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INTRODUCTION

The theory of conability of singlevalued mapping in the sense of J. Durdil [2] M. Fabian [3] is developed in locally convex topological linear spaces. The main results are Theorem 1 and 2, which give the connections between the Gâteaux conability, uniform conability and Frèchet differentiability of mappings.

1. DEFINITIONS AND NOTATIONS

We recall the definition of calibration for a family of locally convex spaces, which was introduced by S. Yamamuro [4]. A calibration for a locally convex space E is a set of continuous seminorms, which induces the topology in E. The set $P(E)$ of all continuous seminorms on E is obviously the largest calibration for E.

Let $E = \{E_\alpha\colon \alpha \in I\}$ be an indexed family of locally convex spaces. A seminorm map on E is a map p defined on I whose value p_E at $\alpha \in I$ belongs to $P(E_\alpha)$. We call a set Γ of semiborm maps on E a calibration for E if for each $\alpha \in I$, the set $\Gamma_{E_\alpha} = \{p_{E_\alpha} : p \in \Gamma\}$ is a calibration for E_α. We shall also say that E is a Γ-family. Throughout this paper E denotes a family of locally convex spaces, Γ is a calibration for E.

For two seminorm maps p, q on E we write $p \leqq q$ if $p_X \leqq q_X$ for all $X \in E$. Let $p \in \Gamma$, $X \in E$, $Y \in E$. Put

$$p_{X\times Y}(x, y) = p_X(x) + p_Y(y) \quad \text{for all} \quad (x, y) \in X \times Y .$$

Throughout this paper we assume that the following assumptions are satisfied:

1. Each normed space $(E, \| \|)$ belongs to E and $p_X = \| \|$ for all $p \in \Gamma$.
2. If $X \in E$, $Y \in E$ and $\{p_{X\times Y} : p \in \Gamma\}$ is a calibration for $X \times Y$ then $X \times Y \in E$.
3. For $X \in E$, $p \in \Gamma$, $q \in \Gamma$ there exists $r \in \Gamma$ such that $p_X \leqq r_X$, $q_X \leqq r_X$.

Definition 1.1. By *a cone in linear space X* we understand every subset C of X such that $C \neq \emptyset$, $C \neq \{0\}$ and $tx \in C$ for all $x \in C$, $t \geqq 0$.

Definition 1.2 (J. Daneš, J. Durdil [1]). Let $X \in E$, C be a cone in X. Put

$$V_{p,\varepsilon}(C) = \{x \in X : \exists c \in C,\ p(x - c) \leqq \varepsilon\, p(x)\} \quad \text{for each} \quad p \in \Gamma\,, \quad \varepsilon > 0\,.$$

Then, of course, $V_{p,\varepsilon}(C)$ is a cone again.

Definition 1.3 (M. Fabian [3]). Let $X \in E$, $Y \in E$. For each cone C in $X \times Y$ we define (taking $1/0 = \infty$, $0/0 = 0$)

$$p(C)_X = \sup \left\{ \frac{p_Y(y)}{p_X(x)} : (x, y) \in C \right\}.$$

Definition 1.4 (S. Yamamuro [4]). Let $X \in E$, $Y \in E$, $p \in \Gamma$. We say that a map f of X into Y is *p-continuous at* $x_0 \in X$ if for each $\varepsilon > 0$ there exists $\delta(p, \varepsilon) > 0$ such that

$$p_Y(f(x) - f(x_0)) < \varepsilon \quad \text{for all} \quad x \in X\,, \quad p_X(x - x_0) < \delta(p, \varepsilon)\,.$$

We say that f is *Γ-continuous at* x_0 if f is p-continuous at x_0 for all $p \in \Gamma$. If T is a linear map of X into Y, then T is p-continuous if and only if

$$p(T) = \sup_{p_X(x) \leqq 1} \{p_Y(Tx)\} < \infty\,.$$

$L_p(X, Y)$ denotes the set of all linear p-continuous maps of X into Y. It is easy to see that a linear map T of X into Y is Γ-continuous if and only if $p(T) < \infty$ for all $p \in \Gamma$. $L_\Gamma(X, Y)$ denotes the set of all linear Γ-continuous maps of X into Y. Then $L_\Gamma(X, Y) = \bigcap_{p \in \Gamma} L_p(X, Y)$. We note that if $T \in L_p(X, Y)$ then $G(T) = \{(x, Tx)\}$ is a cone in $X \times Y$ and $p(G(T))_X = p(T)$.

Definition 1.5. Let X, Y be linear spaces. A map f of X into Y is called *positive homogeneous* if $f(tx) = t\,f(x)$ for all $x \in X$ and $t \geqq 0$.

Definition 1.6. Let $X \in E$, $Y \in E$, $\Omega \subset X$, $x_0 \in \Omega$, $f : \Omega \to Y$, $p \in \Gamma$. A map φ of X into Y is called *a map of good p-approximation for f at* x_0 if φ is p-continuous at 0 and for each $\varepsilon > 0$ there exists $\delta(p, \varepsilon) > 0$ such that $p_Y(f(x_0 + h) - f(x_0) - \varphi(h)) \leqq \varepsilon\, p(h)$ for all $h \in X$, $p_X(h) < \delta(p, \varepsilon)$. A map φ is called *a map of good Γ-approximation for f at* x_0 if φ is a map of good p-approximation for f at x_0 for all $p \in \Gamma$.

Definition 1.7. Let $X \in E$, $Y \in E$, $\Omega \subset X$, $f : \Omega \to Y$. We say that f is *Γ-Frèchet differentiable at* x_0 if there exists a map $\varphi \in L_\Gamma(X, Y)$ such that φ is a map of good Γ-approximation for f at x_0.

Definition 1.8 (J. Durdil [2]). Let $X \in E$ and let I be a net, $\{C_\iota\}_{\iota \in I}$ a family of cones in X. A closed cone C in X is said *to be the conic limit of* $\{C_\iota\}_{\iota \in I}$ if for each $p \in \Gamma$ and each $\varepsilon > 0$ there exists $\varkappa(p, \varepsilon) \in I$ such that

$$C \subseteqq V_{p,\varepsilon}(C_i) \quad \text{and} \quad C_i \subseteqq V_{p,\varepsilon}(C) \quad \text{for all} \quad i \in I\,, \quad i \geqq \varkappa(p, \varepsilon)\,.$$

Notation: $C = \lim_{i \in I} C_i$ or $C_i \to C$.

Let $X \in E$, $Y \in E$, and let C be a cone in $X \times Y$. For each $h \in X$, $h \neq 0$, put $C(h) = \{(x, y) : x = th, t \in R, (x, y) \in C\}$. One can see that $C = \bigcup_{h \in X} C(h)$ if $(0, y) \notin C$ for all $y \in Y$, $y \neq 0$.

Definition 1.9 (J. Durdil [2]). Let $X \in E$, $Y \in E$, $X \times Y \in E$, $\Omega \subseteqq X$, $p \in \Gamma$ and let $f : \Omega \to Y$ be a p-continuous map at $x_0 \in \Omega$. For each $h \in X$, $h \neq 0$ put

$$C^g_{p,r}(f, x_0, h) = \{\lambda(th, f(x_0 + th) - f(x_0)) : \lambda \geqq 0,\ p(th) \leqq r\}\,.$$

We say that a cone C in $X \times Y$ is *a cone of good p-approximation for f at x_0 in a direction h* if $C(h) = C$ and for each $\varepsilon > 0$ there exists $\delta(p, \varepsilon, h) > 0$ such that for all r, $0 < r < \delta(p, \varepsilon, h)$ implies $C^g_{p,r}(f, x_0, h) \subseteqq V_{p,\varepsilon}(C)$ and $C \subset V_{p,\varepsilon}(C^g_{p,r}(f, x_0, h))$. We say that f is *Γ-Gâteaux conable at x_0* if f is Γ-continuous at x_0 and for each $h \in X$, $h \neq 0$ there exists a closed cone $C^g_0(f, x_0, h)$ in $X \times Y$ such that $C^g_0(f, x_0, h)$ is a cone of good p-approximation for f at x_0 in the direction h for all $p \in \Gamma$. Notation:

$$C^g_0(f, x_0) = \bigcup_{h \in X} C^g_0(f, x_0, h)\,.$$

Now we generalize Proposition 1.8 [3].

Proposition 1.1. *Let $X \in E$, $Y \in E$, $X \times Y \in E$, $p \in \Gamma$. Let C be a cone in $X \times Y$ such that $p(C)_X \leqq K$. Then we have*

$$p(V_{p,\varepsilon}(C)) \leqq \frac{K + (1 + K)\,\varepsilon}{1 - (1 - K)\,\varepsilon} \quad \textit{for all} \quad \varepsilon : 0 < \varepsilon < \frac{1}{1 + K}\,.$$

Proof. Let $(x, y) \in V_{p,\varepsilon}(C)$. There exists $(a, b) \in C$ such that $p((x, y) - (a, b)) = p(x - a) + p(y - b) \leqq \varepsilon(p(x) + p(y))$. Further, we have

$$p(y) \leqq p(y - b) + p(b) \leqq p(y - b) + K\,p(a) \leqq p(y - b) + K(p(x - a) + p(x)) \leqq$$
$$\leqq (1 + K)\,(p(x - a) + p(y - b)) + K\,p(x) \leqq (1 + K)\,\varepsilon(p(x) + p(y)) + K\,p(x)\,,$$

hence $[1 - (1 + K)\,\varepsilon]\,p(y) \leqq [K + (1 + K)\,\varepsilon]\,p(x)$ and rewriting it in the form

$$p(y) \leqq \frac{K + (1 + K)\,\varepsilon}{1 - (1 + K)\,\varepsilon}\,p(x)\,,$$

we can see that

$$p(V_{p,\varepsilon}(C)) \leqq \frac{K + (1 + K)\,\varepsilon}{1 - (1 + K)\,\varepsilon}\,.$$

Remark. 1. It is clear that if $X \in E$ and C is a cone in X, then

$$\bigcap_{p \in \Gamma} \bigcap_{\varepsilon > 0} V_{p,\varepsilon}(C) = \bar{C}\,.$$

2. If f is Γ-Gâteaux conable at x_0, then

$$C_0^g(f, x_0, h) = \bigcap_{p\in\Gamma} \bigcap_{r>0} \overline{C_{p,r}^g(f, x_0, h)} \quad \text{for all} \quad h \in X\,, \quad h \neq 0\,.$$

Proof. It is clear that for all $p \in \Gamma$ we obtain

$$C_{p,s}^g(f, x_0, h) \subseteq C_{p,r}^g(f, x_0, h) \quad \text{if} \quad 0 < s < r\,.$$

Definition 1.9 implies that $C_0^g(f, x_0, h) \subset V_{p,\varepsilon}(C_{p,r}^g(f, x_0, h))$ for all $\varepsilon > 0$, $r > 0$. Let $p \in \Gamma$, $q \in \Gamma$. If $p(h) = 0$, then of course we have: $C_{q,r}^g(f, x_0, h) \subseteq C_{p,r}^g(f, x_0, h)$. Hence $C_0^g(f, x_0, h) \subseteq V_{q,\varepsilon}(C_{q,r}^g(f, x_0, h)) \subseteq V_{q,\varepsilon}(C_{p,r}^g(f, x_0, h))$. If $q(h) = 0$, then $q(f(x_0 + th) - f(x_0)) = 0$ for all $t \in R$, so f is q-continuous at x_0. Hence $q(C_{q,r}^g(f, x_0, h))_X = 0$ for all $r > 0$. Since $C_0^g(f, x_0, h) \subseteq V_{q,\varepsilon}(C_{q,r}^g(f, x_0, h))$ for ε, $0 < \varepsilon < 1$ and some $r > 0$ by Proposition 1.1, it follows that $q(C_0^g(f, x_0, h)) \leqq \leqq \varepsilon/(1 - \varepsilon)$. This means that $q(y) = 0$ for all y with $(th, y) \in C_0^g(f, x_0, h)$. Hence $C_0^g(f, x_0, h) \subseteq V_{q,\varepsilon}(C_{p,r}^g(f, x_0, h))$ for all $\varepsilon > 0$ and $r > 0$. If $p(h) > 0$ and $q(h) > 0$, then it is easy to verify that

$$C_{p,r}^g(f, x_0, h) = C_{q,(q(h)/p(h))r}^g(f, x_0, h)\,,$$

which means that

$$C_0^g(f, x_0, h) \subseteq V_{q,\varepsilon}(C_{q,(q(h)/p(h))r}^g(f, x_0, h)) \subseteq V_{q,\varepsilon}(C_{p,r}^g(f, x_0, h))\,.$$

Hence

$$C_0^g(f, x_0, h) \subseteq \bigcap_{p\in\Gamma} \bigcap_{r>0} \bigcap_{q\in\Gamma} \bigcap_{\varepsilon>0} V_{q,\varepsilon}(C_{p,r}^g(f, x_0, h)) \subseteq$$
$$\subseteq \bigcap_{p\in\Gamma} \bigcap_{r>0} \overline{C_{p,r}^g(f, x_0, h)}\,.$$

On the other hand, we obviously have

$$\bigcap_{p\in\Gamma} \bigcap_{r>0} \overline{C_{p,r}^g(f, x_0, h)} \subseteq \bigcap_{p\in\Gamma} \bigcap_{\varepsilon>0} V_{p,\varepsilon}(C_{p,r}^g(f, x_0, h)) = C_0^g(f, x_0, h)\,.$$

2. SOME PROPOSITIONS

Let $X \in E$, $Y \in E$ and let C be a cone in $X \times Y$. Denote $(TC(h))^+ = \{y : (h, y) \in C\}$, $(TC(h))^- = \{y : (-h, -y) \in C\} = -(TC(-h))^+$, $TC(h) = \{y/t : t \neq 0,\ (th, y) \in \in C\} = (TC(h))^+ \, U(TC(h))^-$ for all $h \in X$, $h \neq 0$.

Let A, B be p-bounded subsets of X, $p \in \Gamma$ (i.e. $\sup\{p(x) : x \in A\} < \infty$, $\sup\{p(y) : y \in B\} < \infty\}$. Put

$$d_p(A, B) = \inf\{t > 0,\ A \subseteq B + tS_p,\ B \subseteq A + tS_p\} =$$
$$= \max\{\sup_{x\in A} \inf_{y\in B} p(x - y),\ \sup_{y\in B} \inf_{x\in A} p(y - x)\}\,,$$

where $S_p = \{x \in X : p(x) \leqq 1\}$.

Definition 2.1. Let $X \in E$, $Y \in E$, $p \in \Gamma$ and let $\{C_i\}_{i \in I}$ be a family of cones in $X \times Y$ such that $(0, y) \notin C_i$ for all $y \in Y$, $y \neq 0$ and all $i \in I$. We say that $\{C_i\}_{i \in I}$ *p-uniformly converges to a cone* C_0 if $(0, y) \notin C_0$ for $y \in Y$, $y \neq 0$ and $(T C_i(h))^+ \neq \emptyset$, $(T C_0(h))^+ \neq \emptyset$ for all $h \in X$ and all $i \in I$, and for each $\varepsilon > 0$ there exists $\varkappa \in I$ such that

$$C_i(h) \subseteq V_{p,\varepsilon}(C_0(h)) \quad \text{and} \quad C_0(h) \subseteq V_{p,\varepsilon}(C_0(h)) \quad \text{for ail} \quad i \geqq \varkappa \quad \text{and all } h \in X.$$

Proposition 2.1. *Let $X \in E$, $Y \in E$, $X \times Y \in E$, $p \in \Gamma$. Let C_0 be a cone in $X \times Y$, $\{C_i\}_{i \in I}$ a net of cones in $X \times Y$ and $p(C_i)_X \leqq K$, $p(C_0)_X \leqq K$ for all i. Then $\{C_i\}_{i \in I}$ p-uniformly converges to C_0 if and only if $d_p((T C_i(h))^+, (T C_0(h))^+)$ uniformly converges to 0 on the set $\{h \in X : p(h) \leqq 1\}$. In addition, if for some $\varepsilon > 0$, $\varepsilon < 1$, $i \in I$, $h \in X$ the inclusions*

$$C_i(h) \subseteq V_{p,(\varepsilon/(1+K)^2)}(C_0(h)), \quad C_0(h) \subseteq V_{p,(\varepsilon/(1+K)^2)}(C_i(h))$$

hold, then

$$d_p((T C_i(h))^+, ((T C_0(h))^+) \leqq \varepsilon\, p(h).$$

Proof. 1. Suppose that $\{C_i\}_{i \in I}$ p-uniformly converges to C_0. Let $\varepsilon \in (0, 1)$ be arbitrary. Put $\varepsilon_1 = \varepsilon/(1 + K)^2$. Choose $\varkappa \in I$ such that for all $i \in I$, $i \geqq \varkappa$ implies $C_0(h) \subseteq V_{p,\varepsilon_1}(C_i(h))$ and $C_i(h) \subseteq V_{p,\varepsilon_1}(C_0(h))$. a) If $h \in X$ is such that $p(h) = 0$ then $p(y_i) = p(y) = 0$ for all $y_i \in (T C_i(h))^+$ and $y \in (T C_0(h))^+$. Hence $p(y_i - y) = 0$, which implies that $d_p((T C_i(h))^+, (T C_0(h))^+) = 0$. b) Let $h \in X$ be such that $p(h) > 0$. Let b_i be an arbitrary element of the set $(T C_i(h))^+$. Then there exists $(th, y) \in C_0(h)$, $t \in R$, such that $p((h, b_i) - (th, y)) \leqq \varepsilon_1(p(h) + p(b_i))$, $|1 - t|\, p(h) + p(b_i - y) \leqq \leqq \varepsilon_1(p(h) + K\, p(h))$. Hence $|1 - t| \leqq \varepsilon_1(1 + K) < 1$, which implies that $t > 0$. Then $y/t \in (T C_0(h))^+$ and $p(b_i - y/t) \leqq p(b_i - y) + p(y - y/t) \leqq \varepsilon_1(1 + K\, p(h) + + \varepsilon_1(1 + K) K\, p(h) \leqq \varepsilon_1(1 + K)^2\, p(h) = \varepsilon\, p(h)$. In the same way one can verify that for each $b \in (T C_0(h))^+$ there exists $y_i \in (T C_i(h))^+$ such that $p(b - y_i) \leqq \varepsilon\, p(h)$. Hence $d_p((T C_i(h))^+, (T C_0(h))^+) \leqq \varepsilon\, p(h)$.

2. Suppose that $d_p((T C_i(h))^+, (T C_0(h))^+) \leqq \varepsilon$ for all $h \in X$, $p(h) \leqq 1$. Then $d_p((T C_i(h))^-, (T C_0(h))^-) \leqq \varepsilon$ for all $h \in X$, $p(h) \leqq 1$. a) If $h \in X$ and $p(h) = 0$ then for each $(th, y_i) \in C_i(h)$, $(h, y) \in C_0(h)$ we obtain $p((h, y_i) - (h, y)) = = p(y_i - y) = 0$. Hence $C_i(h) \subseteq V_{p,\varepsilon}(C_0(h))$ and $C_0(h) \subseteq V_{p,\varepsilon}(C_i(h))$ for all $\varepsilon > 0$. b) If $h \in X$ and $p(h) > 0$, then for each $(th, y) \in C_i(h)$, $t \neq 0$, for instance $t > 0$, it follows that $b_i = y/tp(h) \in (TC_i(h/p(h))))^+$ and there exists $b \in (TC_0(h/p(h)))^+$ such that $p(b_i - b) \leqq \varepsilon$. Therefore $p((th, tp(h)\, b_i) - (th, tp(h)\, b) = tp(h)\,. \,. p(b_i - b) \leqq \varepsilon tp(h) \leqq \varepsilon(p(th) + p(y))$. This shows that $(th, y) \in V_{p,\varepsilon}(C_0(h))$. Hence $C_i(h) \subseteq V_{p,\varepsilon}(C_0(h))$. Similarly we have $C_0(h) \subseteq V_{p,\varepsilon}(C_i(h))$. This completes the proof of Proposition 2.1.

Proposition 2.2. *Suppose that $X_i \in E$, $Y_i \in E$, $X_i \times Y_i \in E$, $i = 1, 2$, $p \in \Gamma$. Let $T_1(T_2)$ be a linear p-continuous mapping of $X_1(X_2)$ into $Y_1(Y_2)$ such that there exist positive numbers α, β, a satisfying the inclusions $\alpha\, p(x) \leqq p(T_1 x) \leqq \beta\, p(x)$,*

$p(T_2) \leqq a$. *Then for every cone* C *in* $X_1 \times Y_1$ *such that* $p(C)_{X_1} \leqq b$ *and for each* ε, $0 < \varepsilon < 1/2(b+1)$, *we have* $(T_1 \times T_2)(V_{p,\varepsilon}(C)) \subseteq V_{p,2\alpha^{-1}(\beta+a)(b+1)\varepsilon}(T_1 \times T_2(C))$.

Proof. Let (x, y) be an arbitrary element of $V_{p,\varepsilon}(C)$. Proposition 1.1 implies that

$$p(y) \leqq \frac{(b + (b+1)\varepsilon}{(1-(1+b)\varepsilon)} p(x) .$$

Thus, there exists $(u, v) \in C$ such that $p((x, y) - (u, v)) = p(x - u) + p(y - b) \leqq \leqq \varepsilon(p(x) + p(y))$. Therefore

$$p(T_1 \times T_2(x, y) - T_1 \times T_2(u, v)) \leqq$$

$$\leqq (p(T_1) + p(T_2))(p(x-u) + p(y-b)) \leqq \varepsilon(\beta + a)(p(x) + p(y)) \leqq$$

$$\leqq \varepsilon(\beta + a)\left(1 + \frac{b + (b+1)\varepsilon}{1-(1+b)\varepsilon}\right) p(x) \leqq 2\varepsilon(\beta + a)(1+b)\alpha^{-1} p(T_1 x) \leqq$$

$$\leqq 2\varepsilon(\beta + a)(1+b)\alpha^{-1} p(T_1 \times T_2(x, y)) .$$

Hence $T_1 \times T_2(x, y) \in V_{p,\Delta}(T_1 \times T_2(C))$, where $\Delta = 2\varepsilon(\beta + a)(1+b)\alpha^{-1}$. Therefore $T_1 \times T_2(V_{p,\varepsilon}(C)) \subseteq V_{p,\Delta}(T_1 \times T_2(C))$ and this completes the proof of Proposition 2.2.

Remark. If X is a normed space, $Y \in E$, $\Omega \subseteq X$ and f is a map of Ω into Y, which is Γ-Gâteaux conable at $x_0 \in \Omega$, we write $C_r^g(f, x_0, h)$ instead of $C_{p,r}^g(f, x_0, h)$.

Proposition 2.3. *Let* $X \in E$, $Y \in E$, $X \times Y \in E$, $\Omega \subseteq X$, $p \in \Gamma$ *and let* $f : \Omega \to Y$ *be a* Γ*-Gâteaux conable mapping at* x_0 *and* $p(C_0^g(f, x_0)) \leqq K$. *Then for all* $y' \in \in L_p(Y, R) = Y'_p$, $h \in X$, $h \neq 0$, *the function* $f_{y',h}(t) = \langle y', f(x_0 + th)\rangle$ *is conable at* 0 *and*

$$C_0^g(f_{y',h}, 0) = \overline{(I_h \times y')(C_0^g(f, x_0, h))} ,$$

where $I_h : \{th : t \in R\} \to R$ *is the mapping defined by* $I_h(th) = t$, $t \in R$.

Proof. 1. If $h \in X$ and $p(h) = 0$ then $p(th) = 0$ for all $t \in R$. As f is Γ-continuous at x_0, we have $p(f(x_0 + th) - f(x_0)) = 0$. Moreover, $p(y) \leqq K|t|\, p(h) = 0$ for $(th, y) \in C_0^g(f, x_0, h)$. Hence for all $y' \in Y'_p$,

$$f_{y',h}(t) - f_{y',h}(0) = \langle f(x_0 + th) - f(x_0), y'\rangle = 0 .$$

Then we evidently have

$$C_0^g(f_{y',h}, 0) = \{(t, 0) : t \in R\} = (I_h \times y')(C_0^g(f, x_0, h)) .$$

2. If $s = p(h) > 0$, then it is clear that $C_{r/s}^g(f_{y',h}, 0) = (I_h \times y')(C_{p,r}^g(f, x_0, h))$. Let $\varepsilon > 0$ be arbitrary and put

$$\varepsilon_1 = \frac{\varepsilon s^{-1}}{2(K+1)(p(y') + s^{-1})} = \frac{\varepsilon}{2(K+1)(p(y')s + 1)} .$$

Choose $\delta > 0$ such that $C^g_{p,r}(f, x_0, h) \subseteq V_{p,\varepsilon_1}(C^g_0(f, x_0, h))$ for all $r \in (0, \delta)$. Therefore $C^g_{p,r}(f, x_0, h) \subseteq (V_{p,\varepsilon_1}(C^g_0 f, x_0, h)))(h)$. Then for all $r \in (0, \delta/s)$, we have

$$C^g_r(f_{y',h}, 0) = (I_{h\times}y')((C^g_{p,rs}(f, x_0, h)) \subseteq (I_{h\times}y')((V_{p,\varepsilon_1}(C^g_0(f, x_0, h)))(h)).$$

Using Proposition 2.2 for $\alpha = s^{-1}$, $\beta = s^{-1}$, $a = p(y')$, $b = K$ we have

$$C^g_r(f_{y',h}, 0) \subseteq V_{p,\varepsilon}((I_{h\times}y')(C^g_0(f, x_0, h))) \subseteq V_{p,\varepsilon}\overline{((I_{h\times}y')(C^g_0(f, x_0, h)))}.$$

On the other hand one can see that:

$$(I_{h\times}y')(C^g_0(f, x_0, h)) \subseteq \overline{(I_{h\times}y')(C^g_{p,rs}(f, x_0, h))} =$$
$$= \overline{C^g_r(f_{y',h}, 0)} \subseteq V_\varepsilon(C^g_0(f_{y',h}, 0)) \quad \text{for all} \quad \varepsilon > 0, \quad r > 0.$$

Hence

$$C^g_0(f_{y',h}, 0) = \overline{(I_{h\times}y')(C^g_0(f, x_0, h))},$$

and the proof of Proposition 2.3 is complete.

Let $A \subseteq R$, $B \subseteq R$. We write $A \leqq B$ if for all $a \in A$, $b \in B$ the inequality $a \leqq b$ holds. If C is a cone in $R \times R$, we write TC instead of $TC(1)$.

Proposition 2.4. *Suppose that f is a real continuous function on $(c, d) \supset [a, b]$. Let $\|C^g_0(f, x)\| < +\infty$ for all $x \in [a, b]$. Then there exist points $c_i^+ \in [a, b)$, $c_i^- \in \in (a, b]$, $i = 1, 2$, such that*

$$(TC^g_0(f, c_1^+))^+ \leqq \frac{f(b) - f(a)}{b - a} \leqq (TC^g_0(f, c_2^+))^+,$$

$$(TC^g_0(f, c_1^-))^- \leqq \frac{f(b) - f(a)}{b - a} \leqq (TC^g_0(f, c_2^-))^-.$$

Proof. We can suppose that $f(a) = f(b) = 0$; otherwise we can put

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a)$$

and note that

$$(TC^g_0(f, x))^+ - \frac{f(b) - f(a)}{b - a} = (TC^g_0(g, x))^+$$

and

$$(TC^g_0(f, x))^- - \frac{f(b) - f(a)}{b - a} = (TC^g_0(g, x))^- \quad \text{for all} \quad x \in [a, b].$$

Choose $c_i^+ \in [a, b)$ $c_i^- \in (a, b]$, $i = 1, 2$, such that $f(c_1^+) = \max_{x \in [a,b]} f(x) = f(c_2^-)$, $f(c_1^-) = \min_{x \in [a,b]} f(x) = f(c_2^+)$. Then for all $r > 0$ we have $(TC^g_r(f, c_1^+))^+ \leqq 0$, $(TC^g_r(f, c_2^+))^+ \geqq 0$, $(TC^g_r(f, c_1^-))^- \leqq 0$, $(TC^g_r(f, c_2^-))^- \geqq 0$.

Therefore $(TC_0^g(f, c_1^+))^+ \leqq 0$, $(TC_0^g(f, c_2^+))^+ \geqq 0$, $(TC_0^g(f, c_1^-))^- \leqq 0$, $(TC_0^g(f, c_2^-))^- \geqq 0$.

Corollary 2.1. *Let f be a real continuous function on (a, b) and $\|C_0^g(f, x)\| \leqq K$ for all $x \in (a, b)$. Then*
(i) $|f(s) - f(r)| \leqq K|s - r|$ *for* $s \in (a, b)$, $r \in (a, b)$,
(ii) *f is differentiable almost everywhere on (a, b).*

Proof. Recall that $\|C_0^g(f, x)\| = \sup\{|y| : y \in (TC_0^g(f, x\}$. Then by Proposition 2.4 there exist points c_1, c_2 such that

$$-K \leqq (TC_0^g(f, c_1))^+ \leqq \frac{f(s) - f(r)}{s - r} \leqq TC_0^g(f, c_2) \leqq K\,,$$

which implies that $|f(s) - f(r)| \leqq K|s - r|$. Furthermore, f being Lipschitzian on (a, b) with the constant K, f is differentiable a.e. on (a, b).

Proposition 2.5. *Let $X \in E$, $Y \in E$, $X \times Y \in E$, $\Omega \subseteq X$, $p \in \Gamma$, let $f : \Omega \to Y$ be a Γ-Gâteaux conable mapping on Ω (i.e., f is Γ-Gâteaux conable at every point $x \in \Omega$). Suppose that $p(C_0^g(f, x)) \leqq K$ for all $x \in \Omega$. Then $p(f(x_2) - f(x_1)) \leqq \leqq Kp(x_2 - x_1)$ for $x_1 \in \Omega$, $x_2 \in \Omega$, thus $[x_1, x_2] = \{(1 - t)x_1 + tx_2 : 0 \leqq t \leqq \leqq 1\} \subseteq \Omega$.*

Proof. Put $h = x_2 - x_1$. If $p(h) = 0$ then $p(f(x_2) - f(x_1)) = 0$ as f is p-continuous on Ω. If $p(h) > 0$, put $g_{y'}(t) = \langle f(x_1 + th), y'\rangle$, $t \in (-\delta, 1 + \delta)$, for each $y' \in Y_p'$ and for some $\delta > 0$. By Proposition 2.3, $g_{y'}(t)$ is conable on $(-\delta, 1 + \delta)$ and

$$C_0^g(g_{y'}, t) = \overline{(I_{h\times}y')\,(C_0^g(f, x_t, h))}\,,$$

where $x_t = x_1 + th$. It is clear that $\|C_0^g(g_{y'}, t)\|_R \leqq Kp(y')\,p(h)$. Therefore by Corollary 2.1 we have $|\langle f(x_1 + th) - f(x_1), y'\rangle| \leqq K|t|\,p(y')\,p(h) \leqq Kp(y')\,p(th)$. Eventually, according to the Hahn-Banach Theorem we obtain

$$p(f(x_2) - f(x_1)) = \sup_{\substack{p(y') \leqq 1 \\ y' \in Y_p'}} |\langle f(x_2) - f(x_1), y'\rangle| \leqq Kp(x_2 - x_1)\,.$$

This completes the prof of Proposition 2.5.

3. MAIN THEOREMS

Definition 3.1. Let $X \in E$, $Y \in E$, $\Omega \subseteq X$, and let $C(\cdot)$ be a mapping of Ω into the set of all cones in $X \times Y$ such that $C(x)(h) \neq \emptyset$ for all $x \in \Omega$, $h \in X$ and $p(C(x))_X < < \infty$ for all $p \in \Gamma$, $x \in \Omega$. We say that $C(\cdot)$ is *Γ-continuous at x_0* if for each $\varepsilon > 0$

and $p \in \Gamma$ there exists $\delta(p, \varepsilon) > 0$ such that for all $x \in X$, the relations $p(x - x_0) < \delta$, $h \in X$, $p(h) \leqq 1$, imply

$$C(x)(h) \subseteq V_{p,\varepsilon}(C(x_0)(h)) \quad \text{and} \quad C(x_0)(h) \subseteq V_{p,\varepsilon}(C(x)(h)).$$

Theorem 1. *Let $X \in E$, $Y \in E$, $X \times Y \in E$, $\Omega \subseteq X$. Suppose that $f : \Omega \to Y$ is Γ-Gâteuax conable on Ω and $C_0^g(f, x)$ is Γ-continuous at x_0. Then f is Γ-Frèchet differentiable at x_0 and $df(x_0, h) = (TC_0^g(f, x_0, h))^+$ $(h \in X)$.*

Proof. 1. Suppose that $X = Y = R$, $\Omega = (a, b) \ni x_0$. Let $K = \|C_0^g(f, x_0)\|$ and let $\varepsilon > 0$ be an arbitrary positive number, $\varepsilon < 1/(1 + K)$. Choose $\delta > 0$ such that $|x - x_0| < \delta$ implies $x \in (a, b)$, $C_0^g(f, x) \subseteq V_\varepsilon(C_0^g(f, x_0))$; $C_0^g(f, x_0) \subseteq \subseteq V_\varepsilon(C_0^g(f, x))$. It follows from Proposition 1.1 that $\|C_0^g(f, x)\| \leqq K_1 = = (K + (K + 1)\varepsilon)/(1 - (1 + K)\varepsilon)$. By Corollary 2.1 f is differentiable a.e. on $(x_0 - \delta, x_0 + \delta)$. Take $x_n \in (x_0 - \delta, x_0 + \delta)$, $n = 1, 2, \ldots$, such that f is differentiable at x_n and x_n converges to x_0. Then $C_0^g(f, x_n)$ uniformly converges to $C_0^g(f, x_0)$, so $C_0^g(f, x)$ is Γ-continuous at x_0. By Proposition 2.1, $d((TC_0^g(f, x_n))^\pm, (TC_0^g(f, x_0))^\pm)$ converge to 0. It is clear that $TC_0^g(f, x_n) = f'(x_n)$, $n = 1, 2, \ldots$. Hence for all $y \in \in TC_0^g(f, x_0)$ we have $\lim \|y - f'(x_n)\| = 0$, so that $TC_0^g(f, x_0)$ is a singleton. Let u be the unique point of $TC_0^g(f, x_0)$. By Definition 1.9, $C_r^g(f, x_0)$ converges to $C_0^g(f, x_0)$ if $r \to 0$; then by Proposition 2.1, $d((TC_r^g(f, x_0))^\pm, (TC_0^g(f, x_0))^\pm)$ converge to 0. It is clear that $(f(x_0 + t) - f(x_0))/t \in TC_r^g(f, x_0)$ holds for all t, $0 < |t| < r$. Hence $\lim\limits_{t \to 0} (f(x_0 + t) - f(x_0))/t - u| = 0$, which means that $f'(x_0) = u$.

2. Let $X \in E$, $Y \in E$. We shall prove that $TC_0^g(f, x_0)$ is a singleton. Let $p \in \Gamma$, $K = p(C_0^g(f, x_0))$ and let ε be an arbitrary positive number, $\varepsilon < 1/(1 + K)$. Choose $\delta_0(p, \varepsilon) > 0$ such that for all $x \in X$, $p(x - x_0) < \delta_0$ implies $x \in \Omega$ and $C_0^g(f, x) \subseteq \subseteq \bigcup\limits_{h \in X} V_{p,\varepsilon}(C_0^g(f, x_0, h)) \subseteq V_{p,\varepsilon}(C_0^g(f, x_0))$. Then $p(C_0^g(f, x)_X \leqq K_1 = (K + (K + 1)\varepsilon)/ /(1 - (K + 1)\varepsilon)$. By Proposition 2.5 it follows that $p(C_r^g(f, x_0))_X \leqq K_1$ for all $r < \delta_0$. Let $h \in X$, $h \neq 0$, $p(h) \leqq 1$, $y' \in Y_p'$ and put $g_{y'}(t) = \langle f(x_0 + th), y' \rangle$ for $t \in (-\delta_0, \delta_0)$. It follows from Proposition 2.3 that $g_{y'}$ is conable on $(-\delta_0, +\delta_0)$ and

$$C_0^g(g_{y'}, t) = \overline{(I_h \times y')(C_0^g(f, x_t, h))},$$

where $I_h(th) = t$, $x_t = x_0 + th$. The continuity of $C_0^g(f, x)$ at x_0 implies the continuity of $C_0^g(g_{y'}, t)$ at 0. By the first part of our proof $g_{y'}$ is differentiable at 0 and $TC_0^g(g_{y'}, 0) = y'(TC_0^g(f, x_0, h))$ is a singleton. Hence $y'(TC_0^g(f, x_0, h))$ being singleton for an arbitrary $y' \in \bigcup\limits_{p \in \Gamma} Y_p' = Y'$, $TC_0^g(f, x_0, h)$ is a singleton as well. Let $\varphi(h)$ be the unique element of $TC_0^g(f, x_0, h)$. Then evidently φ is a homogeneous map. Because $C_r^g(f, x_0, h)$ converges to $C_0^g(f, x_0, h)$, it follows from Proposition 2.1 that $d_p((TC_r^g(f, x_0, h))^\pm, (TC_0^g(f, x_0, h))^\pm)$ converge to 0. We have $(h, (f(x_0 + th) - - f(x_0))/t) \in C_r^g(f, x_0, h)$ for all t, $0 < |t| \leqq r$. Hence $\lim p((f(x_0 + th) - f(x_0))/t -$

$- \varphi(h)) = 0$, which means that f is Gâteaux differentiable at x_0 [5] and $Vf(x_0, h) = \varphi(h)$.

3. Now we prove that $\varphi(h)$ is additive and f is Γ-Frèchet differentiable at x_0.

Let $h_1, h_2 \in X$, $p \in \Gamma$ and let ε be an arbitrary positive number $\varepsilon < 1$. As f is Gâteaux differentiable at x_0, there exists $\delta_1(\varepsilon, p) > 0$ such that for all t, $0 < t < \delta_1$, we have $p(th_i) < \delta_1$ for $i = 1, 2$ and

$$\varphi(h_1) = \frac{1}{t}\left[f(x_0 + th_1) - f(x_0)\right] + \alpha_1 ,$$

$$\varphi(h_2) = \frac{1}{t}\left[f(x_0 + th_2) - f(x_0)\right] + \alpha_2 ,$$

$$\varphi(h_1 + h_2) = \frac{1}{t}\left[f(x_0 + th_1 + th_2) - f(x_0)\right] + \alpha_3 ,$$

where $p(\alpha_i) < \frac{1}{4}\varepsilon$ for $i = 1, 2, 3$. Then

$$\begin{aligned} p(\varphi(h_1 + h_2) - \varphi(h_1) - \varphi(h_2)) &\leqq \frac{1}{|t|} p[f(x_0 + th_1 + th_2) - \\ &- f(x_0 + th_1) - f(x_0 + th_2) + f(x_0)] + \tfrac{3}{4}\varepsilon \leqq \\ &\leqq \frac{1}{|t|} p(f(x_0 + th_1 + th_2) - f(x_0 + th_2) - \varphi(th_1)) + \\ &+ \frac{1}{|t|} p(f(x_0 + th_1) - f(x_0) - \varphi(th_1)) + \tfrac{3}{4}\varepsilon . \end{aligned}$$

Choose $0 < \delta_2 < \delta_1$ such that for all $x \in X$, $p(x - x_0) < \delta_2$ implies $C_0^g(f, x, h) \subseteq \subseteq V_{p,\Delta}(C_0^g(f, x_0, h))$ and $C_0^g(f, x_0, h) \subseteq V_{p,\Delta}(C_0^g(f, x, h))$ for all $h \in X$, $h \neq 0$, where $\Delta = \varepsilon/(8(1 + K)^2 (1 + p(h_1) + p(h_2)))$. Proposition 2.1 implies that $d_p(TC_0^g(f, x, h))^+$, $(TC_0^g(f, x_0, h))^+) \leqq \varepsilon\, p(h)/(8(1 + p(h_1) + p(h_2)))$. For $x \in X$, $p(x - x_0) < \frac{1}{3}\delta_2$, $h \in X$, $p(h) < \frac{1}{3}\delta_2$, $y' \in Y_p'$, $p(y') \leqq 1$, put $g_{y'}(s) = \langle f(x - sh) - \varphi(sh), y' \rangle$ for $s \in (-\frac{1}{3}\delta_2, 1 + \frac{1}{3}\delta_2)$. It is easy to verify that

$$TC_0^g(g_{y'}, s) = \overline{y'(TC_0^g(f, x_s, h) - \varphi(h))} ,$$

where $x_s = x + sh$. Therefore

$$\begin{aligned} p(C_0^g(g_{y'}, s)) &= \sup \{|t| : t \in TC_0^g(g_{y'}, s)\} = \\ &= d(y'(TC_0^g(f, x_s, h)), y'(\varphi(h))) \leqq \frac{\varepsilon\, p(h)\, p(y')}{8(1 + p(h_1) + p(h_2))} . \end{aligned}$$

Hence

$$(*)\qquad p(f(x+h)-f(x)-\varphi(h)) = \sup\{|\langle f(x+h)-f(x)-\varphi(h), y'\rangle| : $$
$$: p(y') \leqq 1\} \leqq \frac{\varepsilon\, p(h)}{8(1+p(h_1)+p(h_2))}.$$

Choose $0 < \delta_3 < \delta_2$ such that $0 < |t| < \delta_3$ implies $p(th_1) < \frac{1}{5}\delta_2$, $p(th_2) < \frac{1}{5}\delta_2$. Then for all t, $0 < |t| < \delta_3$, we have

$$p(f(x_0 + th_1 + th_2) - f(x_0 + th_2) - \varphi(th_1)) \leqq$$
$$\leqq \frac{\varepsilon\, p(th_1)}{8(1+p(h_1)+p(h_2))} \leqq \frac{\varepsilon|t|}{8}$$

and

$$p(f(x_0 + th_1) - f(x_0) - \varphi(th_1)) \leqq \frac{\varepsilon\, p(th_1)}{8(1+p(h_1)+p(h_2))} \leqq \frac{|\varepsilon t|}{8}.$$

Hence $p(\varphi(h_1+h_2) - \varphi(h_1) - \varphi(h_2)) < \varepsilon$ for all $\varepsilon > 0$, $p \in \Gamma$. This means that $\varphi(h_1+h_2) = \varphi(h_1) + \varphi(h_2)$, hence $\varphi \in L_\Gamma(X, Y)$. In $(*)$ put $x = x_0$, which shows that f is Γ-Frèchet differentiable at x_0 and $df(x_0, h) = \varphi(h)$. This completes the proof of Theorem 1.

Definition 3.2. Let $X \in E$, $Y \in E$, $X \times Y \in E$, $\Omega \subseteq X$ and let $f : \Omega \to Y$ be Γ-Gâteaux conable on Ω, $p(C_0^g(f, x_0)) < \infty$ for all $p \in \Gamma$. We say that f *is uniformly conable at* x_0 if for each $\varepsilon > 0$, $p \in \Gamma$ there exist $\delta(p, \varepsilon) > 0$, $\eta(\varepsilon, p) > 0$ such that for all $r \in (0, \eta)$ and all $x \in X$, $p(x - x_0) < \delta$, the inclusion $V_{p,\varepsilon}(C_0^g(f, x, h)) \supseteq$ $\supseteq C_{p,r}^g(f, x, h)$ holds for all $h \in X$, $h \neq 0$.

Theorem 2. *Let $X \in E$, $Y \in E$, $X \times Y \in E$, $\Omega \subseteq X$, $x_0 \in \Omega$ and let $f : \Omega \to Y$ be Γ-Gâteaux conable on Ω. Then $C_0^g(f, x)$ is Γ-continuous at x_0 if and only if f is uniformly conable at x_0 and for each $p \in \Gamma$ there exist constants $\alpha_p > 0$ and $K_p > 0$ such that $p((C_0^g(f, x))_X) \leqq K_p$ for all $x : p(x - x_0) \leqq \alpha_p$.*

Proof of necessity. Let $M = p((C_0^g(f, x_0))_X)$ and let $\varepsilon \in (0, 1/(1+M))$ be arbitrary. As $C_0^g(f, x)$ is Γ-continuous at x_0, there exists $\alpha_p > 0$ such that for all x, $p(x - x_0) <$ $< \alpha_p$ implies $C_0^g(f, x, h) \subseteq V_{p,\varepsilon}(C_0^g(f, x_0, h))$. By Proposition 1.1, $p(C_0^g(f, x, h)) \leqq$ $\leqq K_p = (M + (1+M)\varepsilon)/(1 - (1+M)\varepsilon)$ for all $h \in X$. Hence $p(C_0^g(f, x))_X \leqq K_p$. Put $\varepsilon_1 = \varepsilon/2(1+K)^2$. Choose $\delta_1(p, \varepsilon) > 0$ such that $\delta_1(p, \varepsilon) < \alpha_p$ and that $p(x - x_0) < \delta_1$ implies $C_0^g(f, x, h) \subseteq V_{p,\varepsilon_1}(C_0^g(f, x_0, h))$ and $C_0^g(f, x_0, h) \subseteq$ $\subseteq V_{p,\varepsilon_1}(C_0^g(f, x, h))$ for all $h \in X$. By Proposition 2.1 we have $d_p((TC_0^g(f, x_0, h))^\pm$, $(TC_0^g(f, x, h))^\pm) \leqq \frac{1}{2}\varepsilon\, p(h)$. Put $\eta = \delta = \frac{1}{2}\delta_1$. Let $h \in X$, $t \in R$, $t \neq 0$, $p(th) \leqq r$ and $x \in X$, $p(x - x_0) < \delta$; then $(th, f(x + th) - f(x)) \in C_{p,r}^g(f, x, h)$. By Theorem 1, f is Γ-Frèchet differentiable at x_0 and $df(x_0, h) = TC_0^g(f, x_0, h)$. Hence

$p(y - df(x_0, h)) \leqq \frac{1}{2}\varepsilon\, p(h)$ for all y, $(th, ty) \in C_0^g(f, x, h)$. Put $g_{y'}(s) = \langle f(x + sh) - df(x_0, sh), y' \rangle$ for $y' \in Y_p'$, $p(y') \leqq 1$. Then

$$TC_0^g(g_{y'}, s) = \overline{y'(TC_0^g(f, x_s, h) - df(x_0, h))}\,,$$

where $x_s = x + sh$, $s \in [0, t]$.

$$\|C_0^g(g_{y'}, s)\| = \sup \{|t| : t \in TC_0^g(g_{y'}, s)\} =$$

$$= d(y'(TC_0^g(f, x_s, h)), y'(df(x_0, h)) \leqq \frac{\varepsilon}{2} p(y')\, p(h)\,.$$

Hence $p(f(x + th) - f(x) - df(x_0, h)) = \sup\limits_{p(y') \leqq 1} |g_{y'}(t) - g_{y'}(0)| \leqq (\varepsilon/2)\, p(th)$. Hence $p((th, f(x + th) - f(x)) - (th, ty)) \leqq p(f(x + th) - f(x) - df(x_0, th)) + p(df(x_0, th) - ty) \leqq \varepsilon\, p(th)$. This means that $C_{p,r}^g(f, x, h) \subseteq V_{p,\varepsilon} C_0^g(f, x, h)$, which proves that f is uniformly conable at x_0.

Proof of sufficiency. First of all we prove the following two lemmas.

Lemma 1. *Let f be a real continuous function on (a, b) and let $\|C_0^g(f, x)\| \leqq K$ for all $x \in (a, b)$. Suppose that there exist positive numbers $\alpha > 0$, $\delta > 0$ such that for all $x \in (a, b)$ and for all $r \in (0, \delta)$ we have*

$$d((TC_r^g(f, x))^+, (TC_0^g(f, x))^+) \leqq \alpha\,, \quad d((TC_0^g(f, x))^-, (TC_r^g(f, x))^-) \leqq \alpha\,.$$

Then

$$\delta(TC_0^g(f, x)) = \max \{|a_1 - a_2| : a_1, a_2 \in TC_0^g(f, x)\} \leqq \alpha\,.$$

Proof of Lemma 1. Suppose that it is false. Then there exists $x_0 \in (a, b)$ such that $\delta(TC_0^g(f, x_0)) > \alpha$. Let

$$a_1 = \max \{a : a \in TC_0^g(f, x_0)\}\,,$$

$$a_2 = \min \{a : a \in TC_0^g(f, x_0)\}\,.$$

Choose a_1', a_2' such that $a_2 < a_2' < a_1' < a_1$ and $a_1' - a_2' > \alpha$. We know that

$$C_0^g(f, x_0) \subseteq \overline{C_r^g(f, x_0)} \quad \text{for all} \quad r > 0\,.$$

Therefore

$$TC_0^g(f, x_0) \subseteq \overline{TC_r^g(f, x_0)} \quad \text{for all} \quad r > 0\,.$$

Hence for each $n \in N$ there exists $x_n \in (a, b)$ such that $0 < |x_n - x_0| < 1/n$ and $(f(x_n) - f(x_0))/(x_n - x_0) > a_1'$. Let $r \in (0, \delta)$ be fixed. Take $x' \in (a, b)$ such that $0 < |x' - x_0| < r < \delta$ and $(f(x') - f(x_0))/(x' - x_0) < a_2'$. Suppose that $x' > x_0$. Then by Proposition 2.4 there exist points c_n such that $|c_n - x_0| < 1/n$ and $(TC_0^g(f, c_n))^+ \geqq (f(x_n) - f(x_0))/(x_n - x_0) > a_1'$. It is clear that

$$\lim_n \frac{f(x') - f(c_n)}{x' - c_n} = \frac{f(x') - f(x_0)}{x' - x_0} < a_2'\,.$$

Therefore there exists $n_0 \in N$ such that $|x' - c_n| < r$, $c_n < x'$ and $(f(x') - f(c_n)) : (x' - c_n) < a_2'$ for all $n \geqq n_0$. Then $(f(x') - f(c_{n0}))/(x' - c_{n0}) \in (TC_r^g(f, c_{n0}))^+$ and of course $d((TC_r^g(f, c_{n0}))^+, (TC_0^g(f, c_{n0}))^+) > a_1' - a_2' > \alpha$, a contradiction. If $x' < x_0$ we choose c_n such that $(TC_0^g(f, c_n))^- > a_1'$ and in the same way as above we show that there exists a point c_{n_0} such that $d((TC_0^g(f, c_{n0}))^-, (TC_r^g(f, c_{n0}))^-) > \alpha$, a contradiction again. This completes the proof of Lemma 1.

Lemma 2. *Let $X \in E$, $Y \in E$, $X \times Y \in E$, $\Omega \subseteq X$, and let $f : \Omega \to Y$ be a Γ-Gâteaux conable map on Ω which is uniformly conable at x_0, $p \in \Gamma$, $p(C_0^g(f, x)) \leqq K$ for all $x \in \Omega$. Then for all $\varepsilon > 0$ there exists $\delta(p, \varepsilon) > 0$ such that $\delta_p(TC_0^g(f, x, h)) = \sup\{p(y - z) : y,\ z \in TC_0^g(f, x, h)\} \leqq \varepsilon$ for all x with $p(x - x_0) < \delta$ and all $h \in X$, $p(h) \leqq 1$.*

Proof of Lemma 2. Take $s > 0$ such that $\{x \in X : p(x - x_0) < 2s\} \subseteq \Omega$. Put $\Omega_0 = \{x \in X : p(x - x_0) < s\}$. Proposition 2.5 implies $p(f(x + h) - f(x)) \leqq K\, p(h)$ for all $x \in \Omega_0$, $p(h) \leqq r < s$. It is clear that $p(C_{p,r}^g(f, x, h)) \leqq K$ for all $x \in \Omega_0$, $r < s$, $h \in X$. If $p(h) = 0$ then according to our assumption $p(y) \leqq K\, p(h)$ for all $y \in TC_0^g(f, x, h)$. Hence $p(y) = 0$ and then $\delta_p(TC_0^g(f, x, h)) = 0$. Now let $p(h_0) = 1$. Let $\varepsilon > 0$ be arbitrary. Put $\varepsilon_1 = \varepsilon/(4(K + 1)^3)$. Choose δ_1, $\eta > 0$ such that $p(x - x_0) < \delta_1 < s$, $0 < r < \eta < s$ imply $C_{p,r}^g(f, x, h) \subseteq V_{p,\varepsilon_1}(C_0^g(f, x, h))$ for all $h \in X$. Let x be an arbitrary point such that $p(x - x_0) < \delta = \min\{\frac{1}{3}\delta, \frac{1}{3}\eta\}$. Put $g_{y'}(t) = \langle f(x + th_0), y'\rangle$ $t \in (-2\delta, 2\delta)$ for all $y' \in Y_p'$. By Proposition 2.3, $g_{y'}$ is conable and

$$C_0^g(g_{y'}, t) = \overline{(I_h \times y')\,(C_0^g(f, x_t, h_0))}\,,$$

where $x_t = x + th$ and we have $\|C_0^g(g_{y'}, t)\| \leqq K\, p(y')$. It is easy to see that

$$\begin{aligned} C_r^g(g_{y'}, t) &= (I_{h_0} \times y')\,(C_{p,r}^g(f, x_t, h_0)) \subseteq (I_h \times y')\,(V_{p,\varepsilon_1}(C_0^g(f, x_t, h_0))) \subseteq \\ &\subseteq V_{p,2\varepsilon_1}(K + 1)\,(p(y') + 1)\,((I_{h_0} \times y')\,(C_0^g(f, x_t, h_0))) \subseteq \\ &\subseteq V_{p,(\varepsilon/(K+1)^2}(C_0^g(g_{y'}, t))\,, \quad \text{for all} \quad y' : p(y') \leqq 1\,. \end{aligned}$$

Now Proposition 2.1 implies

$$d((TC_r^g(g_{y'}, t))^+, (TC_0^g\{(g_{y'}, t))^+) \leqq \varepsilon\,, \quad d((TC_r^g(g_{y'}, t))^-, (TC_0^g(g_{y'}, t))^-) \leqq \varepsilon$$

and, by Lemma 1, it follows that $\delta(TC_0^g(g_{y'}, 0)) \leqq \varepsilon$ for all $y' \in Y_p'$, $p(y') \leqq 1$. For all $y, z \in TC_0^g(f, x, h_0)$ we have

$$p(y - z) = \sup_{p(y') \leqq 1} |\langle y - z, y'\rangle| \leqq \sup_{p(y') \leqq 1} \delta(TC_0^g(g_{y'}, 0)) \leqq \varepsilon\,.$$

Hence $\delta_p(TC_0^g(f, x, h)) \leqq \varepsilon$ for all x, $p(x - x_0) < \delta$ and all $h \in X$, $p(h) = 1$. If $h \in X$, $0 < p(h) < 1$, then $TC_0^g(f, x, h) = p(h)\, TC_0^g(f, x, h')$, where $h' = (1/p(h))\, h$. Hence $\delta_p(TC_0^g(f, x, h)) \leqq \varepsilon\, p(h) < \varepsilon$ again. This completes the proof of Lemma 2.

Now we return to the proof of sufficiency of Theorem 2. Let $\varepsilon > 0$ be arbitrary and let $p \in \Gamma$, $\Omega_0 = \{x : p(x - x_0) < \alpha_p\} \subset \Omega$. Then $p(C_0^g(f, x)) \leqq K_p$ for all $x \in \Omega_0$. Choose $\delta_1, \eta > 0$ such that $\max\{\delta_1, \eta\} \leqq \frac{1}{2}\alpha_p$ and that $p(x - x_0) < \delta_1$ and $0 < r < \eta$ imply $C_{p,r}^g(f, x, h) \subseteq V_{p,(\varepsilon/5(K_p+1)^2)}(C_0^g(f, x, h))$. Choose $\delta_2 > 0$ such that $p(x - x_0) < \delta_2$ implies $\delta_p(TC_0^g(f, x, h)) \leqq \varepsilon/5$ for all $h \in X$, $p(h) \leqq 1$. Take t_0 such that $0 < |t_0| = r < \eta$ and put $\delta = \min\{\delta_1, \delta_2; (\varepsilon/5K_p)|t_0|\}$. Then for $x_1, x_2 \in \Omega_0$, $p(x_1 - x_2) < \delta$ we have $p(f(x_1) - f(x_2)) \leqq K_p p(x_1 - x_2) \leqq (\varepsilon/5)|t_0|$. By Lemma 2, $\delta_p(TC_0^g(f, x_0, h)) \leqq \varepsilon$ for all $\varepsilon > 0$, $h \in X$, $p(h) \leqq 1$. Hence $\delta_p(TC_0^g(f, x_0, h)) = 0$ for all $p \in \Gamma$, which means that $TC_0^g(f, x_0, h)$ is a singleton for all $h \in X$. Put $\varphi(h) = TC_0^g(f, x_0, h)$. Then φ is homogeneous. As $(f(x + t_0h) - f(x)) : t_0 \in TC_{p,r}^g(f, x, h)$ for x, $p(x - x_0) < \delta$, and $h \in X$, $p(h) \leqq 1$, there exists $b \in TC_0^g(f, x, h)$ such that $(t_0h, t_0b) \in C_0^g(f, x, h)$ and $p((f(x + t_0h) - f(x))/t_0 - b) \leqq (\varepsilon/5)\,p(h)$; in particular, $p((f(x_0 + t_0h) - f(x_0))/t_0 - \varphi(h)) \leqq (\varepsilon/5)\,p(h)$. Therefore, for all $b' \in TC_0^g(f, x, h)$, $p(x - x_0) < \delta$ we have $p(\varphi(h) - b') \leqq p(\varphi(h) - (f(x_0 + t_0h) - f(x_0))/t_0 + p((f(x_0 + t_0h) - f(x_0))/t_0 - (f(x + t_0h) - f(x))/t_0) + p((f(x + t_0h) - f(x))/t_0 - b) + p(b - b') \leqq \varepsilon$. Then $d_p(TC_0^g(f, x, h), \varphi(h)) \leqq \varepsilon$ for all h, $p(h) \leqq 1$. Hence $C_0^g(f, x, h) \subseteq V_{p,\varepsilon}(C_0^g(f, x_0, h))$ and $C_0^g(f, x_0, h) \subseteq V_{p,\varepsilon}(C_0^g(f, x, h))$ for all $h \in X$ and all x, $p(x - x_0) < \delta$. This shows that $C_0^g(f, x)$ is Γ-continuous at x_0 and the proof of Theorem 2 is complete.

Definition 3.3. *Let $X \in E$, $Y \in E$, $X \times Y \in E$, $\Omega \subseteq X$ and, for each $x \in \Omega$, let $\{C_n(x)\}_n$ be a sequence of cones in $X \times Y$ such that $C_n(x)(h) \neq \emptyset$ for all n, all $x \in \Omega$ and all $h \in X$. We say that $\{C_n(x)\}_n$ uniformly converges to $C_0(x)$ on Ω if for each $\varepsilon > 0$ and each $p \in \Gamma$, there exists $n_0(\varepsilon, p)$ such that for all $n \in N$, $n \geqq n_0(\varepsilon, p)$, for all $x \in \Omega$ and for all $h \in X$ the following inclusions hold:*

$$(C_n(x))(h) \subseteq V_{p,\varepsilon}(C_0(x)(h)) \quad \textit{and} \quad C_0(x)(h) \subseteq V_{p,\varepsilon}(C_n(x)(h)) .$$

Theorem 3. *Let $X \in E$, $Y \in E$ and $X \times Y \in E$. Assume that Y is sequentially complete. Let Ω be a convex subset of X and $f_n : \Omega \to Y$, $n \in N$, a family of Γ-Gâteaux conable mappings such that for each $p \in \Gamma$ there exists a constant $K_p > 0$ with $p(C_0^g(f_n, x)) \leqq K_p$ for all $n \in N$, $x \in \Omega$. Let $C_0^g(f_n, x_n)$ uniformly converge to $C_0(x)$ on Ω and suppose that there exists a point $x_0 \in \Omega$ such that $\{f_n(x_0)\}$ converges. Then there exists a Γ-Gâteaux conable mapping $f : \Omega \to Y$ such that $\{f_n\}$ converges to f and $C_0^g(f, x) = C_0(x)$.*

Proof. Put

$$\varepsilon(p, m, n) = \sup\{d_p(TC_0^g(f_n, x, h)\,,\quad TC_0^g(f_m, x, h) : x \in \Omega, h\; p(h) \leqq 1\}$$

for $p \in \Gamma$, $n, m \in N$. We claim that $\varepsilon(p, m, n)$ converges to 0 when $m, n \to \infty$. Let $\varepsilon \in (0, 1)$. Put $\varepsilon_1 = \varepsilon/(2(K_p + 1)^2$. Choose $n_0 \in N$ such that $C_0^g(f_n, x, h) \subseteq V_{p,\varepsilon_1}(C_0(x, h))$ and $C_0(x, h) \subseteq V_{p,\varepsilon_1}(C_0^g(f_n, x, h))$ for all $n \geqq n_0$, $x \in \Omega$, and all $h \in X$. Proposition 2.1 implies $d_p(TC_0^g(f_n, x, h), TC_0(x, h)) \leqq \frac{1}{2}\varepsilon$ for all $x \in \Omega$,

$h \in X$, $p(h) \leqq 1$, whence $d_p(TC_0^g(f_n, x, h), TC_0^g(f_m, x, h)) \leqq \varepsilon$ for all $x \in \Omega$, $h \in X$, $p(h) \leqq 1$, $n, m \in N$, $n \geqq n_0$, $m \geqq n_0$. Hence $\varepsilon(p, m, n) \leqq \varepsilon$. This proves our claim.

a) Let $p \in \Gamma$, $x \in \Omega$. Put $h = x - x_0$. If $p(h) = 0$, then $p(f_n(x) - f_n(x_0)) = 0$ for all $n \in N$. Therefore

$$p(f_n(x) - f_m(x)) \leqq p(f_n(x) - f_n(x_0)) + p(f_n(x_0) - f_m(x_0)) + \\ + p(f_m(x_0) - f_m(x)) \leqq p(f_n(x_0) - f_m(x_0)) .$$

If $p(h) > 0$, put $h_1 = (1/p(h))\, h$, then $p(h_1) = 1$. Put $g_{n,y'}(t) = \langle f_n(x_0 + th_1), y' \rangle$ for each $y' \in Y_p'$, $p(y') \leqq 1$. Proposition 2.3 implies

$$C_0^g(g_{n,y'}, t) = \overline{(I_{h \times} y')\,(C_0^g(f_n, x_t, h_1))}\,, \quad \text{where} \quad x_t = x_0 + th_1 .$$

Hence $\|C_0^g(g_{n,y'}, t)\| \leqq p(C_0^g(f_n, x_t, h_1))\, p(y') \leqq K_p$. By Corollary 2.1, $g_{n,y'}$ is a Lipschitzian function and

$$|g'_{n,y'}(t) - g'_{m,y'}(t)| = d(TC_0^g(g_{n,y'}, t)\,, \quad TC_0^g(g_{m,y'}, t)) \leqq \\ \leqq p(y')\, d_p(TC_0^g(f_n, x_t, h_1),\ TC_0^g(f_m, x_t, h_1)) \leqq \varepsilon(p, m, n)$$

for almost all $t \in [0, p(h)]$. Then

$$|\langle f_n(x) - f_n(x_0) - f_m(x) - f_m(x_0), y' \rangle| = \\ = |g_{n,y'}(p(h)) - g_{n,y'}(0) - g_{m,y'}(p(h)) + g_{m,y'}(0)| \leqq \\ \leqq \int_0^{p(h)} |g'_{m,y'}(t) - g'_{n,y'}(t)|\, \mathrm{d}t \leqq \varepsilon(p, m, n)\, p(h) .$$

Therefore $p(f_n(x) - f_n(x_0) - f_m(x) + f_m(x_0)) \leqq \varepsilon(p, m, n)\, p(h)\ p(f_n(x) - f_m(x)) \leqq \leqq p(f_n(x_0) - f_m(x_0)) + \varepsilon(p, m, n)\, p(x - x_0)$. Hence $\{f_n(x)\}$ is a Cauchy sequence for each $x \in \Omega$. As Y is sequentially complete, there exists $f(x) = \lim f_n(x)$ for all $x \in \Omega$.

b) Let $\varepsilon > 0$ be arbitrary. Choose $n_0 \in N$ such that $d_p(TC_0^g(f_n, x, h), TC_0(x, h)) \leqq \leqq (\frac{1}{6}\varepsilon)\, p(h)$ for all $n \geqq n_0$, for all $x \in \Omega$ and all $h \in X$. Then $d_p(TC_0^g(f_n, x, h), TC_0^g(f_m, x, h)) \leqq \frac{1}{3}\varepsilon\, p(h)$ for $n, m \in N$, $n, m \geqq n_0$, $x \in \Omega$ and $h \in X$. It is easy to see that $p(f(x) - f(x_0) - f_{n_0}(x) + f_{n_0}(x_0)) \leqq \frac{1}{3}\varepsilon\, p(x - x_0)$ for all $x \in \Omega$, $x_0 \in \Omega$. Let $x_1 \in \Omega$, $h \in X$, $h \neq 0$ be arbitrary fixed points. Take $r > 0$ such that $d_p((TC_{p,r}^g(f_{n0}, x_1, h))^+, (TC_0^g(f_{n0}, x_1, h))^+) \leqq \frac{1}{3}\varepsilon\, p(h)$, $d_p((TC_{p,r}^g(f_{n0}, x_1, h))^-, (TC_0^g(f_{n0}, x_1, h))^-) \leqq \leqq \frac{1}{3}\varepsilon\, p(h)$. Then for all $t \neq 0$, $p(th) \leqq r$,

$$\frac{1}{t}\,[f_{n0}(x_1 + th) - f_{n0}(x)] \in TC_{p,r}^g(f_{n0}, x_1, h) .$$

There exist $y_{n_0} \in TC_0^g(f_{n0}, x_1, h)$, $y \in TC_0(x_1, h)$ such that

$$(th, ty_{n_0}) \in C_0^g(f_{n0}, x_1, h)\,, \quad (th, ty) \in C_0(x_1, h) \quad \text{and} \\ p(f_{n0}(x_1 + th) - f_{n0}(x_1) - ty_{n_0}) \leqq \tfrac{1}{3}\varepsilon\, p(h)\,,$$

$$p(ty_{n_0} - ty) \leqq \tfrac{1}{3}\varepsilon\, p(h).$$

Hence $p(f_{n_0}(x_1 + th) - f_{n_0}(x_1) - ty) \leqq \tfrac{2}{3}\varepsilon\, p(ht)$

$$\begin{aligned} p((th, f(x_1 + th) - f(x_1)) - (th, ty)) &= p(f(x_1 + th) - f(x_1) - ty) \leqq \\ &\leqq p(f(x_1 + th) - f(x_1) - f_{n_0}(x_1 + th) + f_{n_0}(x_1)) + \tfrac{2}{3}\varepsilon\, p(h) \leqq \\ &\leqq \tfrac{1}{3}\varepsilon\, p(th) + \tfrac{2}{3}\varepsilon\, p(th) \leqq \varepsilon[p(th) + p(f(x_1 + th) - f(x_1))]. \end{aligned}$$

Hence $C^g_{p,r}(f, x_1, h) \subseteq V_{p,\varepsilon}(C_0(x_1, h))$. The proof of $C_0(x_1, h) \subseteq V_{p,\varepsilon}(C^g_{p,r}(f, x_1, h))$ is similar. This completes the proof.

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