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ON A CHARACTERIZATION OF QUASICONTINUOUS MULTIFUNCTIONS

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Given a function $f : X \to Y$, where X, Y are topological spaces, the quasicontinuity of f may be characterized as follows (see [1]).

Let X, Y be first countable topological spaces and X a Hausdorff space. Then f is a quasicontinuous function at a point $x \in X$ if and only if there exists a nonempty open set $G \subset X$ such that $x \in \bar{G}$ and the restriction $f \mid (G \cup \{x\})$ is continuous at x.

In the literature some attempts have appeared to characterize in a similar way the quasicontinuity of multifunctions (see [9]). We show in this note that under the asumptions given in [9] such a characterization is impossible both for lower and upper semi-quasicontinuity of multifunctions. We show that under some further restrictions on the topological spaces considered such characterization for the upper semi-quasicontinuity may be obtained.

1. A CHARACTERIZATION OF THE UPPER SEMI-QUASICONTINUITY

We introduce some definitions which we shall use. We also present some connections to similar definitions appearing in the literature. To cover various situations we consider mappings from X into Y, or into the potence set of Y, where X is a topological space, but in general we do not suppose that a topology on Y is given. Instead of a topology on Y we suppose that a collection $\mathscr{S}$ on Y is given such that $\bigcup \mathscr{S} = Y$. Given such a collection on Y we say that Y is an $\mathscr{S}$-space (compare also [8]). Evidently, if a topology $\mathscr{G}$ on Y is given, then taking $\mathscr{S} = \mathscr{G}$ we have an example of an $\mathscr{S}$-space.

If a mapping $f : X \to Y$ is given we shall refer to f as to a function or a single-valued mapping of X, into Y. In case a mapping F of X into the set of all nonempty subsets of Y is given we refer to F as to a multifunction. The notation $F : X \to Y$ will be used in this case as well. Usually the capital letter F is used for a multifunction while f stands for a function assuming as values points of the set Y. In all what follows a function f may be considered without misunderstanding as a multifunction assuming as values the sets $\{f(x)\}$ $(x \in X)$.

If X is a topological space and Y is an $\mathscr{S}$-space then a multifunction $F : X \to Y$ is said to be upper (lower) semi-continuous at a point $x \in X$ if for any set $V \in \mathscr{S}$ containing $F(x)$ (for any set $V \in \mathscr{S}$ for which $F(x) \cap V \neq \emptyset$) there exists an open set U containing x such that $F(y) \subset V(F(y) \cap V \neq \emptyset)$ for any $y \in U$.

Under the same assumptions on X and Y the multifunction $F : X \to Y$ is called upper (lower) semi-quasicontinuous at $x \in X$ if for any $V \in \mathscr{S}$ containing $F(x)$ (for any $V \in \mathscr{S}$ for which $F(x) \cap V \neq \emptyset$) and any open set U containing x there exists and open set $G \subset U$, $G \neq \emptyset$ such that $F(y) \subset V(F(y) \cap V \neq \emptyset)$ for any $y \in G$.

The corresponding notions of upper (lower) semi-continuity or upper (lower) semi-quasicontinuity on X are understood as the upper (lower) semi-continuity or upper (lower) semi-quasicontinuity at any $x \in X$.

If Y is a topological space then the above definitions coincide with the definitions of upper and lower semi-continuity (see e.g. [4] p. 393) or upper and lower semi-quasicontinuity (see e.g. [7], [9]). Of course the topology $\mathscr{G}$ of Y is taken instead of $\mathscr{S}$.

If $f : X \to Y$ is a single-valued mapping then the upper as well as the lower semi-continuity at x give the usual continuity at x and conversely. Similarly, the upper as well as the lower semi-quasicontinuity in this case coincide with the quasicontinuity in the sense of Kempisty (see e.g. [3], [6]).

Given an $\mathscr{S}$-space Y and a collection $\mathscr{K}$ of subsets of Y, we say that the space Y is first countable at the collection $\mathscr{K}$ if for any $K \in \mathscr{K}$ there exists a sequence $\{S_n\}_{n=1}^{\infty}$ of elements of $\mathscr{S}$ such that $S_n \supset S_{n+1}$, $S_n \supset K$ for $n = 1, 2, \ldots$ and for any $S \in \mathscr{S}$ for which $S \supset K$ there exists n_0 such that $S_{n_0} \subset S$.

A set $A \subset X$ in a topological space is said to be quasiopen if $A \subset \bar{A}^0$. (The notion of the quasiopen set was introduced under a different name by Levine in [5].)

Theorem 1. *Let X be a first countable Hausdorff topological space. Let $F : X \to Y$ be a multifunction and Y an $\mathscr{S}$-space which is first countable at the collection $\mathscr{K} = \{F(x); x \in X\}$. Then F is upper semi-quasicontinuous at a point $x \in X$ if and only if there exists a quasiopen set A containing x such that $F \mid A$ is upper semi-continuous at x.*

Proof. The "sufficient" part of the theorem can be verified without difficulty. It may be proved without any assumptions on the space X and Y. In fact this part can be proved essentially in the same way as a similar theorem for single-valued functions is proved (see [8]).

Let us also prove this part for the sake of completeness. So let a quasiopen set A exist such that $F \mid A$ is upper semi-continuous at x. Let $S \in \mathscr{S}$, $F(x) \subset S$ and let U be an open set containing x.

The upper semi-continuity of $F \mid A$ implies that an open set $U_1 \subset U$, $x \in U_1$ exists such that $F(y) \subset S$ for any $y \in U_1 \cap A$. Since U_1 is open and $x \in U_1 \cap \bar{A}^0$, the set $G = U_1 \cap A^0$ is nonempty, open and $G \subset U$. Hence $F(y) \subset S$ for any $y \in G$ and the upper semi-quasicontinuity of F at x is proved.

Now let F be upper semi-quasicontinuous at x. If $\{x\}$ is open, then the theorem is proved because it is sufficient to take $A = \{x\}$. Suppose $\{x\}$ is not open. Let $\{U_n\}_{n=1}^{\infty}$ be a non-increasing base of neighbourhoods of the point x and $\{S_n\}_{n=1}^{\infty}$ a non-increasing sequence such that $S_n \supset F(x)$, $S_n \in S$, $n = 1, 2, \ldots$ and for any $S \in \mathscr{S}$ there is S_{n_0} with $S_{n_0} \subset S$. Now, for the set S_1 and for the neighbourhood U_1 there exists an open set $G_1 \subset U_1$, $G_1 \neq \emptyset$, such that $F(y) \subset S_1$ for any $y \in G_1$. Clearly $G_1 \neq \{x\}$. From the fact that X is Hausdorff it follows that there is $n_2 > 1$ such that $G_1 - \overline{U}_{n_2} \neq \emptyset$. Take U_{n_2}. Then again the upper semi-quasicontinuity implies that there exists $G_2 \subset U_{n_2}$ such that $G_2 \neq \emptyset$, G_2 is open and $F(y) \subset S_2$ for $y \in G_2$. Since $G_2 \neq \{x\}$, there exists U_{n_3} $(n_3 > n_2)$ such that $G_2 - \overline{U}_{n_3} \neq \emptyset$. So by induction we can construct a sequence $\{U_{n_k}\}_{k=1}^{\infty}$ such that $n_k < n_{k+1}$ $(k = 1, 2, \ldots)$ and a sequence $\{G_k\}_{k=1}^{\infty}$ of open sets such that $G_k - \overline{U}_{n_{k+1}} \neq \emptyset$, $G_k \subset U_{n_k}$ and $F(y) \subset S_k$ if $y \in G_k$. Evidently, the set

$$A = \left(\bigcup_{k=1}^{\infty} (G_k - \overline{U}_{n_{k+1}})\right) \cup \{x\}$$

is quasiopen.

Now for any S_i take the neighbourhood U_{n_i} of the point x. We have $U_{n_i} \cap A \subset \subset \left(\bigcup_{k=i}^{\infty} G_k\right) \cup \{x\}$ and $F(U_{n_i} \cap A) \subset S_i$. Thus the upper semi-continuity of $F|A$ at x is proved.

Using Theorem 1 we are able to prove a result for the case when Y is a topological space and F a compact-valued multifunction.

Theorem 2. *Let X be a first countable Hausdorff space and Y a second countable topological space. Let $F : X \to Y$ be a compact-valued multifunction. Then F is upper semi-quasicontinuous at a point $x \in X$ if and only if there exists a quasiopen set A containing x, such that $F \mid A$ is upper semi-continuous at x.*

Proof. We shall consider the space Y as an $\mathscr{S}$-space where $\mathscr{S}$ is the topology of Y. To prove our theorem it is sufficient to prove that Y is first countable at any compact set $K \subset Y$ and then to use Theorem 1.

So let $\mathscr{B}$ be a countable base of open sets in Y. Let $\mathscr{C}$ be the collection of all finite unions of the sets from $\mathscr{B}$; $\mathscr{C}$ is countable as well. Let K be compact, let $\{W_k\}_{k=1}^{\infty}$ be the sequence of all $W \in \mathscr{C}$, $W \supset K$. Put $S_k = W_1 \cap W_2 \cap \ldots \cap W_k$, $k = 1, 2, \ldots$. Clearly, $S_k \supset K$, S_k are open for all k. Let $S \supset K$ be an open set. For each $z \in K$, choose $V_z \in \mathscr{B}$ with $z \in V_z \subset S$. The compactness of K implies that some finite union of V_z's covers K, hence there is K such that $S \supset W_k \supset S_k \supset K$. The first countability of S at any compact set K is proved.

The second countability in the preceding theorem may be omitted if a compact-valued multifunction $F : X \to Y$ is considered and Y is supposed to be pseudometric.

Theorem 3. *Let X be a first countable Hausdorff topological space, Y a pseudometric space and $F : X \to Y$ a compact-valued multifunction. Then F is upper semi-quasicontinuous at $x \in X$ if and only if there exists a quasiopen set A containing x, such that $F \mid A$ is upper semi-continuous at x.*

Proof. The proof of Theorem 3 immediately follows from Theorem 1, if we know that the collection $\mathscr{S}$ of open sets is first countable at the collection $\mathscr{K}$ of all compact sets in Y. But if K is any compact set we can take $S_n = \bigcup_{x \in K} (S(x, 1/n))$, where $S(x, 1/n)$ is the sphere with the centre x and radius $1/n$. Evidently $S_n \supset K$, $S_n \supset S_{n+1}$ for $n = 1, 2, \ldots$. If S is any open set such that $S \supset K$, then (see [2] p. 210) there exists n_0 such that $K \subset S_{n_0} \subset S$.

The following corollaries for single-valued functions are evident.

Corollary 1. (See [8].) *Let X be a first countable Hausdorff topological space, Y an $\mathscr{S}$-space which is first countable on the collection of all singletons. Then a single-valued function $f : X \to Y$ is quasicontinuous at $x \in X$ if and only if there exists a quasiopen set A such that $x \in A$ and $f \mid A$ is continuous at x.*

Corollary 2. (See [1].) *Let X be a first countable Hausdorff space, Y a first countable topological space. Then a single-valued function $f : X \to Y$ is quasicontinuous at a point x if and only if there exists a quasiopen set A such that $x \in A$ and $f \mid A$ is continuous at x.*

2. COUNTEREXAMPLES

While the sufficient part in Theorems 1, 2, 3 is true for any two topological spaces X, Y, the necessity, i.e., the existence of a quasiopen set A containing x such that $F \mid A$ is semi-continuous at x is in general not true. It is not true even in the case when X and Y are first countable Hausdorff spaces. So Theorems 3 and 4 in [9] are not valid without further assumptions.

The first of our examples (in which X, Y are separable metric spaces) shows that simultaneous upper and lower semi-quasicontinuity of a multifunction $F : X \to Y$ at a point $x \in X$ does not imply the existence of a quasiopen set A such that $x \in A$ and $F \mid A$ is upper semi-continuous. Examples 2 and 3 are due to referee. They show that neither the condition of the first countability nor the condition that X is Hasudroff may be omitted. Example 4 concerns the lower semi-quasicontinuity. It shows that for $F : X \to Y$ (now X, Y are again separable metric spaces) the simultaneous upper and lower semi-quasicontinuity does not imply the existence of a quasiopen set A such that $x \in A$ and $F \mid A$ is lower semi-continuous.

Example 1. Define a multifunction $F : \langle 0,1) \to R$ in the following way

$F(0) = \{1, 2, 3, \ldots\}$,
$F(x) = \{1, 2, \ldots, n - 1, x + n - 1/(n + 1), n + 1, \ldots\}$ if $x \in \langle 1/(n + 1), 1/n)$.

1) F is upper semi-quasicontinuous at any $x \in \langle 0, 1)$.

a) Let $x \in \langle 1/(n + 1), 1/n)$ and let U, V be open sets such that $x \in U$, $V \supset F(x)$. Since V is open we can choose $\varepsilon > 0$ such that $(x + n - 1/(n + 1) - \varepsilon, x + n - 1/(n + 1) + \varepsilon) \subset V$, further $k \in V$ for $k \neq n$, $k = 1, 2, \ldots$. If we put $G = U \cap (x - \varepsilon, x + \varepsilon) \cap (1/(n + 1), 1/n)$ then evidently for any $y \in G$ we have

$$F(y) = \{1, 2, \ldots, n - 1, y + n - 1/(n + 1), n + 1, \ldots\}.$$

Hence (1) implies that $F(y) \subset V$.

b) Let $x = 0$. Let U, V be open sets such that $V \supset F(0)$ and $0 \in U$. Choose a natural n such that $1/n \in U$. Then $F(1/n) = F(0) \subset V$. By a), there is an open set $G \subset U$ such that $F(y) \subset V$ for any $y \in G$.

2) F is lower semi-quasicontinuous at any $x \in \langle 0, 1)$.

a) Let $x \in \langle 1/(n + 1), 1/n)$. Let U be any open set containing x and V any open set with $F(x) \cap V \neq \emptyset$.

If $x + n - 1/(n + 1) \notin V$, then there exists a point $k \neq n$ such that $k \in V$. Putting $G = U \cap (1/(n + 1), 1/n)$ we have $F(y) \supset \{k\}$ for any $y \in G$, hence $F(y) \cap V \neq \emptyset$.

In the case $x + n - 1/(n + 1) \in V$ we can choose $\varepsilon > 0$ such that $(x + n - 1/(n + 1) - \varepsilon, x + n - 1/(n + 1) + \varepsilon) \subset V$. Then choosing $G = U \cap (1/(n + 1), 1/n) \cap (x - \varepsilon, x + \varepsilon)$ we have $F(y) \cap V \neq \emptyset$ for any $y \in G$.

b) Let $x = 0$. Let V be any open set such that $F(0) \cap V \neq \emptyset$. It means $k \in V$ for some positive integer k. Let U be any open set containing 0. There exists $n > k$ such that $G = (1/(n + 1), 1/n) \subset U$. Then $F(y) \supset \{k\}$ for any $y \in G$, hence $F(y) \cap V \neq \emptyset$. The lower semi-quasicontinuity of F is proved.

3) For any quasiopen set A containing 0 the multifunction $F \mid A$ is not upper semi-continuous at 0.

Suppose A is quasiopen, $0 \in A$. Put $B_n = A^0 \cap (1/(n + 1), 1/n)$ for $n = 1, 2, \ldots$. Each B_n is open and $0 \in A^0$ implies that B_n is non-void for infinitely many n. If $B_n = \emptyset$, put $c_n = 1/2$, if $B_n \neq \emptyset$ choose $c_n > 0$ such that $1/(n + 1) + c_n \in B_n$. Finally, put $V = \bigcup_{n=1}^{\infty} (n - 1/2, n + c_n)$. If U is open and $0 \in U$, there exists n such that $\emptyset \neq B_n \subset U$. Thus $F(A \cap U) \supset F(B_n) \supset F(1/(n + 1) + c_n \ni n + c_n$. But $n + c_n \notin V$. Hence the multifunction $F \mid A$ is not upper semi-continuous at 0.

Example 2. Let X be the space of all ordinal numbers less than or equal to ω_1 with the usual order topology. Of course, X is a compact Hausdorff topological space which is not first countable. If α is an ordinal number, there are a unique non-negative integer n and a limit number β such that $\alpha = \beta + n$. Then, put $f(\alpha) = 1/n$ if $\alpha < \omega_1$, $f(\omega_1) = 0$. Consider the single-valued mapping $f : X \to R$. To prove that f is upper semi-quasicontinuous at the point ω_1 it suffices to observe that in any neighbourhood U of ω_1 and for any $\varepsilon > 0$ there exists $\alpha \in U$ with $0 < f(\alpha) < \varepsilon$ and $\{\alpha\}$ is an open set.

Now, suppose that A is a quasiopen subset of X, $\omega_1 \in A$ and $f \mid A$ is upper semi-continuous (= continuous) at ω_1. If i is a positive integer, there exists a neighbourhood U of ω_1, suppose that $U = \{\lambda \in X; \lambda > \gamma_i\}$, where $\gamma_i \in X$, $\gamma_i < \omega_1$ such that $f(U \cap A) \subset \{t \in R; t < 1/i\}$. Therefore there is no ordinal number $\alpha = \beta + i$ with $\alpha \in A$, $\alpha > \gamma_i$. Put $\gamma = \sup_i \gamma_i$. Again $\gamma < \omega_1$, and all $\alpha \in A$, $\alpha > \gamma$ are limit numbers, hence $A^0 \cap \{\lambda \in X; \lambda > \gamma\} = \emptyset$, thus $\omega_1 \notin \bar{A}^0$, which is a contradiction.

Example 3. Let X consist of all (m, n) where $m, n = 1, 2, \ldots$, and a further element x. Put $X_k = \{(m, n) \in X;\ n = k\}$, $Q_k = \{(m, n) \in X;\ m \leqq k,\ n \leqq k\}$. Define a base neighbourhoods of x as the collection of all sets $\{x\} \cup (X - Q_j)$ where $j = 1, 2, \ldots$, and for $y \in X_k$ as the collection of all sets $\{y\} \cup (X_k - F)$ where F is a finite set. Evidently, X becomes a topological space which is first countable, T_1 but is not Hausdorff. The closure of any neighbourhood of x is X.

Define a single-valued $f : X \to R$ by $f(m, n) = 1/n$, $f(x) = 0$. Let V, U be open sets in R or X, respectively, such that $V \ni 0$, $U \ni x$. We may suppose $V = \{t \in R; |t| < \varepsilon\}$, $U = X - Q_k$, where $\varepsilon > 0$ and k is an integer. Put $G = X_j$, where $j > \max(1/\varepsilon, k)$; then G is open and $f(G) \subset V$.

Now, suppose $A \subset X$ is quasiopen, $x \in A \subset \bar{A}^0$ and $f \mid A$ is upper semi-continuous at x. If i is a positive integer, put $W = \{t \in R; |t| < 1/i\}$. Choose h such that $f((X - Q_h) \cap A) \subset W$. This implies $(X - Q_h) \cap A \cap X_i = \emptyset$, i.e. $A \cap X_i \subset Q_h$, hence $A \cap X_i$ is finite for each i, thus $A^0 = \emptyset$, which is a contradiction.

Example 4. Define a multifunction $F : \langle 0, 1) \to R$ such that

$$F(0) = \{1, 2\}$$

$$F(x) = \begin{cases} \{1\} & \text{if} \quad x \in \langle 1/2n,\ 1/(2n-1)), \quad n = 1, 2, \ldots \\ \{2\} & \text{if} \quad x \in \langle 1/(2n+1),\ 1/2n), \quad n = 1, 2, \ldots \end{cases}$$

1) F is upper semi-quasicontinuous at any $x \in \langle 0, 1)$.

If $x \neq 0$ then this is obvious from the fact that F is constant on any of the intervals $\langle 1/2n, 1/(2n-1))$ or $\langle 1/(2n+1), 1/2n)$.

If $x = 0$ then taking V open such that $V \supset F(0)$ and U any open neighbourhood of 0, we can put $G = U$. Then $F(G) \subset F(0) \subset V$. The upper semi-quasicontinuity of F is proved.

2) F is lower semi-quasicontinuous at any $x \in \langle 0, 1)$.

The lower semi-quasicontinuity at $x \neq 0$ may be proved similarly as the upper semi-quasicontinuity at $x \neq 0$ was.

If $x = 0$, then for any open V for which $F(0) \cap V \neq \emptyset$ and for any open set U containing x, we have $1 \in V$ or $2 \in V$. Let e.g. $1 \in V$. Then we can choose n such that

$G = (1/2n, 1/(2n-1)) \subset U$ and $F(y) \cap V \neq \emptyset$ for any $y \in G$. The lower semi-quasicontinuity of F is proved.

3) There is no quasiopen set A containing 0 for which $F \mid A$ is lower semi-continuous at 0.

Suppose A to be such a set. Let $V_1 = (1/2, 3/2)$ and $V_2 = (3/2, 5/2)$. According to the assumption there exist two open sets U_1, U_2 containing 0 and such that $F(y) \cap \cap V_i \neq \emptyset$ if $y \in U_i \cap A$, $i = 1, 2$. Taking $U_1 \cap U_2 \cap A$, which contains $y \neq 0$, we have that $F(y) = \{1\}$ and simultaneously $F(y) = \{2\}$. This contradicts the definition of F. The multifunction $F \mid A$ is not lower semi-continuous at 0.

A question arises if a characterization of lower semi-quasicontinuity analogous to that given for upper semi-quasicontinuity in Theorems 1—3, is possible. Example 4 gives a negative answer to this question.

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