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ON THE CONTINUITY OF HEAT POTENTIALS

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This note is devoted to a certain analogy of the continuity principle for the heat potentials in R^2. We shall show that if μ is a measure in R^2 such that $\mu(\{[x, t]\}) = 0$ for each $[x, t] \in R^2$ and the support of the measure μ lies on a curve of the form $x = \varphi(t)$, where φ is a $\frac{1}{2}$-Hölder continuous function, then the heat potential of the measure μ is continuous in R^2 if and only if the restriction of this potential on the support of μ is continuous. Further, we shall show that this assertion fails in the case that φ is α-Hölder continuous only for some $\alpha < \frac{1}{2}$.

We deal in this paper with heat potentials in R^2 only. Points in R^2 are denoted $[x, t]$, $[\xi, \tau]$ etc. G will stand for the heat kernel in R^2, that is $G(x, t) = 0$ for $t \leqq 0$ $(x \in R)$,

$$G(x, t) = (\pi t)^{-1/2} \exp\left(-\frac{x^2}{4t}\right) \quad \text{for} \quad t > 0 .$$

For $[x, t] \in R^2$, $c > 0$ let us denote

$$(1) \qquad A(x, t; c) = \{[\xi, \tau] \in R^2;\ G(x - \xi,\ t - \tau) > c\} .$$

If μ is a Borel measure (non-negative and finite — we shall deal only with non-negative and finite measures) with compact support in R^2, then the heat potential U_μ of the measure μ is defined by

$$(2) \qquad U_\mu(x, t) = \int_{R^2} G(x - \xi,\ t - \tau)\, \mathrm{d}\mu(\xi, \tau) \quad ([x, t] \in R^2) .$$

We shall deal in what follows only with continuous measures, that is with measures which vanish on singletons. The following assertion holds (see, for instance, [3], [4], [5]).

1. Proposition. *Let $K \subset R^2$ be a compact set, μ a continuous Borel (non-negative) measure with compact support in R^2. Then the restriction $U_\mu|_K$ is continuous on K if and only if the following condition is fulfilled:*

$$(3) \qquad \lim_{d \to +\infty} \left(\sup \left\{\int_d^\infty \mu(A(x, t; c))\, \mathrm{d}c;\ [x, t] \in K\right\}\right) = 0 .$$

Further, let φ be a continuous function on an interval $\langle a, b\rangle$ ($\langle a, b\rangle$ is supposed to be non-degenerate and compact). Let us denote

$$K = K_\varphi = \{[x, t] \in R^2;\ t \in \langle a, b\rangle,\ x = \varphi(t)\}\,.$$

We shall deal with heat potentials for measures μ with $\operatorname{spt} \mu \subset K$. For the sake of simplicity we shall identify in this note the measure μ with $\operatorname{spt} \mu \subset K$ with a certain measure λ on the interval $\langle a, b\rangle$ in the following way. If μ is a measure in R^2 such that $\operatorname{spt} \mu \subset K$ then we assign to this measure a measure λ on $\langle a, b\rangle$ (that is a measure in R^1 with support contained in $\langle a, b\rangle$) such that for each Borel set $M \subset \langle a, b\rangle$ we put

$$\lambda(M) = \mu(\{[x, t] \in K;\ t \in M\})$$

(roughly speaking the measure λ is a projection of the measure μ on the t-axis). On the other hand, to a Borel measure λ on $\langle a, b\rangle$ we assign a measure μ in R^2 with $\operatorname{spt} \mu \subset K$ such that

$$\mu(M) = \lambda(\{t \in \langle a, b\rangle;\ [\varphi(t), t] \in M\})$$

for any Borel set $M \subset R^2$. In this sense we shall call here the measures μ, λ (on R^2 and $\langle a, b\rangle$, respectively) associated measures (more precisely, associated measures with respect to φ). Further, let $\mathscr{B}^+ = \mathscr{B}^+(\langle a, b\rangle)$ denote the set of all Borel (finite, non-negative) measures on $\langle a, b\rangle$,

$$\mathscr{B}_0^+ = \mathscr{B}_0^+(\langle a, b\rangle) = \{\lambda \in \mathscr{B}^+(\langle a, b\rangle);\ \lambda(\{t\}) = 0 \text{ for each } t \in \langle a, b\rangle\}\,.$$

For $\lambda \in \mathscr{B}^+$ let

$$K_\lambda = \{[x, t] \in K;\ t \in \operatorname{spt} \lambda\}\,.$$

If $\lambda \in \mathscr{B}^+$ and μ is the measure associated with λ (in the above mentioned sense) then $K_\lambda = \operatorname{spt} \mu$. For this pair of associated measures we shall write $U_\lambda = U_\lambda^\varphi = U_\mu$, that is

$$U_\lambda^\varphi(x, t) = U_\mu(x, t) = \int_K G(x - \xi,\ t - \tau)\,\mathrm{d}\mu(\xi, \tau) = \tag{4}$$

$$= \int_a^b G(x - \varphi(\tau),\ t - \tau)\,\mathrm{d}\lambda(\tau) \quad ([x, t] \in R^2)\,.$$

Let us take notice of the following three simple assertions.

2. Lemma. *Let $\lambda \in \mathscr{B}^+(\langle a, b\rangle)$ and let*

$$\lim_{d \to +\infty} \left(\sup\left\{\int_d^\infty \lambda(\langle t - c^{-2}, t)) \,\mathrm{d}c;\ t \in R^1\right\}\right) = 0\,. \tag{5}$$

Then the potential U_λ is continuous (on R^2).

Proof. If $[x, t] \in R^2$, $c > 0$, then

$$A(x, t; c) \subset \left\{[\xi, \tau] \in R^2;\ \tau \in \left(t - \frac{1}{\pi} c^{-2}, t\right),\ \xi \in R^1\right\} \subset$$

$$\subset \{[\xi, \tau] \in R^2;\ \tau \in \langle t - c^{-2}, t\rangle,\ \xi \in R^1\}.$$

If μ is the measure associated with λ then

$$\mu(A(x, t; c)) \leqq \lambda(\langle t - c^{-2}, t\rangle)$$

and it follows from (5) that

$$\lim_{d \to +\infty} \left(\sup \left\{\int_d^\infty \mu(A(x, t; c))\, dc;\ [x, t] \in R^2\right\}\right) = 0.$$

Let us note that we immediately get from (5) that $\lambda(\{t\}) = 0$ for each $t \in R^1$. The assertion follows now from Proposition 1.

3. Lemma. *Let us suppose that the function φ is $\frac{1}{2}$-Hölder continuous on $\langle a, b\rangle$f Then for $\lambda \in \mathscr{B}_0^+(\langle a, b\rangle)$ the restriction $U_\lambda|_{K_\lambda}$ is continuous on K_λ if and only i.*

$$\text{(6)} \qquad \lim_{d \to +\infty} \left(\sup \left\{\int_d^\infty \lambda(\langle t - c^{-2}, t\rangle)\, dc;\ t \in \operatorname{spt} \lambda\right\}\right) = 0.$$

Proof. Let μ be the measure associated with λ (with respect to φ). The restriction $U_\lambda|_{K_\lambda}$ is continuous on K_λ if and only if

$$\text{(7)} \qquad \lim_{d \to +\infty} \left(\sup \left\{\int_d^\infty \mu(A(x, t; c))\, dc;\ [x, t] \in K_\lambda\right\}\right) = 0.$$

It is clear that (6) implies (7) (see the proof of Lemma 2).

Suppose now that the condition (7) is fulfilled. For $t \in \langle a, b\rangle$, $c > 0$ let

$$B(t, c) = \{\tau \in \langle a, b\rangle;\ [\varphi(\tau), \tau] \in A(\varphi(t), t; c)\} =$$

$$= \{\tau \in \langle a, b\rangle;\ G(\varphi(t) - \varphi(\tau);\ t - \tau) > c\}.$$

The function φ is supposed to be $\frac{1}{2}$-Hölder continuous, that is there is a constant k such that

$$|\varphi(t) - \varphi(\tau)| \leqq k \sqrt{|t - \tau|}$$

for $t, \tau \in \langle a, b\rangle$. Let $t, \tau \in \langle a, b\rangle$, $\tau < t$. Then

$$G(\varphi(t) - \varphi(\tau),\ t - \tau) = [\pi(t - \tau)]^{-1/2} \exp\left(-\frac{(\varphi(t) - \varphi(\tau))^2}{4(t - \tau)}\right) \geqq$$

$$\geqq [\pi(t - \tau)]^{-1/2} \exp\left(-\frac{k^2}{4}\right).$$

If $\tau \in (t - c^{-2}, t) \cap \langle a, b\rangle$ then

$$[\pi(t - \tau)]^{-1/2} \exp\left(-\frac{k^2}{4}\right) \geqq [\pi c^{-2}]^{-1/2} \exp\left(-\frac{k^2}{4}\right) = ck_1\,,$$

where

$$k_1 = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{k^2}{4}\right).$$

Hence

$$\langle a, b\rangle \cap (t - c^{-2}, t) \subset B(t, ck_1)$$

for $t \in \langle a, b\rangle$ and thus (as λ is a continuous measure by assumption)

$$\int_d^\infty \lambda(\langle t - c^{-2}, t\rangle \,\mathrm{d}c \leqq \int_d^\infty \lambda(B(t, ck_1))\,\mathrm{d}c = \frac{1}{k_1}\int_{dk_1}^\infty \lambda(B(t, u))\,\mathrm{d}u =$$

$$= \frac{1}{k_1}\int_{dk_1}^\infty \mu(A(\varphi(t), t; u))\,\mathrm{d}u\,.$$

Now we can see that (7) implies (6).

4. Lemma. *Let $\lambda \in \mathscr{B}^+(\langle a, b\rangle)$, $d > 0$. Then*

(8)
$$\sup\left\{\int_d^\infty \lambda(\langle t - c^{-2}, t\rangle)\,\mathrm{d}c;\ t \in R^1\right\} = \sup\left\{\int_d^\infty \lambda(\langle t - c^{-2}, t\rangle)\,\mathrm{d}c;\ t \in \operatorname{spt}\lambda\right\}.$$

Proof. Let $t \in R^1 - \operatorname{spt}\lambda$. If $\operatorname{spt}\lambda \cap (-\infty, t) = \emptyset$ then $\lambda(\langle t - c^{-2}, t\rangle) = 0$ for each $c > 0$ and thus

$$\int_d^\infty \lambda(\langle t - c^{-2}, t\rangle)\,\mathrm{d}c = 0\,.$$

In the case $\operatorname{spt}\lambda \cap (-\infty, t) \neq \emptyset$ let us denote

$$t_0 = \sup\,[\operatorname{spt}\lambda \cap (-\infty, t)]\,.$$

Then

$$\operatorname{spt}\lambda \cap \langle t - c^{-2}, t\rangle \subset \operatorname{spt}\lambda \cap \langle t_0 - c^{-2}, t_0\rangle$$

that is

$$\lambda(\langle t - c^{-2}, t\rangle) \leqq \lambda(\langle t_0 - c^{-2}, t_0\rangle)$$

and hence

$$\int_d^\infty \lambda(\langle t - c^{-2}, t\rangle)\,\mathrm{d}c \leqq \int_d^\infty \lambda(\langle t_0 - c^{-2}, t_0\rangle)\,\mathrm{d}c\,.$$

But $t_0 \in \operatorname{spt}\lambda$ and the assertion follows.

From Lemmas 2, 3, 4 we obtain immediately the following assertion.

5. Theorem. *Let φ be a $\frac{1}{2}$-Hölder continuous function on $\langle a, b\rangle$, $K = \{[\varphi(t), t]; t \in \langle a, b\rangle\}$, μ a continuous measure in R^2 with* spt $\mu \subset K$. *Then the heat potential U_μ is continuous on R^2 if and only if the restriction $U_\mu|_{\operatorname{spt}\mu}$ is continuous on* spt μ.

We shall now show two examples that the assumption that the function φ is $\frac{1}{2}$-Hölder continuous is essential in Lemma 3 as well as in Theorem 5.

6. Example. We shall show that for each $\alpha \in (0, \frac{1}{2})$ there is an α-Hölder continuous function φ on $\langle 0, 1\rangle$ and a continuous measure λ on $\langle 0, 1\rangle$ such that the potential U_λ^φ is continuous even on R^2 but for λ the condition (6) from Lemma 3 is not fulfilled.

Given $\alpha \in (0, \frac{1}{2})$ let $\varphi(\tau) = \tau^\alpha$ for $\tau \in \langle 0, 1\rangle$. Let λ be the measure on $\langle 0, 1\rangle$ defined by the density h (density with respect to the Lebesgue measure on R^1),

$$h(\tau) = \tau^{-\gamma}, \quad \tau \in (0, 1),$$

where

$$\tfrac{1}{2} \leqq \gamma < 1 - \frac{1}{3 - 2\alpha}. \tag{9}$$

Then the measure λ does not fulfil the condition (6). Indeed, if the condition (6) is fulfilled for λ then, choosing for instance $\varphi_0 \equiv 0$, the restriction $U_\lambda^{\varphi_0}|_{K_\lambda}$ is continuous by Lemma 3. But for $t \in (0, 1\rangle$

$$U_\lambda^{\varphi_0}(0, t) = \frac{1}{\sqrt{\pi}} \int_0^t \tau^{-\gamma}(t - \tau)^{-1/2}\, d\tau \geqq \frac{1}{\sqrt{\pi}} \int_0^t [\tau(t - \tau)]^{-1/2}\, d\tau = \sqrt{\pi}$$

and $U_\lambda^{\varphi_0}(0, 0) = 0$ (in the case $\gamma > \frac{1}{2}$ it even holds

$$\lim_{t \to 0+} U_\lambda^{\varphi_0}(0, t) = +\infty).$$

Let us now show that the potential $U_\lambda = U_\lambda^\varphi$ is continuous in R^2. It is evident that U_λ is continuous on $R^2 - \{[0, 0]\}$. $U_\lambda(x, t) = 0$ for $t \leqq 0$ and so it suffices to prove that

$$\lim_{\substack{[x,t] \to [0,0] \\ t > 0}} U_\lambda(x, t) = 0. \tag{10}$$

Choose β such that

$$\frac{1}{2(1 - \gamma)} < \beta < \tfrac{3}{2} - \alpha \tag{11}$$

(it is seen from (9) that there is such a β). Note that $\beta > 1$. Let us estimate the potential U_λ at the points of the form $[(ct)^\alpha, t]$, $t > 0$, $c \in \langle 0, 1\rangle$. If $t \in (0, 1)$ then

$$U_\lambda((ct)^\alpha, t) = \frac{1}{\sqrt{\pi}} \int_0^t \tau^{-\gamma}(t - \tau)^{-1/2} \exp\left(-\frac{((ct)^\alpha - \tau^\alpha)^2}{4(t - \tau)}\right) d\tau = \tag{12}$$

$$= \frac{1}{\sqrt{\pi}} \int_{M_1} \tau^{-\gamma}(t-\tau)^{-1/2} \exp\left(-\frac{((ct)^\alpha - \tau^\alpha)^2}{4(t-\tau)}\right) d\tau +$$

$$+ \frac{1}{\sqrt{\pi}} \int_{M_2} \tau^{-\gamma}(t-\tau)^{-1/2} \exp\left(-\frac{((ct)^\alpha - \tau^\alpha)^2}{4(t-\tau)}\right) d\tau = I_1 + I_2 ,$$

where we put

$$M_1 = (0, t) \cap \{\tau; |\tau - ct| > t^\beta\} ,$$
$$M_2 = (0, t) \cap \{\tau; |\tau - ct| < t^\beta\} .$$

Consider first the integral I_1. Let $0 < \tau \leqq ct$. Then

$$|(ct)^\alpha - \tau^\alpha| \geqq |\tau - ct| \, \alpha (ct)^{\alpha-1} \geqq \alpha |\tau - ct| \, t^{\alpha-1}$$

(for $c \leqq 1$, $\alpha - 1 < 0$). If $ct \leqq \tau \leqq t$ then

$$|(ct)^\alpha - \tau^\alpha| \geqq |\tau - ct| \, \alpha \tau^{\alpha-1} \geqq \alpha |\tau - ct| \, t^{\alpha-1} .$$

So in any case

$$|(ct)^\alpha - \tau^\alpha| \geqq \alpha |\tau - ct| \, t^{\alpha-1}$$

for $\tau \in (0, t)$. Consider $\tau \in (0, t)$ such that $|\tau - ct| \geqq t^\beta$. Then

$$\frac{((ct)^\alpha - \tau^\alpha)^2}{4(t-\tau)} \geqq \frac{\alpha^2 (\tau - ct)^2 \, t^{2\alpha-2}}{4(t-\tau)} \geqq \frac{\alpha^2 t^{2\beta} t^{2\alpha-2}}{4t} = \frac{\alpha^2}{4} t^{2(\alpha+\beta)-3} .$$

It is $2(\alpha + \beta) - 3 < 0$ by (11). Hence we obtain

$$(13) \qquad I_1 \leqq \frac{1}{\sqrt{\pi}} \exp\left(-\frac{\alpha^2}{4} t^{2(\alpha+\beta)-3}\right) \int_0^t \tau^{-\gamma}(t-\tau)^{-1/2} \, d\tau \leqq$$

$$\leqq \frac{1}{\sqrt{\pi}} \exp\left(-\frac{\alpha^2}{4} t^{2(\alpha+\beta)-3}\right) \left\{(\sqrt{2}) \, t^{-1/2} \int_0^{t/2} \tau^{-\gamma} \, d\tau + 2^\gamma t^{-\gamma} \int_{t/2}^t (t-\tau)^{-1/2} \, d\tau\right\} =$$

$$= \frac{1}{\sqrt{\pi}} \exp\left(-\frac{\alpha^2}{4} t^{2(\alpha+\beta)-3}\right) \left\{\frac{\sqrt{2}}{1-\gamma} \, t^{-1/2} \left(\tfrac{1}{2}t\right)^{1-\gamma} + 2^{\gamma+1} t^{-\gamma} \left(\tfrac{1}{2}t\right)^{1/2}\right\} =$$

$$= \frac{1}{\sqrt{\pi}} \, t^{1/2-\gamma} \exp\left(-\frac{\alpha^2}{4} t^{2(\alpha+\beta)-3}\right) \left\{\frac{2^{\gamma-1/2}}{1-\gamma} + 2^{\gamma+1/2}\right\} \to_{(t \to 0+)} 0 .$$

The terms in (13) are independent of $c \in \langle 0, 1 \rangle$.

Now let us consider the integral I_2. First, we have

$$(14) \qquad I_2 \leqq \frac{1}{\sqrt{\pi}} \int_{\max\{ct - t^\beta, 0\}}^{\min\{ct + t^\beta, t\}} \tau^{-\gamma}(t-\tau)^{-1/2} \, d\tau .$$

Let us suppose that $t^{\beta-1} < \frac{1}{4}$ and consider the following four cases:

1) $0 \leqq c \leqq 2t^{\beta-1}$, 2) $2t^{\beta-1} < c \leqq \frac{1}{2}$, 3) $\frac{1}{2} < c \leqq 1 - 2t^{\beta-1}$,
4) $1 - 2t^{\beta-1} < c \leqq 1$.

In the case 1) we have

$$(15) \quad I_2 \leqq \frac{1}{\sqrt{\pi}} \int_0^{3t^\beta} \tau^{-\gamma}(t-\tau)^{-1/2}\, d\tau \leqq \frac{1}{\sqrt{\pi}} (t - 3t^\beta)^{-1/2} \frac{1}{1-\gamma} (3t^\beta)^{1-\gamma} = $$
$$= \frac{3^{1-\gamma}}{\sqrt{\pi}(1-\gamma)} (1 - 3t^{\beta-1})^{-1/2}\, t^{\beta(1-\gamma)-1/2} \to_{(t\to 0+)} 0\,,$$

since $\beta(1-\gamma) - \frac{1}{2} > 0$ by (11). The last term in (15) is independent of c.

In the case 2) we have

$$(16) \quad I_2 \leqq \frac{1}{\sqrt{\pi}} \int_{ct-t^\beta}^{ct+t^\beta} \tau^{-\gamma}(t-\tau)^{-1/2}\, d\tau \leqq \frac{1}{\sqrt{\pi}} t^{-\beta\gamma}(t - ct - t^\beta)^{-1/2}\, 2t^\beta =$$
$$= 2[\pi(1 - c - t^{\beta-1})]^{-1/2}\, t^{\beta-\beta\gamma-1/2} \leqq 2[\pi(\tfrac{1}{2} - t^{\beta-1})]^{-1/2}\, t^{\beta(1-\gamma)-1/2} \to_{(t\to 0+)} 0\,;$$

the last term is independent of c.

In the case 3) we have

$$(17) \quad I_2 \leqq \frac{1}{\sqrt{\pi}} \int_{ct-t^\beta}^{ct+t^\beta} \tau^{-\gamma}(t-\tau)^{-1/2}\, d\tau \leqq (ct - t^\beta)^{-\gamma} [\pi(t - (t - t^\beta))]^{-1/2}\, 2t^\beta \leqq$$
$$\leqq \frac{2}{\sqrt{\pi}} (\tfrac{1}{2} - t^{\beta-1})^{-\gamma}\, t^{\beta/2-\gamma} \to_{(t\to 0+)} 0\,;$$

it suffices to note here that if $\gamma \in \langle\frac{1}{2}, 1\rangle$ then $1/2(1-\gamma) \geqq 2\gamma$ and thus $\frac{1}{2}\beta - \gamma > 0$. The last term in (17) is independent of c.

At last we obtain in the case 4)

$$(18) \quad I_2 \leqq \frac{1}{\sqrt{\pi}} \int_{t-3t^\beta}^{t} \tau^{-\gamma}(t-\tau)^{-1/2}\, d\tau \leqq \frac{1}{\sqrt{\pi}} (t - 3t^\beta)^{-\gamma} \int_{t-3t^\beta}^{t} (t-\tau)^{-1/2}\, d\tau =$$
$$= \frac{2\sqrt{3}}{\sqrt{\pi}} (1 - 3t^{\beta-1})^{-\gamma}\, t^{\beta/2-\gamma} \to_{(t\to 0+)} 0\,.$$

The last term is also independent of c. We get immediately from (13), (15), (16), (17), (18) that

$$(19) \quad \lim_{\substack{[x,t]\to[0,0] \\ t>0,\, 0\leqq x\leqq t^\alpha}} U_\lambda(x, t) = 0\,.$$

If $x \leqq 0$, $\tau > 0$ then $(x - \tau^\alpha)^2 \geqq \tau^{2\alpha}$. Hence

$$(20) \quad U_\lambda(x, t) \leqq U_\lambda(0, t)$$

for $x \leqq 0$, $t > 0$. Similarly $(x - \tau^\alpha)^2 \geqq (t^\alpha - \tau^\alpha)^2$ for $t > \tau > 0$, $x \geqq t^\alpha$ and thus

(21) $$U_\lambda(x, t) \leqq U_\lambda(t^\alpha, t)$$

for $t > 0$, $x \geqq t^\alpha$. Finally, it follows from (20), (21) and (19) that (10) holds.

7. **Remark.** In a similar way one can easily show that the restriction $U_\lambda|_{K_\lambda}$ is continuous on K_λ (where $K_\lambda = \{[\tau^\alpha, \tau];\ \tau \in \langle 0, 1\rangle\}$) whenever $\gamma < 1 - \alpha$. We have just shown that U_λ is continuous on R^2 if $\gamma < 1 - (1/(3 - 2\alpha))$. But $1 - (1/(3 - 2\alpha)) < < 1 - \alpha$ for $\alpha < \frac{1}{2}$ and thus a question arises if the potential U_λ is continuous on R^2 in the case $1 - (1/(3 - 2\alpha)) \leqq \gamma < 1 - \alpha$. I do not know the answer.

8. **Example.** We shall show in this example that for each $\alpha < \frac{1}{2}$ $(\alpha > 0)$ there is an α-Hölder continuous function φ on $\langle 0, 1\rangle$ and a continuous measure λ on $\langle 0, 1\rangle$ such that the heat potential U_λ^φ is not continuous on R^2 while its restriction $U_\lambda^\varphi|_{K_\lambda}$ is continuous on K_λ $(K_\lambda = \{[\varphi(t), t];\ t \in \operatorname{spt} \lambda\})$. It is thus seen from this example that the constant $\frac{1}{2}$ in Theorem 5 is exact.

Choose $0 < \xi < \frac{1}{4}$ and let $D \subset \langle 0, 1\rangle$ be the standard "symmetric" set of the Cantor type obtained from the interval $\langle 0, 1\rangle$ so that the "middle" interval of the length $1 - 2\xi$ is removed in the first step, two intervals of the length $\xi(1 - 2\xi)$ are removed in the second step etc. Let φ be the corresponding Cantor function (see, for instance, [9] – under the notation used in [9] we choose $d = 1$). So D is the set of all real numbers of the form

(22) $$t = (1 - \xi) \sum_{k=1}^{\infty} i_k \xi^{k-1},$$

where $i_k = 0, 1$. For t of this form we have

(23) $$\varphi(t) = \sum_{k=1}^{\infty} \frac{i_k}{2^k}.$$

It is well known that φ is a monotonic continuous function on $\langle 0, 1\rangle$. Further, the function φ is an α-Hölder continuous function, where

(24) $$\alpha = \frac{\ln 2}{-\ln \xi} = \frac{1}{2} \frac{\ln 4}{-\ln \xi}$$

(see [9] for example). We suppose $\xi < \frac{1}{4}$ and thus $\alpha < \frac{1}{2}$ (and for any given $\alpha_1 \in \in (0, \frac{1}{2})$ one can choose $\xi < \frac{1}{4}$ such that $\alpha = \alpha_1$).

Now let m be a given integer, $m > 1$. Let us denote by D_m the set of all $t \in D$ of the form (22) such that for each integer $k \geqq 1$ there is a $\nu \in \{0, 1, \ldots, m\}$ with $i_{k+\nu} = 1$. It is easily seen that D_m is a compact uncountable set. Denote further

$$K_m = \{[\varphi(t), t];\ t \in D_m\}.$$

Thr heat kernel in R^2 can be regarded as a function on $R^2 \times R^2$ if we write

$$G_1(x, t, \xi, \tau) = G(x - \xi, t - \tau).$$

Let us take notice of the property of K_m that the restriction of the kernel G_1 on $K_m \times K_m$ is continuous (and bounded for K_m is compact). For the sake of simplicity one can consider a function H defined on $D_m \times D_m$ by .

$$H(t, \tau) = G(\varphi(t) - \varphi(\tau), t - \tau), \quad (t, \tau \in D_m).$$

Let us show that H is continuous on $D_m \times D_m$. H is clearly continuous on the set

$$\{[t, \tau] \in D_m \times D_m, t \neq \tau\}$$

(that is, outside the diagonal). It suffices to prove that H is continuous at thepoints of the form $[t_0, t_0]$, $t_0 \in D_m$. We have

$$H(t_0, t_0) = 0.$$

If $[t, \tau] \in D_m \times D_m$, $\tau \geqq t$, then $H(t, \tau) = 0$. Let $[t, \tau] \in D_m \times D_m$, $\tau < t$ and let

$$t = (1 - \xi) \sum_{k=1}^{\infty} i_k \xi^{k-1}, \quad \tau = (1 - \xi) \sum_{k=1}^{\infty} j_k \xi^{k-1}.$$

Since $\tau < t$ there is an integer k_0 such that $i_\nu = j_\nu$ for $\nu = 1, 2, \ldots, k_0 - 1$, $i_{k_0} = 1$, $j_{k_0} = 0$. Then

$$(1 - 2\xi)\, \xi^{k_0-1} \leqq t - \tau = (1 - \xi)\,(\xi^{k_0-1} + \sum_{k=k_0+1}^{\infty} (i_k - j_k)\, \xi^{k-1}) \leqq \leqq (1 - \xi) \sum_{k=k_0}^{\infty} \xi^{k-1} = \xi^{k_0-1}. \tag{25}$$

Further

$$\varphi(t) - \varphi(\tau) = \frac{1}{2^{k_0}} + \sum_{k=k_0+1}^{\infty} \frac{i_k - j_k}{2^k}.$$

There is a $\nu \in \{0, 1, \ldots, m\}$ (by the definition of D_m) such that $i_{k_0+1+\nu} = 1$ and thus $i_{k_0+1+\nu} - j_{k_0+1+\nu} \neq -1$. Hence

$$\varphi(t) - \varphi(\tau) \geqq \frac{1}{2^{k_0}} - \sum_{k=k_0+1}^{\infty} \frac{1}{2^k} + \frac{1}{2^{k_0+1+\nu}} \geqq \frac{1}{2^{k_0+m+1}}. \tag{26}$$

We obtain from (25), (26) that

$$H(t, \tau) = [\pi(t - \tau)]^{-1/2} \exp\left(-\frac{(\varphi(t) - \varphi(\tau))^2}{4(t - \tau)}\right) \leqq$$

$$\leqq [\pi(1 - 2\xi)\, \xi^{k_0-1}]^{-1/2} \exp[-(4.2^{2(k_0+m+1)}\xi^{k_0-1})^{-1}] =$$

$$= [\pi(1 - 2\xi)]^{-1/2}\, \xi^{(1-k_0)/2} \exp\left[-(4\xi)^{-k_0} \frac{\xi}{4^{m+2}}\right] \to_{(k_0 \to +\infty)} 0,$$

as $4\xi < 1$. The last term is independent of the choice of $t \in D_m$ (that term depends on k_0, that is on the distance of the points t, τ – see (25)). Now it is seen that H is continuous on $D_m \times D_m$.

Let $\lambda \in \mathscr{B}^+(\langle 0, 1\rangle)$ be arbitrary but such that $\operatorname{spt} \lambda \subset D_m$. For $t \in D_m$ we have

$$U_\lambda^\varphi(\varphi(t), t) = \int_0^1 G(\varphi(t) - \varphi(\tau), t - \tau)\, d\lambda(\tau) = \int_{D_m} H(t, \tau)\, d\lambda(\tau) = I(t)\,.$$

As the function H is continuous on $D_m \times D_m$ the integral I is continuous on D_m and so the restriction $U_\lambda^\varphi|_{K_m}$ is continuous. In other words for any measure μ in R^2 such that $\operatorname{spt} \mu \subset K_m$ the restriction $U_\mu|_{K_m}$ is continuous (this is an analogue of the trivial fact that the heat potential of any measure with support contained in the x-axis vanishes on the x-axis).

Now it suffices to find a continuous measure λ with $\operatorname{spt} \lambda \subset D_m$ for which the potential U_λ^φ is not continuous. We shall show a little more – that the heat potential U_λ^φ is discontinuous for any non-trivial measure λ on $\langle 0, 1\rangle$ with $\operatorname{spt} \lambda \subset D$.

Let $\lambda \in \mathscr{B}^+(\langle 0, 1\rangle)$, $\operatorname{spt} \lambda \subset D$ and let $\lambda(\langle 0, 1\rangle) > 0$. First we show that the following assertion holds:

There exists a constant $k > 0$ such that for each $\varepsilon > 0$ there are $t \in (0, 1)$, $0 < \delta < \varepsilon$ with $\langle t - \delta, t + \delta\rangle \subset \langle 0, 1\rangle$ such that

$$\lambda(\langle t - \delta,\ t + \delta\rangle) \geqq k\delta^\alpha$$

(α is defined by (24)).

Suppose that this assertion is not valid. Then for each $k > 0$ there is an $\varepsilon > 0$ such that for any $t \in (0, 1)$, $0 < \delta < \varepsilon$ with $\langle t - \delta, t + \delta\rangle \subset \langle 0, 1\rangle$ it holds

$$\lambda(\langle t - \delta,\ t + \delta\rangle) < k\delta^\alpha\,.$$

It is well-known that the α-dimensional Hausdorff measure of the set D is finite. It is seen from the definition of the α-dimensional Hausdorff measure that there is a constant M such that for each $\varepsilon > 0$ there are intervals $I_1, I_2, \ldots \subset \langle 0, 1\rangle$ such that $\operatorname{diam} I_\nu < \varepsilon$ $(\nu = 1, 2, \ldots)$,

$$\bigcup_{\nu=1}^{\infty} I_\nu \supset D \quad \text{and} \quad \sum_{\nu=1}^{\infty} (\operatorname{diam} I_\nu)^\alpha \leqq M\,.$$

Hence

$$\lambda(\langle 0, 1\rangle) = \lambda(D) \leqq \sum_{\nu=1}^{\infty} \lambda(I_\nu) \leqq \sum_{\nu=1}^{\infty} k \left(\frac{\operatorname{diam} I_\nu}{2}\right)^\alpha =$$

$$= k\, 2^{-\alpha} \sum_{\nu=1}^{\infty} (\operatorname{diam} I_\nu)^\alpha \leqq k\, 2^{-\alpha} M.$$

As $k > 0$ is arbitrary, we have $\lambda(\langle 0, 1\rangle) = 0$ which contradicts the assumption that the measure λ is not trivial. (Note that the mentioned assertion follows immediately from some much more general assertions concerning the so-called upper h-derivative with respect to the function $h(t) = t^\alpha$ – see, for instance, [8] or [6], ch. 3, § 3. It is perhaps of interest to note here that it may happen in the case $\alpha < 1$ that a non-trivial measure λ has its support contained in a set of zero α-dimensional Hausdorff measure but the lower h-derivative with respect to the function $h(t) = t^\alpha$ vanishes everywhere – see [8], p. 20.)

It is seen from the mentioned assertion that there are $k > 0$, $t_i \in (0, 1)$, $\delta_i > 0$ $(i = 1, 2, \ldots)$ such that $\delta_i \to 0$ for $i \to \infty$, $\langle t_i - \delta_i, t_i + \delta_i \rangle \subset \langle 0, 1 \rangle$ and

$$\lambda(\langle t_i - \delta_i, t_i + \delta_i \rangle) \geqq k\delta_i^\alpha .$$

The function φ is an α-Hölder continuous function, that is, there is a k_1 such that

$$|\varphi(t) - \varphi(\tau)| \leqq k_1 |t - \tau|^\alpha , \quad t, \tau \in \langle 0, 1 \rangle .$$

Consider i sufficiently large such that $\delta_i^{1-2\alpha} \leqq \frac{1}{2}$. For $\tau \in \langle t_i - \delta_i, t_i + \delta_i \rangle$ we then have

$$|\varphi(t_i) - \varphi(\tau)| \leqq k_1 |t_i - \tau|^\alpha \leqq k_1 \delta_i^\alpha ,$$

$$|t_i + \delta_i^{2\alpha} - \tau| \geqq \delta_i^{2\alpha} - |t_i - \tau| \geqq \delta_i^{2\alpha} - \delta_i = \delta_i^{2\alpha}(1 - \delta_i^{1-2\alpha}) \geqq \tfrac{1}{2}\delta_i^{2\alpha}$$

and hence

$$\frac{(\varphi(t_i) - \varphi(\tau))^2}{4(t_i + \delta_i^{2\alpha} - \tau)} \leqq \tfrac{1}{2}k_1^2 .$$

Further

$$|t_i + \delta_i^{2\alpha} - \tau| \leqq \delta_i^{2\alpha} + \delta_i = \delta_i^{2\alpha}(1 + \delta_i^{1-2\alpha}) \leqq \tfrac{3}{2}\delta_i^{2\alpha} .$$

We obtain from the last two inequalities that

$$G(\varphi(t_i) - \varphi(\tau), t_i + \delta_i^{2\alpha} - \tau) = [\pi(t_i + \delta_i^{2\alpha} - \tau)]^{-1/2} \exp\left(- \frac{(\varphi(t_i) - \varphi(\tau))^2}{4(t_i + \delta_i^{2\alpha} - \tau)} \right) \geqq$$

$$\geqq (\tfrac{3}{2}\pi)^{-1/2} \exp(-\tfrac{1}{2}k_1^2)\, \delta_i^{-\alpha} = k_0 \delta_i^{-\alpha}$$

for $\tau \in \langle t_i - \delta_i, t_i + \delta_i \rangle$ (if i is sufficiently large). Thus we see that

$$\{[\varphi(\tau), \tau];\ \tau \in \langle t_i - \delta_i, t_i + \delta_i \rangle\} \subset A(\varphi(t_i), t_i + \delta_i^{2\alpha}; c)$$

for each $0 < c < k_0 \delta_i^{-\alpha}$. If μ is the measure in R^2 associated with λ (with respect to φ) then we have

$$\mu(A(\varphi(t_i), t_i + \delta_i^{2\alpha}; c)) \geqq \lambda(\langle t_i - \delta_i, t_i + \delta_i \rangle) \geqq k\delta_i^\alpha$$

and so for $d > 0$

$$\int_d^\infty \mu(A(\varphi(t_i), t_i + \delta_i^{2\alpha}; c))\, \mathrm{d}c \geqq \int_d^{k_0\delta_i^{-\alpha}} k\delta_i^\alpha\, \mathrm{d}c = k\delta_i^\alpha(k_0\delta_i^{-\alpha} - d) =$$

$$= kk_0 - kd\delta_i^\alpha \to_{(i \to +\infty)} kk_0 .$$

In the end we obtain that for any $d > 0$

$$\sup \left\{ \int_d^\infty \mu(A(\varphi(t_i), t_i + \delta_i^{2\alpha}; c))\, \mathrm{d}c;\ i > 0 \text{ integer} \right\} \geqq kk_0 > 0$$

which implies that the heat potential $U_\lambda^\varphi = U_\mu$ is not continuous in R^2 (note that if $t_i \to t_0$, then the potential U_λ^φ is not continuous at the point $[\varphi(t_0), t_0]$, for instance).

Now it suffices to note that $D_m \subset D$ is an uncountable compact set and thus there are non-trivial continuous measures λ with spt $\lambda \subset D_m$ (see, for example, [6], theorem 35). It follows from the first part of this example that if λ is any measure with spt $\lambda \subset D_m$ then the restriction $U_\lambda^\varphi|_{K_\lambda}$ is continuous. On the other hand, by the second part, the potential U_λ^φ is not continuous in R^2 whenever spt $\lambda \subset D$ and λ is not trivial.

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