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ON THE CROSSING NUMBERS OF $S_m \times P_n$ AND $S_m \times C_n$

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Let G be a *graph* (in the sense of Harary [4]) with V and E the sets of *vertices* and *edges*, respectively. A *drawing* is a mapping of a graph into a surface. The vertices go into distinct points, *nodes*. An edge and its incident vertices map into a homeomorphic image of the closed interval $[0, 1]$ with the relevant nodes as endpoints and the interior, an *arc*, containing no node. A *good drawing* is one in which no two arcs incident with a common node have a common point; and no two arcs have more than one point in common. A common point of two arcs is a *crossing*. An optimal drawing in a given surface is such that exhibits the least possible number of crossing. Optimal drawings are good. This least possible number is the *crossing number* of the graph for the surface. We denote the crossing number of G for the plane by $\nu(G)$.

For a detailed account of problems and results concerning this topic, the reader is referred to Erdös and Guy [1], Guy [2, 3], Harary [4] or Koman [6]. There are few homeomorphism classes of nonplanar graphs with $\nu(G)$ determined for every member G. Only two families of graphs with arbitrarily large crossing numbers for the plane are known. Kleitman [5] determined $\nu(K_{m,n})$ of complete bipartite graphs for min $\{m, n\} \leqq 6$. Ringeisen and Beineke [7] proved that the crossing number of the *Cartesian product* $C_3 \times C_n$ of *cycles* C_3 and C_n is n. (For a definition of Cartesian product see [4].)

Let S_m be the *star* $K_{1,m}$ and P_n the *path* of a length n. The purpose of this article is to find the crossing numbers of the Cartesian products $S_m \times P_n$ and $S_m \times C_n$ for $m = 3$. In the case $m \geqq 3$ we obtain an upper bound for $\nu(S_m \times P_n)$ and for $\nu(S_m \times C_n)$.

Let the vertex of degree m of S_m be denoted by label 0 and the other vertices of S_m (having degree 1) by labels $1, 2, \ldots, m$. Let the vertices of the path P_n be labelled successively by $0, 1, \ldots, n$ so that the end vertices have labels 0 and n, respectively; the vertex i is adjacent to the vertices $i - 1$ and $i + 1$ for all i, $i = 1, 2, \ldots, n - 1$. The vertices of the cycle C_n are analogously denoted by $0, 1, \ldots, n$. The Cartesian product $S_m \times P_n$ has $(m + 1)(n + 1)$ vertices (i, j) for $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$. In $S_m \times P_n$ there are adjacent pairs of vertices $(0, j)$ and (i, j) for

$i = 1, 2, \ldots, m, j = 0, 1, \ldots, n$; (i, j) and $(i, j + 1)$ for $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n - 1$. In $S_m \times C_n$ containing $n(m + 1)$ vertices (i, j) for $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, n - 1$, there are adjacent pairs of vertices $(0, j)$ and (i, j) for $i = 1, 2, \ldots, m$; $j = 0, 1, \ldots, n - 1$ and the pairs (i, j), $(i, j + 1)$ for $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, n - 1$. (The second coordinates are taken modulo n.)

Theorem 1. *If* $m \geqq 1$, $n \geqq 1$, *then*

$$v(S_m \times P_n) \leqq (n - 1)\left[\frac{m}{2}\right]\left[\frac{m - 1}{2}\right].$$

Proof. We shall construct a good drawing D of $S_m \times P_n$ with the number of crossings equal to the upper bound. Place the node (i, j) of $S_m \times P_n$ for $i = 1, 2, \ldots$ $\ldots, m, j = 0, 1, \ldots, n$ to the point in the plane with coordinates $((-1)^i\, i, j)$; the node $(0, 0)$ to the point $(0, -1)$ and the node $(0, j)$ for $j = 1, 2, \ldots, n$ to the point $(0, j + 1)$. Join the corresponding nodes by line segments. In the obtained drawing D the segment $(0, j + 1)((-1)^i\, i, j)$ for every $j = 1, 2, \ldots, n - 1$ contains $[(i - 1)/2]$ crossings. On the segments incident with the node $(0, j + 1)$ for $j = 1, 2, \ldots, n - 1$ there are

$$\sum_{i=1}^{m}\left[\frac{i - 1}{2}\right] = \left[\frac{m}{2}\right]\left[\frac{m - 1}{2}\right]$$

crossings. This immediately yields the upper bound in Theorem 1.

Theorem 2. *If* $m \geqq 1$, $n \geqq 3$, *then*

$$v(S_m \times C_n) \leqq n\left[\frac{m}{2}\right]\left[\frac{m - 1}{2}\right].$$

Proof. If we join $(i, 0)$ to $(i, n - 1)$ for $i = 0, 1, \ldots, m$ by a suitable arc of the circuit with the centre $(2^{mn}, n/2)$ in the good drawing D of $S_m \times P_{n-1}$ from Theorem 1, we obtain a good drawing of $S_m \times C_n$ with the required number of crossings.

In the remainder of this paper we determine the precise values of the crossing numbers of graphs $S_3 \times P_m$, $S_4 \times P_2$ and $S_3 \times C_n$. For our convenience let a_i, b_i, c_i and d_i denote the vertices $(0, i)$, $(1, i)$, $(2, i)$ and $(3, i)$, respectively, in the graph $S_3 \times P_n$ or $S_3 \times C_n$. Let S^i denote the induced subgraph of $S_3 \times P_n$ $(S_3 \times C_n)$ having vertices a_i, b_i, c_i and d_i. Let us remark that S^i is isomorphic to S_3. Denote the induced subgraph of $S_3 \times P_n$ $(S_3 \times C_n)$ having vertices $a_i, b_i, c_i, d_i, a_{i+1}, b_{i+1}, c_{i+1}, d_{i+1}$ by H^i. (H^i has S^i and S^{i+1} as subgraphs.) The cycle induced by the vertices $a_0, a_1, \ldots, a_{n-1}$ of the graph $S_3 \times C_n$ will be called the *a-cycle*. In the same way we define the *b-cycle*, the *c-cycle* and the *d-cycle*.

For the sake of simplicity we identify sometimes the graph with its drawing.

Lemmas 1 and 2 will be very important in what follows.

Lemma 1. *If D is a good drawing of* $S_3 \times P_n$, $n \geqq 2$, *in which no star* S^i, $i = 0, 1, \ldots, n$, *has a crossed arc, then D has at least* $n - 1$ *crossings.*

Lemma 2. *If D is a good drawing of* $S_3 \times C_n$, $n \geqq 3$, *in which no star* S^i, $i = 0, 1, \ldots, n - 1$, *has a crossed arc, then D has at least n crossings.*

Proof of Lemmas 1 and 2. Every arc of $S_3 \times P_n$ which belongs to no S^j is in exactly one H^i (we shall say that it is a *nonstar arc*). We show that in every drawing D^i of H^i induced by D, $i = 0, 1, \ldots, n - 2$, there is at least one nonstar arc which is crossed. If there are two nonstar arcs of D^i that are mutually crossed, then the assertion is valid. Suppose that no two arcs of D^i cross each other. Such a drawing D^i induces a map with two hexagonal regions and two quadrangular regions, see Fig. 1.

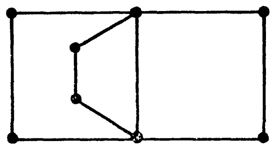


Fig. 1.

Consider the drawing of S^{i+2} induced by D. By the assumption of Lemma 1 it must lie entirely in one of the regions of D^i, say, in the region ω. However, at least one of the nodes $b_{i+1}, c_{i+1}, d_{i+1}$ lies outside the region ω. Then there is an arc connecting one of them with the corresponding one of the drawing of S^{i+2} which crosses some arc on the boundary of the region ω; this crossing is formed by a nonstar arc of D^i and a nonstar arc of D^{i+1}. Since i runs through $0, 1, \ldots, n - 2$, the drawing D has at least $n - 1$ crossings.

Similarly we can prove Lemma 2 – in the graph $S_3 \times C_n$ there are n suitable pairs of subgraphs H^i and S^{i+2}.

Theorem 3. $\nu(S_3 \times P_n) = n - 1$ *for* $n \geqq 1$.

Proof. In accordance with Theorem 1 it is sufficient to prove that $\nu(S_3 \times P_n) \geqq n - 1$. We proceed by induction on n. The case $n = 1$ is trivial. The inequality in the case $n = 2$ follows from the fact that $S_3 \times P_2$ has a subgraph homeomorphic to $K_{3,3}$, and by Kuratowski's theorem it is not planar.

Assume that the result is valid for $n = k$, $k \geqq 1$. Let D be a good drawing of $S_3 \times P_{k+1}$ with less than k crossings. By Lemma 1 in D there exists a star S^j with a crossed arc. By the removal of all edges of this star we obtain a graph homeomorphic to $S_3 \times P_k$ drawn with less than $k - 1$ crossings. This contradicts the induction hypothesis. This completes the proof.

Theorem 4. $\nu(S_4 \times P_2) = 2$.

Proof. By Theorems 1 and 2 we have $1 \leqq \nu(S_4 \times P_2) \leqq 2$. We eliminate the case $\nu(S_4 \times P_2) = 1$. Let F_i, $i = 1, 2, 3, 4$, be a subgraph of the graph $S_4 \times P_2$ with the set of vertices $\{(0, 0), (0, 1), (0, 2), (i, 0), (i, 1), (i, 2)\}$ and the set of edges $\{(0, 0)(i, 0), (0, 1)(i, 1), (0, 2)(i, 2), (i, 0)(i, 1), (i, 1)(i, 2)\}$. Let D be a good drawing of $S_4 \times P_2$ with one crossing. This crossing must lie on some F_i because the arcs $(0, 0)(0, 1)$ and $(0, 1)(0, 2)$ cannot be mutually crossed. The deletion of all arcs of F_i and the nodes $(i, 0)$, $(i, 1)$ and $(i, 2)$ from the drawing D gives a good drawing of $S_3 \times P_2$ without a crossing. This contradiction with Theorem 3 finishes the proof of Theorem 4.

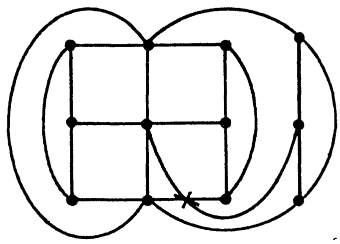
Note. Theorems 3 and 4 lead to the conjecture that

$$\nu(S_m \times P_n) = (n - 1)\left[\frac{m}{2}\right]\left[\frac{m - 1}{2}\right].$$

Theorem 5.

$$\nu(S_3 \times C_3) = 1\,,$$
$$\nu(S_3 \times C_4) = 2\,,$$
$$\nu(S_3 \times C_5) = 4\,.$$

Proof. The graf $S_3 \times C_3$ ($S_3 \times C_4$) has the graph $S_3 \times P_2$ ($S_3 \times P_3$) as a subgraph and by Theorem 3, $\nu(S_3 \times C_3) \geqq 1$ ($\nu(S_3 \times C_4) \geqq 2$) holds. In Fig. 2 there



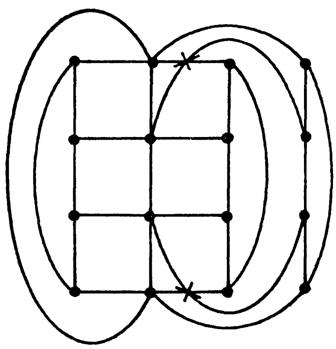


Fig. 2.

are good drawings of $S_3 \times C_3$ and $S_3 \times C_4$ with one and two crossings, respectively.

The graph $S_3 \times C_5$ has the graph $S_3 \times P_4$ as a subgraph, therefore $\nu(S_3 \times C_5) \geqq \geqq 3$. We shall show that three crossings are not sufficient. Assume the opposite. Let D be a good drawing of $S_3 \times C_5$ with three crossings. The drawing D has the following properties:

Property (1). None of the arcs a_ia_{i+1}, b_ib_{i+1}; c_ic_{i+1}, d_id_{i+1} for $i = 0, 1, 2, 3, 4$ is crossed.

In the opposite case we remove these edges from $S_3 \times C_5$ and obtain a good drawing of $S_3 \times P_4$ with at most two crossings.

Property (2). For all j, $j = 0, 1, 2, 3, 4$, the star S^j has at most one crossing.

If some S^j contains at least two crossings, then by the deletion of all edges of S^j from $S_3 \times C_5$ we obtain a good drawing of the graph homeomoıphic to $S_3 \times C_4$ with one crossing.

Let D^* be a subdrawing of D induced by nodes of the a-cycle and the b-cycle. According to the properties (1) and (2), D^* divides the plane to two pentagonal and five quadrangular regions. One of the pentagonal regions is bounded by the a-cycle and the other by the b-cycle. Every quandrangular region is incident with two nodes of the a-cycle and two of the b-cycle. There are precisely two nonstar arcs on the boundary of every quandrangular region. From the property (1) it follows that the c-cycle lies entirely in the pentagonal region bounded by the a-cycle. If it lay in some quadrangular region, then at least three arcs joining its nodes with the corresponding nodes of the a-cycle would cross the boundary of this region in contradiction with the property (2) permitting at most two crossings. From the property (1) it follows that the c-cycle cannot lie in the second pentagonal region.

A subdrawing D^{**} of D induced by the nodes of the a-cycle, the b-cycle and the c-cycle divides the plane to ten quadrangular and two pentagonal regions. One pentagonal region is bounded by the b-cycle and the other by the c-cycle. Every quadrangular region is incident with precisely two nodes of the a-cycle. Similarly as above we can show that the d-cycle belongs neither to the interior of a quadrangular region, nor to the interior of a pentagonal region. This contradicts the hypothesis $\nu(S_3 \times C_5) = 3$. In Fig. 3, an optimal diagram of $S_3 \times C_5$ with four crossings is shown.

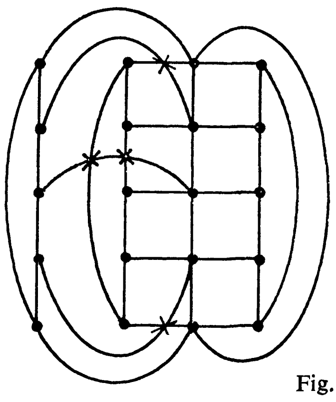


Fig. 3.

Theorem 6. *$\nu(S_3 \times C_n) = n$ for all $n \geqq 6$.*

Proof. The proof of this Theorem for $n = 6$ is similar to that of the identity $\nu(S_3 \times C_5) = 4$. By Theorem 2 we have $\nu(S_3 \times C_6) \leqq 6$. We assume that D is a good drawing of $S_3 \times C_6$ with at most five crossings. For $i = 0, 1, 2, 3, 4, 5$, let E^i denote the subgraph of $S_3 \times C_6$ consisting of the vertices $a_i, b_i, c_i, d_i, a_{i+1}, b_{i+1}, c_{i+1}, d_{i+1}$ and the edges $a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1}, d_id_{i+1}$. (The indices are taken modulo 6.) We can successively show that for all i, $i = 0, 1, 2, 3, 4, 5$, the drawing D has the following properties:

(1) The star S^i has at most one crossing.

(2) The subgraph E^i has exactly one crossing.

(3) The star S^i has at least one crossing.

From the properties (1), (2) and (3) we derive a contradiction.

For $n \geqq 7$ the proof proceeds by induction on n in the same way as in Theorem 3 (using Theorem 2 and Lemma 2).

Note (added in November 1981). In the paper "*On the crossing numbers of products of cycles and graphs of order four*", J. Graph Theory 4 (1980), 145–155, L. W. Beineke and R. D. Ringeisen determined the crossing numbers of the graphs $G \times C_n$ when G is any graph of order four different from the star S_3. The exact values of crossing number for this case are determined by Theorems 5 and 6 of the present paper.

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