) = x_1^{\delta_1 - 2} x_2^{\delta_2 - 2}$ and $\boldsymbol{M} = \boldsymbol{M}_1 \cup \boldsymbol{M}_2$ leads to the result (for $\delta_1 \neq 1$, $\delta_2 \neq 1$).

So far as the conditions (1.34) are fulfilled it suffices to choose $\boldsymbol{M}_1 = \{(1,1)\}$ (i.e., we have $\boldsymbol{M}_1 \subset \boldsymbol{M}_0$, $\boldsymbol{M}_1 \neq \boldsymbol{M}_0$), $\boldsymbol{M}_2 = \{(0,0)\}$ with the weight function $w_{(0,0)}$ as above, and $\boldsymbol{M} = \boldsymbol{M}_1 \cup \boldsymbol{M}_2$.

If only the first or the second pair of conditions in (1.34) is fulfilled, we arrive at the result even with the choice $\boldsymbol{M}_1 = \{(0,1), (1,1)\}$ or $\boldsymbol{M}_1 = \{(1,0), (1,1)\}$, respectively, provided we choose $\boldsymbol{M}_2$ and $w_{(0,0)}$ in the same way as above and put again $\boldsymbol{M} = \boldsymbol{M}_1 \cup \boldsymbol{M}_2$.

4. CONCLUDING REMARKS

4.1. One more example. On the half plane $\Omega = \{(x_1, x_2);\ x_2 > 0\}$ let us consider the operator

$$Au = -\Delta u - \frac{\lambda}{x_2} \frac{\partial u}{\partial x_2}, \quad \lambda \text{ real} .$$

Here $a_{(1,0)(1,0)}(x) = a_{(0,1)(0,1)}(x) \equiv 1$, $a_{(0,0)(0,1)}(x) = -\lambda/x_2$. Hence $\boldsymbol{M}_0 = \boldsymbol{M}_1 = \{(1,0), (0,1)\}$. Further,

$$a(u,u) = \iint_\Omega \left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right] \mathrm{d}x_1\, \mathrm{d}x_2 - \lambda \iint_\Omega \frac{\partial u}{\partial x_2}\, u \frac{1}{x_2}\, \mathrm{d}x_1\, \mathrm{d}x_2 .$$

The first integral on the right-hand side is the seminorm $|||u|||^2_{\boldsymbol{M}_1}$; the second integral we estimate by the Hölder inequality and by Hardy's inequality with respect to x_2 for $p = 2$ and $\varepsilon = 0$:

$$\left|\iint_\Omega \frac{\partial u}{\partial x_2}\, u \frac{1}{x_2}\, \mathrm{d}x_1\, \mathrm{d}x_2\right| \leqq \left(\iint_\Omega \left(\frac{\partial u}{\partial x_2}\right)^2 \mathrm{d}x_1\, \mathrm{d}x_2\right)^{1/2} \left(\iint_\Omega u^2 \frac{1}{x_2^2}\, \mathrm{d}x_1\, \mathrm{d}x_2\right)^{1/2} \leqq$$

$$\leqq 2 \iint_\Omega \left(\frac{\partial u}{\partial x_2}\right)^2 \mathrm{d}x_1\, \mathrm{d}x_2 ,$$

so that

$$a(u, u) \geqq (1 - 2|\lambda|) |||u|||^2_{M_1} .$$

In this way we are able to prove that for $|\lambda| < \frac{1}{2}$ condition **B.5** is fulfilled. Let us show that condition **B.5** is also fulfilled for every $\lambda \leqq 0$.

Indeed, if $u \in C_0^\infty(\Omega)$, then

$$\frac{\partial}{\partial x_2}\left(u^2 \frac{1}{x_2}\right) = 2u \cdot \frac{\partial u}{\partial x_2}\frac{1}{x_2} - u^2 \frac{1}{x_2^2}$$

and consequently

$$0 = \iint_\Omega \frac{\partial}{\partial x_2}\left(u^2 \frac{1}{x_2}\right) \mathrm{d}x_1 \, \mathrm{d}x_2 = 2 \iint_\Omega u \cdot \frac{\partial u}{\partial x_2}\frac{1}{x_2} \mathrm{d}x_1 \, \mathrm{d}x_2 - \iint_\Omega u^2 \frac{1}{x_2^2} \mathrm{d}x_1 \, \mathrm{d}x_2 .$$

This implies

$$\iint_\Omega \frac{\partial u}{\partial x_2} u \cdot \frac{1}{x_2} \mathrm{d}x_1 \, \mathrm{d}x_2 \geqq 0$$

for every function $u \in C_0^\infty(\Omega)$, which for $\lambda \leqq 0$ yields

$$a(u, u) \geqq |||u|||^2_{M_1} .$$

Further, it can be seen that there is a number $\bar{\lambda} \geqq \frac{1}{2}$ such that condition **B.5** is not valid for all λ, $\lambda \geqq \bar{\lambda}$. Hence our theory cannot be applied to the case $\lambda \geqq \bar{\lambda}$.

However, this difficulty can be avoided, if we consider, instead of the operator A, the operator B given by the formula

$$Bu = -\frac{\partial}{\partial x_1}\left(x_2^\lambda \frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(x_2^\lambda \frac{u}{x_2}\right).$$

This operator is very simply connected with the operator A:

$$Bu = x_2^\lambda \, Au .$$

Moreover, here $M_0 = M_1 = \{(1, 0), (0, 1)\}$ and the bilinear form b corresponding to the operator B satisfies

$$b(u, u) = |||u|||^2_{M_1} = \iint_\Omega \left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right] x_2^\lambda \, \mathrm{d}x_1 \, \mathrm{d}x_2 ,$$

so that condition **B.5** presents no difficulties. If we choose, in addition, $M_2 = \{(0, 0)\}$ with $w_{(0,0)}(x) = x_2^{\lambda - 2}$, then the solvability of the Dirichlet problem for the operator B is guaranteed in the space $W^{1,2}(\Omega, S)$ with $M = M_1 \cup M_2$ and $S = \{x_2^{\lambda-2}, x_2^\lambda, x_2^\lambda\}$, provided $\lambda \neq 1$: for such λ also condition **B.6** holds, since as a consequence of Hardy's inequality (with respect to x_2 and for $p = 2$ and $\varepsilon = \lambda$) we obtain

$$\|u\|^2_{2,w(0,0)} \leqq \frac{4}{(\lambda-1)^2} \iint_\Omega \left|\frac{\partial u}{\partial x_2}\right|^2 x_2^\lambda \, dx_1 \, dx_2 \leqq \frac{4}{(\lambda-1)^2} |||u|||^2_{M_1},$$

which is inequality (3.5).

4.2. Weakening the conditions on the set M_0. Let again the operator A from (2.1) be given and let us assume that some of the "diagonal" coefficient $a_{\delta\delta}$ has the following property:

$$(4.1) \qquad \begin{aligned} a_{\delta\delta}(x) &> 0 \quad \text{for} \quad x \in \Omega_0, \quad \Omega_0 \subsetneqq \Omega, \\ a_{\delta\delta}(x) &= 0 \quad \text{for} \quad x \in \Omega - \Omega_0; \end{aligned}$$

we assume moreover that the set $\Omega - \Omega_0$ has a *positive measure.* Then $a_{\delta\delta}$ *does not belong to* $\mathscr{W}(\Omega)$ and therefore $\delta \notin M_0$; however, since $a_{\delta\delta}(x) \equiv 0$ does not hold, either, we necessarily have $\delta \in M_2$ and hence

$$(4.2) \qquad \int_{\Omega_0} |D^\delta u(x)|^2 \, a_{\delta\delta}(x) \, dx \leqq \tilde{c}_\delta |||u|||^2_{M_1}.$$

This inequality is a consequence of conditions **B.7** and **B.6**: according to **B.7** there is $w_\delta \in \mathscr{W}(\Omega)$ such that $|a_{\delta\delta}(x)| \leqq c_{\delta\delta} \, w_\delta(x)$, and according to **B.6** we have $\|D^\delta u\|^2_{2,w_\delta} \leqq$ $\leqq c_\delta |||u|||^2_{M_1}$.

If condition (4.2) is not satisfied, our theory is inapplicable; nevertheless, it can be modified in the following way: the multiindex δ is included in the set M_0 and the set M_1 is again formed with help of conditions **B.2** (for $\alpha = \delta$ condition (4.1) implies that on the left hand side of (3.1) we actually have only the integral over Ω_0). The further steps are the same as above, with our modification of the set M_0 manifesting itself in the other conditions **B.3**—**B.8**: for instance, condition **B.7** implies that the coefficients $a_{\alpha\delta}$ and $a_{\delta\alpha}$ vanish for $x \in \Omega - \Omega_0$ (cf. (3.6) with $w_\delta = a_{\delta\delta}$).

As an example of an operator for which condition (4.2) is violated let us introduce the operator

$$Au = \frac{\partial^4 u}{\partial x_1^4} + \frac{\partial^2}{\partial x_2^2}\left(a(x)\frac{\partial^2 u}{\partial x_2^2}\right),$$

where $a(x)$ is a function of type (4.1) with the following property: there is a point $x_0 \in \Omega_0$, a neighbourhood $U(x_0) \subset \Omega_0$ and a positive constant c such that $a(x) \geqq c$ for $x \in U(x_0)$. According to the above introduced modification of our theory we then have $M_1 = M_0 = \{(2, 0), (0, 2)\}$ and

$$|||u|||^2_{M_1} = \int_\Omega \left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 dx + \int_{\Omega_0} \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2 a(x) \, dx.$$

4.3. Boundary conditions. In the weak formulation of the Dirichlet problem the boundary condition is substituted by condition (i) from Definition 0.8, i.e. the condition

$$u - u_0 \in W_0^{k,2}(\Omega, S).$$

The interpretation of this condition is closely connected with the characterization of *traces of functions from weighted spaces*. Until now, the knowledge of the latter has been very incomplete. This is why the interpretation we are acquainted with from the classical Sobolev spaces (see the end of Sec. 0.2) can be transferred to the weight spaces only to a very limited extent.

Let us illustrate this fact on the operator

$$Au = -\frac{\partial}{\partial x_1}\left(x_1^\varepsilon \frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(x_1^\varepsilon \frac{\partial u}{\partial x_2}\right) + x_1^\varepsilon u$$

considered on a plane domain $\Omega = \{(x_1, x_2); 0 < x_1 < 1, 0 < x_2 < 1\}$. The weighted space corresponding to this operator is $W^{1,2}(\Omega, S)$ with the collection $S = \{x_1^\varepsilon, x_1^\varepsilon, x_1^\varepsilon\}$ and the existence of a weak solution of the Dirichlet problem in this space is guaranteed for every $\varepsilon \in \mathbb{R}$.

Concerning the trace of the function $w \in W^{1,2}(\Omega, S)$ on the boundary $\partial\Omega$ we can assert the following facts (cf. [4], [8]):

(a) the trace $w|_{\partial\Omega}$ exists for all $\varepsilon < 1$ (as a function from $L^2(\partial\Omega)$), and for $\varepsilon \leqq -1$ *vanishes* on $\Gamma = \{(0, x_2); 0 < x_2 < 1\}$, i.e. $w|_\Gamma = 0$ for $\varepsilon \leqq -1$.

(b) for $\varepsilon \geqq 1$, the trace exists on $\partial\Omega - \Gamma$ (again as a function from $L^2(\partial\Omega - \Gamma)$), while on Γ *its existence is not guaranteed.*

Consequently, our "boundary condition" $u - u_0 \in W^{1,2}(\Omega, S)$ means that

(i) $u = u_0$ on $\partial\Omega - \Gamma$ for every $\varepsilon \in \mathbb{R}$,

(ii) $u = u_0$ and $u = 0$ on Γ for $\varepsilon \in (-1, 1)$ and $\varepsilon \leqq -1$, respectively,

and there is *no condition* given on Γ for $\varepsilon \geqq 1$ [if $\varepsilon \leqq -1$, condition (ii) is a consequence of the identity $u_0|_\Gamma = 0$ that is necessarily fulfilled in view of (a)]. Hence we can say that only on $\partial\Omega - \Gamma$ the situation is "normal".

We see that even in this simple case the problem of interpretation of the boundary conditions is difficult. Evidently it will be still more complicated for higher order equations and more general boundary functions.

Let us go back to our space $W^{1,2}(\Omega, S)$. We have stated for $\varepsilon \geqq 1$ the existence of the trace is not guaranteed; indeed, we can construct such functions $u(x_1, x_2) \in W^{1,2}(\Omega, S)$ that u is not bounded in a neighbourhood of the set Γ, i.e.

$$\lim_{x_1 \to 0+} |u(x_1, x_2)| = \infty \quad \text{for a.e.} \quad x_2 \in (0, 1).$$

Nonetheless, the behavior of the function u in the neighbourhood of Γ can be described more precisely: namely, for every $u \in W^{1,2}(\Omega, S)$ we have

$$\lim_{x_1 \to 0+} x_1^\lambda u(x_1, x_2) = 0 \quad \text{for a.e.} \quad x_2 \in (0, 1), \tag{4.3}$$

where

$$\lambda > \frac{\varepsilon - 1}{2}. \tag{4.4}$$

This concerns the case $\varepsilon \geqq 1$; however, relation (4.3) holds even for $\varepsilon \leqq -1$ with λ satisfying the inequality

$$\frac{\varepsilon - 1}{2} < \lambda < 0 . \tag{4.5}$$

4.4. Remark. In Sec. 4.3 we have met weight functions and coefficients of the following type:

$$w(x) = [\operatorname{dist}(x, \Gamma)]^{\varepsilon}, \quad \Gamma \subset \partial\Omega . \tag{4.6}$$

Equations with such coefficients and spaces with such weights occur most frequently in applications. For $\varepsilon < 0$ they grow to infinity in a neighbourhood of Γ and therefore an equation with coefficients of this type is said to have a *singularity* on Γ. On the contrary, for $\varepsilon > 0$ the function $w(x)$ converges to zero for $x \to x_0 \in \Gamma$ and an equation with such coefficients is said to *degenerate* on Γ. Since coefficients of the type (4.6) "get spoiled" solely on $\Gamma \subset \partial\Omega$, there are no difficulties with condition of the type **A.2**.

Quite another situation occurs if $\Gamma \subset \Omega$. Under the conditions meas $\Gamma = 0$ the condition $w \in L^1_{\text{loc}}(\Omega)$ is fulfilled for $\varepsilon > 0$, but for negative ε's only provided $\varepsilon \in (-\varepsilon_0, 0)$, where $\varepsilon_0 > 0$ depends on the dimension of the set Γ; similarly the condition $1/w \in L^1_{\text{loc}}(\Omega)$ is not fulfilled for $\varepsilon > \varepsilon_0$. Hence we can say that for certain sufficiently large $|\varepsilon|$ conditions of the type **A.2** are not fulfilled; we speak about *strong singularity* or *strong degeneration* on Γ (i.e. *inside* Ω) and our theory is inapplicable. Nevertheless, our method can be modified to suit even such cases. Roughly speaking, it is necessary to consider the domain $\Omega_0 = \Omega - \Gamma$ instead of Ω, so that Γ then becomes part of the boundary $\partial\Omega_0$. We shall resume the study of these problems, as well as of "non-Dirichlet" boundary problems and of nonlinear equations in the next paper.

4.5. Bibliographical notes. There is extensive literature on equations of the above described type. Above all, degenerate equations are frequently the object of study (see e.g. [13], where a survey of results till 1966 can be found; more recent literature is represented e.g. by [2], [6], [8], [12]). Most of these works consider degeneration of the type "a power of the distance of the point $x \in \Omega$ from the set $\Gamma \subset \partial\Omega$". Our approach is more general: we admit simultaneously both degenerations and singularities (and not only of the power type); the coefficients may "get spoiled" not only on the boundary $\partial\Omega$ but inside the domain Ω as well, and even the case of degeneration on sets of positive measure us possible — cf. Sec. 4.2. Moreover, we impose no restrictions on the domain Ω. An approach similar to ours in its generality (including the method, i.e. the use of the Lax-Milgram Lemma) can be found in [12]; in comparison with [12], we admit operators that are substantially more general, and our approach is a little more systematical.

It is the intention of the authors to develop the present topics in further papers, including the linear equations (coefficients with a strong singularity and strong degeneration inside Ω, Neumann's problem and further types of boundary conditions) as well the nonlinear ones.

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