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TWO-VALUED MEASURES ON σ-CLASSES

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INTRODUCTION

We bring a partial solution to an open question of the mathematical foundations of quantum mechanics. The question was posed by S. Gudder in the paper [3] (and repeated in [4], [5] and [6]). One asks whether the integral of the sum of two (bounded measurable summable) functions on a σ-class is the sum of the respective integrals. We investigate the question for two-valued measures (= two-valued states). We show that the answer is no in general while as soon as we assume at least one of the functions finitely valued, the answer is yes.

1. BASIC NOTIONS

The motivation of the question we shall pursue in the sequel comes from quantum mechanics (see e.g. [6]). We ask whether the expectation of the sum of two (possibly noncompatible) observables is always the sum of the expectations. The explicit form of the question was introduced by S. Gudder who also indicated the interpretation of the potential results (see [4], [6]). Our investigation is purely mathematical, we only include the physical terminology in some places for the physically oriented reader to make the interpretation easier.

Definition 1. A *σ-class* (*a concrete logic of a quantum system*) is a pair (Ω, Δ) where Ω is a set and Δ is a collection of subsets of Ω subject to the following conditions:

(1) $\phi \in \Delta$,
(2) if $A \in \Delta$ then $\Omega - A \in \Delta$,
(3) if $\{A_i \mid i \in N\} \subset \Delta$ is a *mutually disjoint* family then $\bigcup_{i=1}^{\infty} A_i \in \Delta$.

One easily sees that if $A, B \in \Delta$, $B \supset A$ then $B - A \in \Delta$.

Definition 2. A *measurable mapping* $f : (\Omega, \Delta) \to R$ (an *observable*) is a bounded countably valued mapping such that $f^{-1}(a, b) \in \Delta$ for any real numbers $a, b \in R$. Two measurable mappings $f, g : (\Omega, \Delta) \to R$ are called summable if $f + g$ is again measurable on (Ω, Δ).

Observe that any two measurable mappings on a σ-class are summable iff the σ-class is a σ-algebra.

Definition 3. A mapping $m : \Delta \to \langle 0, 1 \rangle$ is called a *measure* (a *state*) on a σ-class (Ω, Δ) if $m(\Omega) = 1$ and $m(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} m(A_i)$ for any mutually disjoint collection $\{A_i \mid i \in N\} \subset \Delta$.

Let $f : (\Omega, \Delta) \to R$ be a measurable function. Denote by $B(R)$ the σ-algebra of Borel subsets of R. It is easy to see that the set $A_f = \{f^{-1}(C) \mid C \in B(R)\}$ is a σ-algebra and, moreover, $A_f \subset \Delta$. If $m : (\Omega, \Delta) \to \langle 0, 1 \rangle$ is a measure then we can restrict m to A_f and the symbol $\int f \, dm$ has the obvious meaning – the Lebesgue integral of f on the measurable space (Ω, A_f, m).

Suppose now that we have two summable measurable functions $f, g : (\Omega, \Delta) \to R$. Suppose that m is a measure on (Ω, Δ). It is known (see [3]) that the assumption of summability of f, g does not imply that there is a σ-algebra Σ, $\Sigma \subset \Delta$ for which all the functions $f, g, f + g$ are measurable. This fact prevents us from showing the equality $\int f \, dm + \int g \, dm = \int (f + g) \, dm$ in a standard manner but the question still remains whether the latter equality holds at all. We clarify the situation for two-valued measures.

Remark. A certain effort has been made to prove the above equality (see [2], [3], [4], [8]). Nevertheless, only fairly unsatisfactory results have been obtained so far.

There is another natural question (posed in [4]): Does the inequality $\int f \, dm \leqq \int g \, dm$ always hold provided $f \leqq g$? The question was answered affirmatively in [8] and [9] (independently and by different methods).

2. TWO-VALUED MEASURES ON σ-CLASSES

A measure $m : \Delta \to \langle 0, 1 \rangle$ is called two-valued if $m(\Delta) \subset \{0, 1\}$. In what follows, a measure is automatically understood two-valued unless the contrary is explicitly stated.

We start with a theorem which gives us the direction of our efforts in proving (or disproving) the desired equality. Before doing that, let us introduce a convention. Suppose we are given two (bounded countably valued) functions $f, g : \Omega \to R$. Let us denote by $\Delta_{f,g}$ ($\Delta^+_{f,g}$) the least σ-class on Ω which contains the set $A_f \cup A_g$ ($A_f \cup A_g \cup A_{f+g}$, respectively). Obviously, if $f, g : (\Omega, \Delta) \to R$ are measurable (and summable) then $\Delta_{f,g} \subset \Delta$ ($\Delta^+_{f,g} \subset \Delta$, respectively).

Theorem 1. *Let $f, g : (\Omega, \Delta) \to R$ be measurable summable functions and let m be a two-valued measure on (Ω, Δ). Then the following statements (1), (2) are equivalent and (3) implies (1):*

(1) $\int f \, dm + \int g \, dm = \int (f + g) \, dm$,

(2) *the measure m restricted to $\Delta_{f,g}^{+}$ is a concentrated measure (i.e., there exists a point $x \in \Omega$ such that, for any set $A \in \Delta_{f,g}^{+}$, $m(A) = 1$ iff $x \in A$).*

(3) *for all $a, b \in R$, the set $K(a, b) = (f^{-1}(-\infty, a) \cap g^{-1}(-\infty, b)) \cup$ $\cup (f^{-1}(-\infty, a) \cap (f + g)^{-1} \langle a + b, +\infty)) \cup (g^{-1}(-\infty, b) \cap (f + g)^{-1}$ . . $\langle a + b, +\infty))$ belongs to Δ.*

Proof. (1) $\Rightarrow$ (2). Let $\int f \, dm = a$, $\int g \, dm = b$. Then $\int (f + g) \, dm = a + b$ and we have $m(f^{-1}\{a\}) = 1$, $m(g^{-1}\{b\}) = 1$ and $m((f + g)^{-1} \{a + b\}) = 1$. Obviously, $f^{-1}\{a\} \cap g^{-1}\{b\} \neq \emptyset$. Take a point $x \in f^{-1}\{a\} \cap g^{-1}\{b\}$. It follows that the measure m restricted to $\Delta_{f,g}^{+}$ must be concentrated at x because if two measures agree on all generators of a σ-class, they have to agree on the entire σ-class.

(2) $\Rightarrow$ (1). Trivial.

(3) $\Rightarrow$ (1). Suppose the contrary. Then, for some $a, b, c \in R$, $m(f^{-1}\{a\}) = 1$, $m(g^{-1}\{b\}) = 1$, $m((f + g)^{-1} \{c\}) = 1$ and $a + b \neq c$. Consider the sets $M_1 = K(c - b, b)$, $M_2 = f^{-1}\{c - b\}$, $M_3 = \Omega - (M_1 \cup M_2)$. Since $M_1 \cap g^{-1}\{b\} = \emptyset$, $M_2 \cap f^{-1}\{a\} = \emptyset$ and $M_3 \cap (f + g)^{-1} \{c\} = \emptyset$, we obtain that $m(M_1) = m(M_2) = m(M_3) = 0$. This is a contradiction because $\Omega = M_1 \cup M_2 \cup M_3$ and M_1, M_2, M_3 are mutually disjoint.

Let us first observe that the equality (1) does not always hold.

Example. We refine an idea of the paper [2]. Let Q be the set of all rational numbers of the interval $(0, 1)$. Put $\Omega = Q \times Q$ and define the functions $f, g : \Omega \to (0, 1)$ by setting $f(x, y) = x$, $g(x, y) = y$. Put $\Delta = \Delta_{f,g}^{+}$. Then f and g become measurable summable functions on (Ω, Δ).

Define a measure $m : \Delta \to \{0, 1\}$ by requiring $m(A) = 1$ iff A contains at least one of the following three sets B, C, D : $B = f^{-1}\{1/3\}$, $C = g^{-1}\{1/3\}$, $D = (f + g)^{-1}$ . . $\{1/2\}$. For checking that m is indeed a measure, one only needs to realize that $\Delta_{f,g}^{+} = \Delta_{f,f+g} \cup \Delta_{g,f+g}$. This is not difficult. Since $B \cap C \cap D = \emptyset$, the measure m is not concentrated and it follows that $\int f \, dm + \int g \, dm \neq \int (f + g) \, dm$.

Theorem 2. *Let $f, g : (\Omega, \Delta) \to R$ be measurable summable functions and let m be a two-valued measure on (Ω, Δ). Let the number of values of f be finite. Then $\int f \, dm + \int g \, dm = \int (f + g) \, dm$.*

Proof. We shall show that $K(a, b) \in \Delta$ for any $a, b \in R$. This is sufficient in view of Theorem 1.

Suppose that f attains values $a_1, a_2, \ldots, a_n$ and suppose further that $a_1 < a_2 < \ldots$ $\ldots < a_n$. If $a \leqq a_1$ then $K(a, b)$ obviously belongs to Δ because $K(a, b) =$

$= g^{-1}(-\infty, b) \cap (f + g)^{-1} \langle a + b, +\infty) = (f + g)^{-1} \langle a + b, +\infty) -$
$- g^{-1}\langle b, +\infty)$. We shall now proceed by induction on the values $a_1, a_2, \ldots, a_n$. More precisely, we shall prove that for any $a', a'' \in R$, $a' < a''$, the assumption $K(a', b) \in \Delta$ and $f^{-1}(a', a'') = \emptyset$ implies $K(a'', b) \in \Delta$. This will establish that any $K(a, b)$ belongs to Δ.

Assume that $K(a', b) \in \Delta$. Since $K(a', b) \cap f^{-1}\{a'\} = \emptyset$ then the set $L = K(a', b) \cup \cup f^{-1}\langle a', a'')$ belongs to Δ. But $L = (f^{-1}(-\infty, a'') \cap g^{-1}(-\infty, b)) \cup \cup (f^{-1}(-\infty, a'') \cap (f + g)^{-1} \langle a' + b, +\infty)) \cup (g^{-1}(-\infty, b) \cap (f + g)^{-1} . . \langle a' + b, +\infty)$ and this implies that $(f + g)^{-1} \langle a' + b, a'' + b) \subset L$. Therefore the set $L - (f + g)^{-1} \langle a' + b, a'' + b) = K(a'', b)$ belongs to Δ and the proof is complete.

Let us make two comments on the above results. It is a natural question whether we can generalize Theorem 2 to more then two functions. We shall show by the following simple example that we cannot. Put $\Omega = \{1, 2, 3, 4, 5, 6\}$ and pick up the sets $M_1 = \{1, 2, 3\}$, $M_2 = \{3, 4, 5\}$, $M_3 = \{5, 6, 1\}$, $M_4 = \{2, 4, 6\}$. Then the least σ-class on Ω containing M_1, M_2, M_3, M_4 consists of the sets: $\emptyset$, Ω, M_i and $\Omega - M_i$, $i = 1, 2, 3, 4$. Denote this σ-class by Δ. Define a measure $m : \Delta \to \{0, 1\}$ by setting $m(\Omega) = 1$, $m(M_i) = 0$, $i = 1, 2, 3, 4$. Finally, let f_1, f_2, f_3 be the respective characteristic functions of M_1, M_2, M_3. Then f_1, f_2, f_3 are clearly measurable on (Ω, Δ) and so is the function $f_1 + f_2 + f_3$. On the other hand, we have $\int f_1 \, dm + \int f_2 \, dm + + \int f_3 \, dm = 0 \neq \int (f_1 + f_2 + f_3) \, dm = 2$.

Our last comment involves again the initial problem of S. Gudder. We have solved it for two-valued measures and that would give us the solution in general (and in the affirmative under the assumption of Theorem 2) if we were able to show that any pure measure on a σ-class is two-valued. (A measure is called pure if the equality $m = = \alpha m_1 + (1 - \alpha) m_2$, $0 < \alpha < 1$, implies $m = m_1 = m_2$). Unfortunately this is not the case. Put $\Omega = \{1, 2, 3, 4, 5, 6\}$ and take for Δ the set of all subsets of Ω with an even number of elements. Obviously, (Ω, Δ) is a σ-class (see [1], [3]). Define a measure m on (Ω, Δ) by putting $m\{1, a\} = 0$ whenever $a \neq 1$, $m\{a, b\} = 1/2$ whenever $a \neq 1$, $b \neq 1$. One can easily check that the above requirement defines a unique measure. We claim that m is a pure measure. Indeed, if $m = \alpha m_1 + + (1 - \alpha) m_2$, $0 < \alpha < 1$, then $m_1\{1, a\} = 0$. We shall now show that m_1 extends uniquely to m. If $1 < a < b < c < d$ then $m_1\{a, b, c, d\} = m_1\{a, b\} + m_1\{c, d\} = = 1$ and we only need to obtain the equality $m_1\{a, b\} = m_1\{c, d\}$. This will follow as soon as we show that $m_1\{a, b\} = m_1\{a, c\}$ for any fixed $a \in \Omega$. But this is obvious for $m_1\{1, a, b, c\} = m_1\{1, b\} + m_1\{a, c\} = m_1\{1, c\} + m_1\{a, b\}$ and therefore $m_1\{a, c\} = m_1\{a, b\}$.

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