Gabriela Pringerová

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ON SEMISIMPLE CLASSES OF ABELIAN LINEARLY ORDERED GROUPS

Gabriela Pringerová, Prešov

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Radical classes and semisimple classes of linearly ordered groups were defined and investigated by Chehata and Wiegandt [2]. Jakubík [4] studied radical classes of abelian linearly ordered groups; cf. also [6].

In this paper the notion of semisimple class of linearly ordered groups will be modified in order to obtain the possibility of working with the abelian case only. Some questions concerning the lattice $\mathscr{S}_a$ of all semisimple classes of abelian linearly ordered groups will be dealt with; analogous questions for radical classes of abelian linearly ordered groups were studied in [4].

Let S_0 and G_a be the least and the greatest element of $\mathscr{S}_a$ (i.e., $S_0 = \{\{0\}\}$ and G_a is the class of all abelian linearly ordered groups). We denote by $\mathscr{S}_p$ the collection of all principal semisimple classes of abelian linearly ordered groups. For $X \in \mathscr{S}_a$ let $a(X)$ and $a'(X)$ be the collection of all elements of $\mathscr{S}_a$ which cover X or which are covered by X, respectively. Sample results:

Let G_1 and G_2 be non-zero non-isomorphic archimedean linearly ordered groups and let X be the semisimple class generated by $\{G_1, G_2\}$. Then the interval $[S_0, X]$ of $\mathscr{S}_a$ fails to be modular. Let $G \in G_a$, $G \neq \{0\}$ and let Y be the semisimple class generated by G. Then there exists $C \subset [S_0, Y] \cap \mathscr{S}_p$ such that (i) C is a proper collection and (ii) C is linearly ordered. A collection $C_1 \subset \mathscr{S}_a$ is constructed such hat C_1 is a proper collection and an antichain (i.e., any two distinct elements of C are incomparable). For $X \in \mathscr{S}_a$ a sufficient condition is found under which $a'(X) = \emptyset$ (there are infinitely many such semisimple classes X). A proper collection $A_1 \subseteq \mathscr{S}_p$ is constructed such that $a(X) = \emptyset$ for each $X \in A_1$. A proper collection $A_2 \subset G_a$ is described having the property that if X is a principal semisimple class generated by some $G \in A_2$, then $a'(X)$ is a one-element collection; as a corollary we obtain that the collection of all prime intervals $[Y, Z]$ of the lattice $\mathscr{S}_a$ such that Z is principal, is a proper collection.

1. BASIC PROPERTIES OF THE LATTICE OF SEMISIMPLE CLASSES

In this paper we shall deal with objects belonging to some type of the following hierarchy:

sets and their elements; classes of linearly ordered groups; collections of classes of linearly ordered groups.

A collection C will be called proper if there exists an injective mapping of the class of all cardinals into C. Greek letters will denote ordinals (unless they are explicitely defined to have a different meaning).

By considering a subclass X of G_a we always assume that X is closed under isomorphisms and that $\{0\} \in X$.

For the terminology concerning lattice and linearly ordered groups cf. G. Birkhoff [1] and L. Fuchs [3]. The group operation in a linearly ordered group will by written additively.

All linearly ordered groups dealt with in this paper are assumed to be abelian; the words 'linearly ordered group' will always mean 'abelian linearly ordered group'.

Let S be a nonempty class of linearly ordered groups. S is said to be c-hereditary, if for each $G \in S$ and each convex subgroup H of G we have $H \in S$.

Let $H_1, H_2, \ldots, H_\alpha, \ldots (\alpha < \delta)$ be linearly ordered groups. Let G be a linearly ordered group and let

$$(*) \qquad G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_\alpha \supseteq \ldots (\alpha < \delta)$$

be a descending chain of convex subgroups of G such that

$$(\textstyle\bigcap_{\gamma<\beta} G_\gamma)/G_\beta$$

is isomorphic to H_β for each $1 < \beta < \delta$. Assume that $\bigcap_{\alpha<\delta} G_\alpha = \{0\}$. Then the linearly ordered group $G/\bigcap_{\alpha<\delta} G_\alpha$ is called a transfinite co-extension of linearly ordered groups H_α $(\alpha < \delta)$.

Let G_a be the class of all linearly ordered groups.

1.1. Definition. A nonempty class S of linearly ordered groups is said to be a *semisimple class* if

(a*) S is c-hereditary, and

(b*) S is closed under transfinite co-extensions.

(The analogous notion concerning the case when we do not suppose the commutativity of the linearly ordered groups under consideration was studied in [2]; cf. Definition 4 and Theorem 2 of [2].)

Let $\mathcal{S}_a$ be the collection of all semisimple classes of linearly ordered groups. $\mathcal{S}_a$ is partially ordered by inclusion. Under this partial order, G_a is the greatest element of $\mathcal{S}_a$, and the class $S_0 = \{\{0\}\}$ is the least element of $\mathcal{S}_a$. If S_1 is a nonempty subcollection of $\mathcal{S}_a$, then the intersection S of all semisimple classes belonging to S_1

fulfils the condition (a*) and (b*), thus S is the meet (in the sense of the given partial order) of S_1 in $\mathscr{S}_a$; this yields

1.2. Theorem. *$\mathscr{S}_a$ is a complete lattice.*

For each subclass $X \neq \emptyset$ of G_a we denote by

$T_s X$ – the intersection of all semisimple classes S with $X \subseteq S$;

$\operatorname{Sub} X$ – the class of all convex subgroups of linearly ordered groups belonging to X;

co-Ext X – the class of all transfinite co-extensions of linearly ordered groups belonging to X.

It is obvious that $T_s X$ is the smallest semisimple class containing X as a subclass; $T_s X$ is said to be the semisimple class generated by X. If $G \in G_a$ and $S = \{G\}$, then we also write $T_s X = T_s(G)$ and $T_s(G)$ is said to be a principal semisimple class. We denote by $\mathscr{S}_p$ the collection of all principal semisimple classes.

1.3. Theorem. *Let X be a subclass of G_a. Then* $T_s X =$ co-Ext Sub X.

Proof. Denote $Y =$ co-Ext Sub X. Because $T_s X$ is a semisimple class with $X \subseteq T_s X$, we obtain from 1.1 that $Y \subseteq T_s X$ is valid. Further, the relation co-Ext $Y = Y$ obviously holds. Therefore we have only to verify that Sub $Y = Y$.

Let $G \in Y$. There exists a descending chain $\{G_\alpha\}$ $(\alpha < \delta)$ of convex subgroups of G with $G_1 = G$ such that

$$\bigcap_{\alpha<\delta} G_\alpha = \{0\},$$

and for each $\beta < \delta$, the linearly ordered group

$$\bar{G}_\beta = \left(\bigcap_{\gamma<\beta} G_\gamma\right)/G_\beta$$

belongs to Sub X. Let H be a convex subgroup of G, $H \neq \{0\}$. There exists a least $\beta < \delta$ with $G_\beta \subset H$. Consider the descending chain

$$H \supset G_\beta \supseteq G_{\beta+1} \supseteq \ldots \supseteq G_\alpha \supseteq \ldots (\beta \leqq \alpha < \delta)$$

of convex subgroups of H. We have clearly

$$H \cap \left(\bigcap_{\beta\leqq\alpha<\delta} G_\alpha\right) = \{0\}.$$

The linearly ordered group H/G_β is a subgroup of $\bar{G}_\beta$, hence H/G_β belongs to Sub X. Thus $H \in Y$ and therefore Sub $Y = Y$.

We denote by $\wedge$ and $\vee$ the lattice operations in the complete lattice $\mathscr{S}_a$. (If $X, Y \in \mathscr{S}_a$ and $X \subseteq Y$, then we write also $X \leqq Y$.) In fact, the operation $\wedge$ coincides with the intersection of classes. The operation $\vee$ in $\mathscr{S}_a$ is constructively described by the following

1.4. Theorem. *Let I be a nonempty class and for each $i \in I$ let X_i be a semisimple class. Then* $\bigvee_{i\in I} X_i =$ co-Ext $\bigcup_{i\in I} X_i$.

Proof. This follows immediately from 1.3 and from the fact that $\mathrm{Sub}\, X_i = X_i$.

If $X_1, X_2 \in \mathscr{S}_a$, $X_1 \leqq X_2$, then $[X_1, X_2]$ denotes the collection of all $X \in \mathscr{S}_a$ with $X_1 \leqq X \leqq X_2$.

We denote by $G_1 \circ G_2$ the lexicographic product of G_1 and G_2 (cf. e. g. [3] for this notation).

1.5. Lemma. *Let G_1 and G_2 be non-zero archimedean linearly ordered groups such that G_1 is not isomorphic to G_2. Then $T_s(G_1 \circ G_2) \wedge T_s(G_2) = S_0$.*

Proof. By way of contradiction, assume that there is $G \in T_s(G_1 \circ G_2) \wedge T_s(G_2)$ with $G \neq \{0\}$. Let us denote by $c(G)$ the system of all convex subgroup of G partially ordered by inclusion; in fact, under this partial order, $c(G)$ is linearly ordered.

From $G \in T_s(G_2)$ it follows that there is $K_2 \in c(G)$ such that G/K_2 is isomorphic to G_2; let f be an isomorphism of G/K_2 onto G_2. Assume that $H \in c(G)$, $K_2 \subseteq H \subseteq G$. Then H/K_2 is a convex subgroup of G/K_2 and thus $f^{-1}(H/K_2)$ is a convex subgroup of G_2. Since G_2 is archimedean, we have either $f^{-1}(H/K_2) = \{0\}$ or $f^{-1}(H/K_2) = G_2$. Hence either $H = K_2$ or $H = G$. This implies that K_2 is covered by G in $c(G)$. Similarly, from $G \in T_s(G_1 \circ G_2)$ we obtain that there is $K_3 \in c(G)$ such that either (i) G/K_3 is isomorphic to G_1, or (ii) G/K_3 is isomorphic to $G_1 \circ G_2$. If (i) is valid, then K_3 is covered by G in the chain $c(G)$, whence $K_3 = K_2$; in such a case G_1 would be isomorphic to G_2, which is a contradiction. Thus (ii) holds and hence the interval $[K_3, G]$ of $c(G)$ (being isomorphic to the interval $[\{0\}, G_1 \circ G_2]$ of $c(G_1 \circ G_2)$) has exactly three elements; hence K_3 is covered by K_2 in $c(G)$; moreover, K_2/K_3 is isomorphic to G_1. We have $K_2 \neq \{0\}$ and $K_2 \in T_s(G_2)$; thus there is $K_2' \in c(K_2)$ such that K_2/K_2' is isomorphic to G_2. This implies that $K_2' = K_3$ and therefore G_2 is isomorphic to G_1, which is a contradiction.

1.6. Lemma. *Let G_1 and G_2 be as in 1.5. Then $T_s(G_1) \wedge T_s(G_2) = S_0$.*

Proof. By way of contradiction, assume that there exists $\{0\} \neq G \in T_s(G_1) \wedge T_s(G_2)$. From $G \in T_s(G_i)$ it follows that there are $K_i \in c(G)$ such that G/K_i is isomorphic to G_i ($i = 1, 2$). Both K_1 and K_2 are covered by G in $c(G)$, hence $K_1 = K_2$ and thus G_1 is isomorphic to G_2, which is a contradiction.

1.7. Theorem. *Let G_1, G_2 be non-zero archimedean linearly ordered groups. Assume that G_1 is not isomorphic to G_2. Then the interval $[S_0, T_s(G_1) \vee T_s(G_2)]$ fails to be modular.*

Proof. Put

$$S_1 = T_s(G_1)\,, \quad S_2 = T_s(G_2)\,, \quad S_3 = T_s(G_1 \circ G_2)\,.$$

Clearly $S_0 < S_i$ ($i = 1, 2, 3$). Because G_1 and G_2 are archimedean, we have

$$\mathrm{Sub}\,\{G_i\} = \{\{0\}, G_i\} \ (i = 1, 2)\,,$$

$$\mathrm{Sub}\,\{G_1 \circ G_2\} = \{\{0\}, G_1, G_1 \circ G_2\}\,.$$

Hence according to 1.3,

$$(1)\quad S_i = \text{co-Ext}\,\{\{0\}, G_i\}\ (i = 1, 2)\,,$$

$$(2)\quad S_3 = \text{co-Ext}\,\{\{0\}, G_1, G_1 \circ G_2\}\,.$$

From (1) and (2) we conclude that $S_1 < S_3$. According to 1.5 and 1.6,

$$(3)\quad S_1 \wedge S_2 = S_3 \wedge S_2 = S_0$$

is valid. Moreover, in view of 1.3 we have $S_3 \leqq S_1 \vee S_2$ (because of $G_1 \circ G_2 \in \in \text{co-Ext}\,\{G_1, G_2\}$), thus

$$(4)\quad S_1 \vee S_2 = S_3 \vee S_2\,.$$

From (3) and (4) it follows that the lattice $[S_0, S_1 \vee S_2]$ is not modular.

1.8. Corollary. *The lattice $\mathcal{S}_a$ is not modular.*

2. ON THE INTERVALS $[S, X]$

Let α be an infinite cardinal. We denote by $\omega(\alpha)$ the least ordinal having the property that the power set of all ordinals less then $\omega(\alpha)$ is α. Let $J(\alpha)$ be the linearly ordered set dual to $\omega(\alpha)$.

For each $G \in G_a$ we put

$$G_\alpha^1 = \Gamma_{j \in J(\alpha)}\, G_j\,,$$

where each G_j is isomorphic to G. (The symbol Γ denotes the operation of the lexicographic product, cf. e. g. [6].) Further, let G_α^2 be the linearly ordered group consisting of all $f \in G_\alpha^1$ such that the set $\{j \in J(\alpha) : f(j) \neq 0\}$ is finite. If $G \neq \{0\}$ and $\alpha > \text{card}\, G$, then $\text{card}\, G_2^\alpha = \alpha$. Moreover, for each non-zero convex subgroup G' of G_α^2 we also have $\text{card}\, G' = \alpha$. Because G_α^2 is a transfinite co-extension of G, we infer

2.1. Lemma. *For each $G \in G_a$ and for each infinite cardinal α, G_α^2 belongs to $T_s(G)$.*

2.2. Lemma. *Let $G \in G_a$, $G \neq \{0\}$ and let α be a cardinal with $\alpha >$ card G. Then G does not belong to $T_s(G_\alpha^2)$.*

Proof. By way of contradiction, assume that G belongs to $T_s(G_\alpha^2)$. Hence $G \in \in \text{co-Ext Sub}\,\{G_\alpha^2\}$. Thus there exists a convex subgroup H of G with $H \neq G$ such that G/H is isomorphic to a convex subgroup H_1 of G_α^2. Then $H_1 \neq \{0\}$ and according to the construction of G_α^2 we have $\text{card}\, H_1 = \alpha$. Therefore $\text{card}\, G \geqq \text{card}\, G/H = \alpha$, which is impossible.

2.3. Corollary. *The lattice $\mathcal{S}_a$ does not contain any atom.*

Proof. Let $X \in S_a$, $X \neq S_0$. Hence there exists $G \in X$ with $G \neq \{0\}$. Thus $T_s(G) \leqq X$. Let α be a cardinal with $\alpha >$ card G. In view of 2.1 we have $T_s(G_\alpha^2) \leqq T_s(G)$. From 2.2 we obtain $T_s(G_\alpha^2) \neq T_s(G)$, hence $T_s(G_\alpha^2) < T_s(G)$. Because of $G_\alpha^2 \neq \{0\}$ we have $T_s(G_\alpha^2) \neq S_0$. Therefore $S_0 < T_s(G_\alpha^2) < X$ and thus X fails to be an atom in $\mathcal{S}_a$.

Corollary 2.3 can be sharpened by using the following consideration.

2.4. Lemma. *Let $\{0\} \neq G \in G_a$. Let α, β be infinite cardinals with* card $G < \alpha < \beta$. *Then $S_0 \neq T_s(G_\beta^2) < T_s(G_\alpha^2)$.*

Proof. Since $G_\beta^2 \neq \{0\}$, we have $S_0 \neq T_s(G_\beta^2)$. From the construction of the linearly ordered groups G_α^2, G_β^2 and from $\alpha < \beta$ it follows that G_β^2 is an infinite co-extension of G_α^2, hence $G_\beta^2 \in T_s(G_\alpha^2)$. Hence $T_s(G_\beta^2) \leqq T_s(G_\alpha^2)$. Now it suffices to verify that G_α^2 does not belong to $T_s(G_\beta^2)$. By way of contradiction, assume that $G_\alpha^2 \in T_s(G_\beta^2) =$ co-Ext Sub $\{G_\beta^2\}$. Thus there exists a convex subgroup H of G_α^2 with $H \neq G_\alpha^2$ such that G_α^2/H is isomorphic to some $H_1 \in$ Sub $\{G_\beta^2\}$. Thus $H_1 \neq \{0\}$ and hence card $H_1 = \beta$ implying card $G_\alpha^2/H = \beta$, which is a contradiction.

2.5. Theorem. *Let $X \in \mathcal{S}_a$, $X \neq S_0$. There exists a subcollection C of $[S_0, X]$ such that* (i) *C is linearly ordered,* (ii) *C is a proper collection, and* (iii) *$C \subset \mathcal{S}_p$.*

Proof. There exists $G \in X$ with $G \neq \{0\}$. For each cardinal $\alpha >$ card G we construct $X_\alpha = T_s(G_\alpha^2)$; let C be the class of all such X_α. Each G_α^2 is an infinite co-extension of G, hence $G_\alpha^2 \in T_s(G) \subseteq X$, and hence $X_\alpha \in [S_0, X]$. From 2.4 it follows that (i) and (ii) are valid. The assertion (iii) obviously holds.

2.6. Corollary. *Let $X \in \mathcal{S}_a$, $X \neq S_0$. Then the interval $[S_0, X]$ is a proper collection.*

It remains an open question whether the condition (i) can be replaced by the condition (i′) S is an antichain. The following weaker result is valid:

2.7. Theorem. *There exists a subcollection S of $\mathcal{S}_p$ such that* (i) *S is an antichain, and* (ii) *S is a proper collection.*

Proof. If I_1 and I_2 are disjoint linearly ordered sets, then we denote by $I_1 \oplus I_2$ the set $I = I_1 \cup I_2$ which is linearly ordered in such a way that for pairs of elements belonging to the same I_i $(i = 1, 2)$ the linear order in I is the same as in I_i, and for each pair $i_1 \in I_1$, $i_2 \in I_2$ we put $i_1 < i_2$. Let $G \in G_a$, $G \neq \{0\}$, $\alpha >$ card G. Let $I(\alpha)$ be a linearly ordered set dually isomorphic to $J(\alpha)$; we shall assume that $I(\alpha) \cap J(\alpha) = \emptyset$. Put $I^*(\alpha) = J(\alpha) \oplus I(\alpha)$ and for each $i \in I^*(\alpha)$ let G_i be a linearly ordered group isomorphic to G. Put

$$K'_\alpha = \Gamma_{i \in I^*(\alpha)} G_i \,.$$

Let K_α be the set of all $f \in K'_\alpha$ such that $\{i \in I^*(\alpha) : f(i) \neq 0\}$ is finite. The construction of K_α implies:

(a) If K_0 is a non-zero convex subgroup of K_α, then card $K_0 = \alpha$.

(b) If H_0 is a convex subgroup of K_α with $K_\alpha \neq H_0$, then card $K_\alpha / H_0 = \alpha$.

Now let β be a cardinal with $\beta > \alpha$. From (a) and (b) it follows that neither $K_\alpha \in T_s(K_\beta)$ nor $K_\beta \in T_s(K_\alpha)$ can hold. Hence $T_s(K_\alpha)$ is incomparable with $T_s(K_\beta)$. Let S be the class of all $T_s(K_\alpha)$, where α runs over the the class of all cardinals larger than card G. Then S fulfils the conditions (i) and (ii).

3. SEMISIMPLE CLASSES HAVING NO DUAL COVERS

Let α be an infinite cardinal and let $\omega(\alpha)$ be as in § 2. For each $G \in G_a$ we put

$$G_\alpha = \Gamma_{i \in I(\alpha)} G_i ,$$

where $I(\alpha)$ is a linearly ordered set isomorphic to $\omega(\alpha)$ and G_i is isomorphic to G for each $i \in I(\alpha)$.

3.1. Lemma. *Let $G \in G_a$, $G \neq \{0\}$, $\alpha >$ card G. Then G_α does not belong to $T_s(G)$.*

Proof. By way of contradiction, assume that G_α belongs to $T_s(G)$. Thus according to 1.3, $G_\alpha \in$ co-Ext Sub $\{G\}$. Hence there exists a convex subgroup H_1 of G_α and a convex subgroup $\{0\} \neq H_2$ of G such that G_α/H_1 is isomorphic to H_2. From the structure of G_α it follows that card $G_\alpha/H_1 = \alpha$; because of card $H_2 < \alpha$, we arrived at a contradiction.

Let X be a semisimple class and let $G \in G_a$. Let $\{H_i\}_{i \in I}$ be the set of all convex subgroups of G having the property that G/H_i belongs to X. We denote

$$X(G) = \bigcap_{i \in I} H_i .$$

3.2. Lemma. *$G/X(G)$ belongs to X.*

Proof. If we consider the set $\{H_i\}_{i \in I}$ as partially ordered by inclusion, then this set is linearly ordered. Axiom of Choice implies that there exists a well-ordered set of indices J such that $\{H_j\}_{j \in J}$ is a subset of $\{H_i\}_{i \in I}$, $\bigcap_{j \in J} H_j = X(G)$, and for each pair $j_1, j_2 \in J$ we have

$$j_1 \leqq j_2 \Leftrightarrow H_{j_1} \supseteq H_{j_2} .$$

Let $j \in J$. Then

$$\bar{H}_j = \bigcap_k H_k / H_j \quad (k \in J,\ k < j)$$

is a convex subgroup of the linearly ordered group G/H_j, hence $\bar{H}_j \in X$. Thus $G/X(G)$ is a co-extension of linearly ordered groups belonging to X.

3.3. Lemma. $X(X(G)) = X(G)$.

Proof. By way of contradiction, assume that $H = X(X(G))$ is a proper subset of $X(G)$. Hence $X(G)/H$ belongs to X. Thus G/H is a co-extension of linearly ordered groups belonging to X and therefore $G/H \in X$. Hence $H \in \{H_i\}_{i \in I}$ which implies $H \supseteq X(G)$, and so we arrived at a contradiction.

Remark. For the analogous result concerning semisimple classes of linearly ordered groups which need not be abelian cf. [2].

3.4. Lemma. *Let $A, B \in \mathcal{S}_a$ and suppose that A covers B. Then there is $H \in A \setminus B$ such that $B \vee T_s(H) = A$ and $B(H) = H$.*

Proof. Since $B \subset A$ there is $H_1 \in A \setminus B$. Put $H = B(H_1)$. Since A is c-herediatary, we have $H \in A$. According to 3.3, $B(H) = H$. If $H \in B$, then H_1 is a transfinite co-extension of linearly ordered groups belonging to B, whence $H_1 \in B$, which is a contradiction; thus $H \notin B$. Therefore $B < B \vee T_s(H) \leqq A$. Since B is covered by A we infer that $B \vee T_s(H) = A$.

3.5. Theorem. *Let $A \in \mathcal{S}_a$. Assume that for each $G \in A$ and each cardinal α we have $G_\alpha \in A$. Then no semisimple class is covered by A.*

Proof. By way of contradiction, suppose that there exists a semisimple class B such that B is covered by A. Let H be as in 3.4 and let α be a cardinal, $\alpha >$ card H. According to the assumption, $H_\alpha \in A$. Thus $T_s(H_\alpha) \leqq A$. Since $H \in T_s(H_\alpha)$, the linearly ordered group H_α does not belong to B.

In view of 3.4 we have $H_\alpha \in B \vee T_s(H)$. This together with 1.4 implies that there exists a convex subgroup K of H_α with $K \neq H_\alpha$ such that either (i) H_α/K belongs to B, or (ii) H_α/K is isomorphic to a convex subgroup of H. Since card $H_\alpha/K = \alpha >$ > card H, the condition (ii) cannot hold; thus (i) is valid.

In view of the structure of H_α there are linearly ordered groups P, Q such that H_α/K is isomorphic to $P \circ Q$, where Q is isomorphic to H_α and either (i_1) $P = \{0\}$ or (ii_1) there is a convex subgroup P_1 of H with $P_1 \neq H$ such that P is isomorphic to H/P_1. Now $H_\alpha/K \in B$ implies $P \circ Q \in B$, hence $P \in B$; if (ii_1) holds, then $H/P_1 \in B$, which implies $B(H) \subseteq P_1 \neq H$ and this is a contradiction (cf. 3.4). Therefore (i_1) is valid and hence H_α/K is isomorphic to H_α. This yields $H_\alpha \in B$, which is a contradiction.

From 3.5 we obtain immediately:

3.6. Corollary. *The lattice $\mathcal{S}_a$ contains no dual atoms.*

For $H \in G_a$ we denote by $U_1(H)$ the class of all linearly ordered groups G such that no convex subgroup of G is isomorphic to H.

3.7. Lemma. *Let $H \neq \{0\}$ be an archimedean linearly ordered group. Then $U_1(H)$ is a semisimple class.*

Proof. Put $U_1(H) = X$. Then we have $\operatorname{Sub} X = X = \text{co-Ext}\, X$. Hence in view of 1.3, $X = T_s X$.

3.8. Lemma. *Let H be as in 3.7 and let $G \in U_1(H)$. Then $G_\alpha \in U_1(H)$ for each cardinal α.*

This is an immediate consequence of the definition of $U_1(H)$.

From 3.7, 3.8 and 3.5 we conclude:

3.9. Corollary. *Let $H \neq \{0\}$ be an archimedean linearly ordered group. Then the semisimple class $U_1(H)$ does not cover any linearly ordered group.*

If H_1, H_2 are non-zero archimedean linearly ordered groups and if H_1 is not isomorphic to H_2, then $H_1 \in U_1(H_2)$, $H_2 \in U_1(H_1)$, whence $U_1(H_1) \neq U_1(H_2)$. Therefore we have:

3.10. Corollary. *The class of all semisimple classes X having the property that X does not cover any semisimple class is infinite.*

The above considerations can be sharpened as follows.

3.11. Lemma. *Let $A, B \in \mathscr{S}_a$, $B < A$. Assume that $H \in A \setminus B$, $B(H) = H$. Let α be an infinite cardinal. Then $B(H_\alpha) = H_\alpha$.*

Proof. By way of contradiction, assume that $B(H_\alpha) \subset H_\alpha$. Hence $\{0\} \neq H_\alpha/B(H_\alpha) \in B$. Now we may apply the same procedure as in the proof of 3.5 (with K replaced by $B(H_\alpha)$) and we arrive at a contradiction.

3.12. Lemma. *Let $H \in G_a$, $H \neq \{0\}$. Let α, β be cardinals with $\alpha < \beta$. Then $(H_\alpha)_\beta = H_\beta$.*

This follows immediately from the definition of H_α and H_β.

3.13. Lemma. *Let A be as in 3.5 and let B be a semisimple class with $B \subset A$. Let $H \in A \setminus B$ and let α be a cardinal with $\alpha >$ card H. Then H_α does not belong to $B \vee T_s(H)$.*

The proof (by way of contradiction) is similar to that of 3.5 and therefore it will be omitted.

From 3.1, 3.11, 3.12 and 3.13 we obtain:

3.14. Corollary. *Let A, B, H and α be as in 3.13. Let β be a cardinal with $\beta > \alpha$. Then $B < B \vee T_s(H_\alpha) < B \vee T_s(H_\beta) < A$.*

The following theorem is a consequence of 3.14:

3.5.1. Theorem. *Let $A, B \in \mathscr{S}_a$, $B < A$. Assume that for each $G \in A$ and each cardinal α we have $G_\alpha \in A$. Then the interval $[B, A]$ of $\mathscr{S}_a$ contains a subcollection C such that* (i) *C is linearly ordered, and* (ii) *C is a proper collection.*

3.5.2. Corollary. *Let $B \in \mathscr{S}_a$, $B \neq G_a$. Then the interval $[B, G_a]$ of $\mathscr{S}_a$ is a proper collection.*

4. THE CLASS $\mathscr{A}_1$

In this section we will describe a rather large collection of semisimple classes X which have no cover in the lattice $\mathscr{S}_a$.

We denote by A_1 the class of all linearly ordered groups G which have the following property: there exists a convex subgroup G' of G such that G' is non-zero and archimedean.

Let $\mathscr{A}_1$ be the collection of all semisimple classes $X = T_s(G)$ with $G \in A_1$. It will be proved that $\mathscr{A}_1$ is a proper collection and that for each $X \in \mathscr{A}_1$, X has no cover in the lattice $\mathscr{S}_a$.

Let us remark that if G_1 is a non-zero archimedean linearly ordered group and G_2 is any linearly ordered group, then $G_1 \circ G_2$ belongs to A_1.

If we have a lexicographic product $P = \Gamma_{i \in I} G_i$, then we denote by $\Gamma'_{i \in I} G_i$ the linearly ordered group consisting of all elements $f \in P$ such that the set $\{i \in I : f(i) \neq 0\}$ is finite.

4.1. Theorem. *Let $G \in A_1$, $B = T_s(G)$. Then B has no cover in $\mathscr{S}_a$.*

Proof. By way of contradiction, assume that there is $A \in \mathscr{S}_a$ such that A covers B. According to 3.4, there is $H \in A \setminus B$ such that $B(H) = H$ and $B \vee T_s(H) = A$. Let α be a cardinal, $\alpha > \operatorname{card} H$. Consider the semisimple class $T_s(H_\alpha^2)$. Because $H_\alpha^2 \in T_s(H)$, we have $T_s(H_\alpha^2) \leqq T_s(H)$. First assume that $H_\alpha^2 \notin B$. Then $B \vee T_s(H_\alpha^2) = A$, hence $H \in B \vee T_s(H_\alpha^2)$. Thus in view of 1.4, $H \in \text{co-Ext}\,(B \cup T_s(H_\alpha^2))$. Therefore there is a convex subgroup K of H with $H \neq H$ such that either (i) $H/K \in B$, or (ii) H/K is isomorphic to a convex subgroup of H_α^2. The validity of (i) is impossible because $B(H) = H$; the validity of (ii) is impossible as well since from the definition of H_α^2 we obtain $\operatorname{card} H/K = \alpha > \operatorname{card} H$, which cannot hold. Thus $H_\alpha^2 \in B = \text{co-Ext Sub}\,\{G\}$.

There exists a convex subgroup $K_1 \neq \{0\}$ of G such that K_1 is isomorphic to a homomorphic image K_2 of H_α^2. From the structure of H_α^2 it follows that K_2 can be expressed as

$$K_2 = P \circ Q$$

so that

a) either (i) $P = \{0\}$ or (ii) $P \neq \{0\}$, P is not isomorphic to H and P is a homomorphic image of H;

b) either (i_1) $Q = \{0\}$, or (ii_1) there is a linearly ordered set I such that I is dually well-ordered and Q is isomorphic to $\Gamma'_{i \in I} H_i$, where each H_i is isomorphic to H.

If (ii) is valid, then $K_1 \in B$ implies that $K_2 \in B$ and thus $P \in B$; but in this case we should have $B(H) \subset H$, which is a contradiction. Thus $P = \{0\}$ and hence $Q \neq \{0\}$;

therefore (ii_1) is valid and K_2 is isomorphic to Q. Now we distinguish two cases. If I has the least element i_0, then H_{i_0} is a convex subgroup of K_2 and hence H_{i_0} belongs to B, thus H belongs to B, which is a contradiction. If I has no least element, then no non-zero convex subgroup of K_2 is o-simple, hence the same holds for K_1, which is a contradiction.

4.2. Lemma. *Let G and G' be non-zero non-isomorphic archimedean linearly ordered groups. Let α, β be cardinals with* $\max\{\text{card } G, \text{card } G'\} < \alpha < \beta$, $H_1 = G' \circ G_\alpha^2$, $H_2 = G' \circ G_\beta^2$. *Then* $T_s(H_1) \neq T_s(H_2)$.

Proof. By way of contradiction, assume that $H_1 \in T_s(H_2)$. Hence there is a homomorphic image $K_1 \neq \{0\}$ of H_1 such that K_1 is isomorphic to a convex subgroup K_2 of H_2. Since $K_2 \neq \{0\}$, it contains a convex subgroup G_1 isomorphic to G'; since G' is lexicographically indecomposable, it follows from Mal'cev's theorem on the existence of isomorphic refinements of two lexicographic decompositions ([5], cf. also [3] p. 42, Thm. 9) that K_1 also contains a convex subgroup isomorphic to G'. Therefore $K_1 = H_1$ and hence $K_2 \neq G'$; but in this case we have

$$\text{card } K_1 = \alpha < \beta = \text{card } K_2 ,$$

which is a contradiction.

Since both H_1 and H_2 belong to A_1, we obtain immediately from 4.2:

4.3. Theorem. *$\mathscr{A}_1$ is a proper collection.*

5. EXISTENCE OF PRIME INTERVALS IN $\mathscr{S}_a$

An interval $[X, Y]$ of $\mathscr{S}_a$ is called prime if $X < Y$ and if there exists no $Z \in \mathscr{S}_a$ with $X < Z < Y$. Until now we have established only negative results concerning the existence of prime intervals in $\mathscr{S}_a$ (cf. 2.5, 3.5, 3.5.1, 4.1).

In this section it will be shown that there exist infinitely many prime intervals in the lattice $\mathscr{S}_a$ (in fact, the class of all prime intervals of $\mathscr{S}_a$ is a proper class).

Let A_2 be the class of all linearly ordered groups G which have the following property: there exists a convex subgroup G' of G such that

(i) G' is non-zero and archimedean;

(ii) if $K, K' \in c(G)$, $\{0\} \neq K \subseteq K'$, then K'/K is not isomorphic to G'.

It is easy to verify that if $G \in A_2$, then its convex subgroup G' fulfilling (i) is uniquely determined.

5.1. Lemma. *Let $G \in A_2$ and let B_G be the class of all linearly ordered groups $H \in T_s(G)$ such that no convex subgroup of H is isomorphic to G. Then B_G is a semi-simple class.*

Proof. It is obvious that B_G is c-hereditary; hence it suffices to verify that B_G is closed with respect to transfinite co-extensions. By way of contradiction, assume that there exists $\{0\} \neq H' \in$ co-Ext B_G such that H' does not belong to B_G. Because $H' \in T_s(G)$ there exists a convex subgroup G_1 of H' such that G_1 is isomorphic to G. Further, there exist convex subgroups H_α $(\alpha < \delta)$ of H' such that

$$H' = H_1 \supset H_2 \supset \ldots \supset H_\alpha \supset \ldots (\alpha < \delta),$$

$$\bigcap_{\alpha<\delta} H_\alpha = \{0\},$$

and for each $1 < \beta < \delta$, the linearly ordered group

$$(\bigcap_{\gamma<\beta} H_\gamma)/H_\beta$$

belongs to B_G. We distinguish two cases:

a) δ is a limit ordinal. Then no convex sugroup of H' is o-simple, hence no convex subgroup of H' is isomorphic to G', which is a contradiction (in view of the existence of G_1).

b) δ is non-limit, $\delta = \varkappa + 1$. Hence $H_\varkappa = \{0\}$. If $\varkappa$ is a limit ordinal, then we have the same conclusion as in a). Let $\varkappa$ be non-limit, $\varkappa = \tau + 1$. Then $H_\tau = H_\tau/H_{\tau+1}$ belongs to B_G, thus we cannot have $G_1 \subseteq H_\tau$. Therefore $H_\tau \subset G_1$.

The linearly ordered group G_1/H_τ is a convex subgroup of H'/H_τ. If τ is a non-limit ordinal, then H'/H_τ has a convex subgroup isomorphic to G', hence the same holds for G_1/H_τ, which is a contradiction (cf. (ii) above). Hence τ is a limit ordinal. Thus there are infinitely many pairs P_i, Q_i of convex subgroups of G_1 such that $P_i \subset Q_i$ and Q_i/P_i is isomorphic to G', which is a contradiction (again, cf. (ii)). Thus B_G is closed with respect to transfinite extensions.

5.2. Lemma. *Let G and B_G be as in 5.1. Let X be a semisimple class with $X < T_s(G)$. Then $X \leqq B_G$.*

Proof. Let $H \in X$. If H has a convex subgroup isomorphic to G, then $G \in X$ and thus $T_s(G) \leqq X$, which is a contradiction. Therefore $H \in B_G$, which implies $X \leqq B_G$.

Since $G \in T_s(G) \setminus B_G$, we obtain from 5.2 as a corollary:

5.3. Theorem. *Let $G \in A_2$. Then $[B_G, T_s(G)]$ is a prime interval of the lattice $\mathcal{S}_a$.*

5.4. Lemma. *Let $G, G' \in A_2$. Assume that $T_s(G) < T_s(G')$. Then $B_G < B_{G'}$.*

Proof. According to 5.3 we have $B_G < T_s(G) \leqq B_{G'}$.

Let G and G' be non-isomorphic non-zero archimedean linearly ordered groups. For each cardinal α we put

$$K(\alpha) = G \circ (G')_\alpha .$$

5.5. Lemma. *Let α, β be cardinals,* max {card G, card G'} $< \alpha < \beta$. *Then $T_s(K(\alpha)) < T_s(K(\beta))$.*

Proof. $K(\alpha)$ is isomorphic to a convex subgroup of $K(\beta)$, hence $K(\alpha) \in T_s(K(\beta))$ and thus $T_s(K(\alpha)) \leqq T_s(K(\beta))$. On the other hand, no nontrivial homomorphic image of $K(\beta)$ is isomorphic to a convex subgroup of $K(\alpha)$; thus $K(\beta)$ does not belong to $T_s(K(\alpha))$. Therefore $T_s(K(\alpha)) < T_s(K(\beta))$.

From 5.4 and 5.5 we obtain (since $K(\alpha) \in A_2$):

5.6. Lemma. *Let G, α, β be as above. Then $[B_{K(\alpha)}, K(\alpha)]$ and $[B_{K(\beta)}, K(\beta)]$ are distinct prime intervals of the lattice $\mathcal{S}_a$.*

As a corollary, we infer:

5.7. Theorem. *The collection of all prime intervals of the lattice $\mathcal{S}_a$ is a proper collection.*

References

[1] *G. Birkhoff:* Lattice theory, third edition, Providence 1967.

[2] *C. G. Chehata, R. Wiegandt:* Radical theory for fully ordered groups. Rév. Anal. Numer. Théor. Approx. *20* (43), 1979, 143—157.

[3] *Л. Фукс:* Частично упорядоченные алгебраические системы, Москва 1972.

[4] *J. Jakubík:* On radical classes of abelian linearly ordered groups. (Submitted).

[5] *А. И. Мальцев:* Об упорядоченных группах. Изв. Акад. наук СССР, сер. матем., *13*, 1949, 473—482.

[6] *G. Pringerová:* Covering condition in the lattice of radical classes of linearly ordered groups, Mathem. Slovaca (submitted).

Author's address: 081 00 Prešov, Leninova 24 (Výpočtové stredisko).