Elena Wisztová

Paths in powers of graph

Časopis pro pěstování matematiky, Vol. 105 (1980), No. 3, 292--301

Persistent URL: http://dml.cz/dmlcz/118072

PATHS IN POWERS OF GRAPH

ELENA WISZTOVÁ, Žilina

(Received February 7, 1978)

1. Introduction. By a graph we shall mean a finite undirected graph with no loop of multiple edge (i.e. a graph in the sense of monographs [1] or [2]). If G is a graph, then we denote by $V(G)$, $V_1(G)$, and $E(G)$ the vertex set of G, the set of vertices of degree one in G, and the edge set of G, respectively. The distance between vertices u and v of G will be denoted by $d(u, v, G)$. By the n-th power G^n of G (where $n \geqq 1$) we mean the graph with the properties that $V(G^n) = V(G)$ and that vertices u and v are adjacent in G^n if and only if $1 \leqq d(u, v, G) \leqq n$. If $n \geqq 1$ and u is a vertex of G, then we denote by $G(u, n)$ the set of vertices which are adjacent to u in G^n.

If G_1 and G_2 are graphs, then we denote by $G_1 \cup G_2$ the graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Let G be a graph. A path connecting vertices u and v in G is referred to as a $u - v$ path in G. In the present paper a path in G will be regarded as a subgraph of G. A path P in G is called *hamiltonian* if $V(P) = V(G)$. We say that G is *hamiltonian* if it contains a hamiltonian path.

Let G be a nontrivial graph. We say that it is *hamiltonian-conneceted* if for every pair of distinct vertices u and v of G, there exists a hamiltonian $u - v$ path in G. Hamiltonian properties of powers of graphs, especially of the second and third powers, were studied very intensively: see, for example, SEKANINA and CHARTRAND-KAPOOR. Some further references can be found in LESNIAK [7].

In the present paper we shall study a certain general modification of hamiltonian connectedness for higher powers of graphs.

Let G be *a* graph. For every positive integer i, we denote by $\mathscr{D}_i(G)$ the set of all ordered pairs (U_1, U_2) with the properties that U_1 and U_2 are disjoint subsets of $V(G)$, and $|U_1| = |U_2| = i$. Denote

$$\mathscr{D}(G) = \bigcup_{i=1}^{\infty} \mathscr{D}_i(G).$$

Let $(U_1, U_2) \in \mathscr{D}(G)$. We shall say that a set $\mathscr{P}$ of paths in G is a (U_1, U_2)-*path system in* G, if

(i) given $P \in \mathscr{P}$, then one end-vertex of P belongs to U_1, and the other belongs to U_2,

(ii) $|\mathscr{P}| = |U_1|$,

(iii) every vertex of G belongs to at most one path in $\mathscr{P}$. We shall say that $\mathscr{P}$ is a (U_1, U_2)-path system on G, if it is a (U_1, U_2)-path system in G, and every vertex of G belongs to at least one path in $\mathscr{P}$. Let G be a tree, and let $\mathscr{P}$ be a (U_1, U_2)-path system in (on) G^n, where $n \geqq 1$. We shall say that $\mathscr{P}$ is n-good for G if for every $P \in \mathscr{P}$ and every pair of distinct vertices v and w of P it holds that if $d(v, w, G) \leqq n$ and no $u \in V(P - v - w)$ belongs to the $v - w$ path in G, then $vw \in E(P)$.

Let G be a graph, and let i be a positive integer. We shall say that G is i-traceable if $|V(G)| \geqq 2i$ and for every $(U_1, U_2) \in \mathscr{D}_i'(G)$, there exists a (U_1, U_2)-path system on G. It is obvious that a nontrivial graph is 1-traceable if and only if it is hamiltonian-connected. In the present paper we shall prove that if G is a connected graph with at least $2i$ vertices, where $i \geqq 3$, then G^{i+1} is i-traceable. We recall four theorems which will be very useful for this purpose.

Theorem A (J.-L. Jolivet [3]). *If G is a connected graph with at least $n \geqq 1$ vertices, then G^n is n-connected.*

Theorem B (see Theorem 5.14 in Harary [1]). *A graph with at least $2n$ vertices ($n \geqq 1$) is n-connected if and only if for every $(U_1, U_2) \in \mathscr{D}_n(G)$, there exists a (U_1, U_2)-path system in G.*

Theorem C (M. Sekanina [5]). *If G is a nontrivial connected graph, then G^3 is hamiltonian-connected.*

Theorem D (M. Sekanina [6]). *Let a, b, c and be distinct vertices of a connected graph G. Then there exist a $a - b$ path P_1 in G^4 and a $c - d$ path P_2 in G^4 such that $\{P_1, P_2\}$ is a $(\{a, c\}, \{b, d\})$-path system on G^4.*

Corollary 1. *Let G be a connected graph. If $|V(G)| \geqq 2$, then G^3 is 1-traceable; if $|V(G)| \geqq 4$, then G^4 is 2-traceable.*

2. Results. We first prove five lemmas.

Lemma 1. *Let G be a connected graph with $p \geqq 2$ vertices. Then for an arbitrary pair of distinct vertices x and y of G there exists a hamiltonian $x - y$ path P in G^3 with the property that there exists $s \in G(x, 2)$ such that $xs \in E(P)$.*

Proof. We prove the lemma by using induction on p. If $p = 2$, the result is obvious. Assume that $p \geqq 3$, and that the result is proved for every nontrivial connected graph with at most $p - 1$ vertices. Let x and y be distinct vertices of G. Since G is connected, there exists a spanning tree T of G. There exists exactly one vertex r of G such that $ry \in E(T)$, and that r belongs to the $x - y$ path in T. Clearly, $T - ry$ consists of two components, say T_x and T_y, where $x \in V(T_x)$ and $y \in V(T_y)$. Obviously, at least one of the trees T_x and T_y is nontrivial.

First, let T_x be trivial. Then there exists $s \in V(T_y)$ such that $sy \in E(T_y)$. According to Theorem C there exists a hamiltonian $s - y$ path P_y in $(T_y)^3$. If we denote by P the path $P_y + xs$, then we get the result of the lemma.

Next, let T_x be nontrivial. Then there exists $t \in V(T_x - x)$ such that $t \in T(y, 2)$. By the induction assumption there exists a hamiltonian $x - t$ path P_x in $(T_x)^3$ with the property that there exists $s \in T_x(x, 2)$ such that $xs \in E(P_x)$. If T_y is trivial and we denote by P the path $P_x + ty$, then we get the result. Assume that T_y is nontrivial, and consider $z \in V(T_y)$ such that $yz \in E(T_y)$. According to Theorem C there exists a hamiltonian $z - y$ path P_y in $(T_y)^3$. Obviously, $d(t, z, T) \leqq 3$. If we denote by P the path $(P_x \cup P_y) + tz$, then we get the result of the lemma, which completes the proof.

Corollary 2. *Let G be a connected graph with at least three vertices, and let $u \in V(G)$. Then there exist vertices x_u and y_u of $G - u$ such that $x_u \in G(u, 1)$, $y_u \in G(u, 2)$, and that there exists a hamiltonian $x_u - y_u$ path in $G^3 - u$.*

Corollary 2 immediately implies the following result, which is due to Chartrand and Kapoor [4]: If G is a connected graph with at least four vertices and $u \in V(G)$, then $G^3 - u$ is hamiltonian.

Lemma 2. *Let T be a tree with at least $2i$ vertices, where $i \geqq 1$, and let $(U_1, U_2) \in \mathscr{D}_i(T)$. Then there exists a (U_1, U_2)-path system in T^i which is i-good for T.*

Proof. According to Theorem A, T^i is i-connected. From Theorem B it follows that there exists a (U_1, U_2)-path system in T^i.

Consider a (U_1, U_2)-path system $\mathscr{P}$ in T^i which has the following property: if $P \in \mathscr{P}$, then there exists no path P' such that $V(P') \subseteq V(P)$, $|V(P')| < |V(P)|$, and that $(\mathscr{P} - \{P\}) \cup \{P'\}$ is a path system in T^i. We shall show that $\mathscr{P}$ is i-good for T.

On the contrary, we assume that $\mathscr{P}$ is not i-good for T. From the definition of an i-good path system it follows that there exists $P_0 \in \mathscr{P}$ such that there exist distinct $v, w \in V(P_0)$ with the properties that $vw \notin E(P_0)$, $d(v, w, T) \geqq i$, and that no $u \in V(P_0 - v - w)$ belongs to the $v - w$ path in T. Since v and w are distinct vertices of P_0, we have that there exists a $v - w$ path Q in T^i which is a subgraph of P_0. Since $vw \notin E(P_0)$, we have $|V(Q)| \geqq 3$. We denote by P' the path $P_0 - V(Q - v - w)$. Since P' and P_0 have the same end vertices, we have that $(\mathscr{P} - \{P_0\}) \cup \{P'\}$ is a (U_1, U_2)-path system in T^i, which is a contradiction. Hence the lemma follows.

Let T be a nontrivial tree, and let $(U_1, U_2) \in \mathscr{D}(T)$. We denote by $T(U_1, U_2)$ the minimum subtree T' of T with the property that $U_1 \cup U_2 \subseteq V(T')$. Obviously, $V_1(T(U_1, U_2)) \subseteq U_1 \cup U_2$.

We shall say that T is (U_1, U_2)-primitive if there exists no $v \in V(T(U_1, U_2)) - (U_1 \cup U_2)$ with the property that each component T_0 of $T - v$ satisfies $(V(T_0) \cap U_1, V(T_0) \cap U_2) \in \mathscr{D}(T_0)$. It is obvious that if T is (U_1, U_2)-primitive, then $T(U_1, U_2)$ is also (U_1, U_2)-primitive.

Lemma 3. *Let T be a tree with at least $2i$ vertices, where $i \geqq 1$, and let $(U_1, U_2) \in \mathscr{D}_i(T)$. Assume that T is identical with $T(U_1, U_2)$, and that T is (U_1, U_2)-primitive. Then there exists a (U_1, U_2)-path system on T^i which is i-good for T.*

Proof. According to Lemma 2, there exists a (U_1, U_2)-path system $\mathscr{P}_0$ in T^i which is i-good for T.

If $\mathscr{Q}$ is a (U_1, U_2)-path system in T^i, then we denote

$$V(\mathscr{Q}) = \bigcap_{Q \in \mathscr{Q}} V(Q) .$$

Assume that $\mathscr{P}$ is a (U_1, U_2)-path system in T^i which is i-good for T, and that there exists a vertex $v \in V(T) - V(\mathscr{P})$. Since T is (U_1, U_2)-primitive, there exists a component T_1 of $T - v$ such that $(V(T_1) \cap U_1, V(T_1) \cap U_2) \notin \mathscr{D}(T_1)$. Therefore, $|V(T_1) \cap U_1| \neq |V(T_1) \cap U_2|$. Since $|U_1| = |U_2|$ there exists a component T_2 of $T - v$ such that T_2 is different from T_1 and $|V(T_2) \cap U_1| \neq |V(T_2) \cap U_2|$. This implies that there exists a path $P \in \mathscr{P}$ with the property that there exists $v_1, v_2 \in V(P)$ such that $v_1 v_2 \in E(P)$, and that v belongs to the $v_1 - v_2$ path in T. We denote by P' the path obtained from $P - v_1 v_2$ by adding the vertex v and the edges $v_1 v$ and $v v_2$. It is easy to see that $(\mathscr{P} - \{P\}) \cup \{P'\}$ is a(U_1, U_2)-path system in T^i which is i-good for T, and that $V(\mathscr{P} - \{P\}) \cup \{P'\} = V(\mathscr{P}) \cup \{v\}$.

If $V(\mathscr{P}_0) = V(T)$, then $\mathscr{P}_0$ is a (U_1, U_2)-path system on T^i. Assume that $V(\mathscr{P}_0) \neq V(T)$; if we reiterate the above procedure, then from $\mathscr{P}_0$ we can construct a (U_1, U_2)-path system on T^i which is i-good for T.

Hence the lemma follows.

Let T be a nontrivial tree, and let $(U_1, U_2) \in \mathscr{D}(T)$. If $v \in V(T(U_1, U_2))$, then we denote by $T(v, U_1, U_2)$ the component of $T - E(T(U_1, U_2))$ which contains v. Further, we denote by $m(T, U_1, U_2)$ the number of vertices $v \in V(T(U_1, U_2)) - V_1(T(U_1, U_2))$ with the property that $T(v, U_1, U_2)$ is nontrivial.

Lemma 4. *Let T be a tree with at least $2i$ vertices, where $i \geqq 3$, and let $(U_1, U_2) \in \mathscr{D}_i(T)$. Assume that T is (U_1, U_2)-primitive and that $m(T, U_1, U_2) = 0$. Then there exists a (U_1, U_2)-path system on T^{i+1}.*

Proof. We denote the tree $T(U_1, U_2)$ by S. If $v \in V_1(S)$, then we denote $T(v, U_1, U_2)$ by $T(v)$. Moreover, we denote

$$W = \{w \in V_1(S);\ T(w) \text{ is nontrivial}\} .$$

Corollary 2 implies that for every $w \in W$ there exist $x_w, y_w \in V(T(w) - w)$ such that $x_w \in T(w, 1)$, $y_w \in T(w, 2)$, and that there exists a hamiltonian $x_w - y_w$ path in $(T(w))^3 - w$, say a hamiltonian path $P(w)$. According to Lemma 3, there exists a (U_1, U_2)-path system on S^i which is i-good for S, say a (U_1, U_2)-path system $\mathscr{P}$.

We distinguish two cases:

1. There exists no $P_0 \in \mathscr{P}$ with the following properties:

(i) P_0 contains only two vertices, say a and b;

(ii) $a, b \in W$; and

(iii) $d(a, b, T) = i$.

2. There exists $P_0 \in \mathscr{P}$ with the properties (i)–(iii).

Case 1. Let P be an arbitrary path in $\mathscr{P}$, and let u and v be the end vertices of P. There exist vertices u' and v' such that $uu', vv' \in E(P)$. Obviously, P is a path in T^i. If $u \in W$, then $(P \cup P(u)) - uu' + uy_u + x_u u'$ is a path in T^{i+1}. Let $u, v \in W$; then either $|V(P)| \geqq 3$ or $d(u, v, T) < i$; this means that $(P \cup P(u) \cup P(v)) - uu' - vv' + uy_u + x_u u' + v'x_v + y_v v$ is a path in T^{i+1}. This observation yields that the paths of $\mathscr{P}$ can be extended to a (U_1, U_2)-path system on T^{i+1}.

Case 2. Without loss of generality we assume that $a \in U_1$ and $b \in U_2$. We denote by Z the set of all vertices of the $a - b$ path in T which do not belong to $U_1 \cup U_2$. Since S is (U_1, U_2)-primitive, we have that there exists no $x \in V(S - a - b) - ((U_1 - \{a\}) \cup (U_2 - \{b\})) - Z$ such that every component S_0 of $S - a - b - x$ satisfies $(V(S_0) \cap U_1,\ V(S_0) \cap U_2) \in \mathscr{D}(S_0)$. Consider an arbitrary vertex $c \in Z$. We denote by S_a or S_b the component of $S - c$ which contains a or b, respectively. Assume that c has the following properties:

(1) Every component $S_0 \neq S_a, S_b$ of $S - c$ satisfies

$$(V(S_0) \cap U_1,\ V(S_0) \cap U_2) \in \mathscr{D}(S_0)$$

(2) either

$$|V(S_a) \cap U_1| = |V(S_a) \cap U_2| + 1\,,$$
$$|V(S_b) \cap U_1| = |V(S_b) \cap U_2| - 1$$

or

$$|V(S_a) \cap U_1| = |V(S_a) \cap U_2| - 1\,,$$
$$|V(S_b) \cap U_1| = |V(S_b) \cap U_2| + 1\,.$$

Then every component S_0' of $S - a - b - c$ satisfies

$$(V(S_0') \cap U_1,\ V(S_0') \cap U_2) \in \mathscr{D}(S_0')\,.$$

We denote by Z' the set of all $c \in Z$ which have the properties (1) and (2). Moreover, we denote $Z_0 = Z' \cup \{a, b\}$. Then every component S' of $S - Z_0$ satisfies

$$(U_1 \cap V(S'),\ U_2 \cap V(S')) \in \mathscr{D}(S')\,,$$

$$S' \text{ is } (U_1 \cap V(S'),\ U_2 \cap V(S'))\text{-primitive}$$

$$\text{and } S' \text{ is identical with } S'(U_1 \cap V(S'),\ U_2 \cap V(S'))\,.$$

According to Lemma 3, for each component S' of $S - Z_0$ there exists a $(U_1 \cap V(S'), U_2 \cap V(S'))$-path system $\mathscr{P}_{S'}$ on $(S')^{i-1}$ which is $(i-1)$-good for S'. Denote

$$\mathscr{P}_0 = \bigcup \mathscr{P}_{S'}\,, \quad \text{over all components } S' \text{ of } S - Z_0\,.$$

Subcase 2.1. Let $|Z_0| \geqq 3$. Then there exists an $a - b$ path P_0 in T^{i-1} such that $V(P_0) = Z_0$ and that $\mathscr{P}_0 \cup \{P_0\}$ is a (U_1, U_2)-path system on S^{i-1} which is $(i-1)$-

good for S. If we denote $\mathscr{P} = \mathscr{P}_0 \cup \{P_0\}$, we have a (U_1, U_2)-path system on S^i which is i-good for S and which fulfils the condition of Case 1.

Subcase 2.2. Let $|Z_0| < 3$. Then $Z_0 = \{a, b\}$. We denote by P_0 the graph with $V(P_0) = \{a, b\}$ and $E(P_0) = \{ab\}$. It is clear that $S - a - b$ has exactly one component. This implies that $\mathscr{P}_0$ is a $(U_1 - \{a\}, U_2 - \{b\})$-path system on $(S - a - b)^{i-1}$ which is $(i - 1)$-good for $S - a - b$ (and therefore for S). Denote $\mathscr{P}' = \mathscr{P}_0 \cup \{P_0\}$.

Subcase 2.2.1. Assume that there exists $P_1 \in \mathscr{P}' - \{P_0\}$ with the property that at least two vertices of P_1, say vertices v and w, belong to the $a - b$ path in S. We can assume that $d(a, v, S) < d(a, w, S)$; for an illustration see Fig. 1. Obviously,

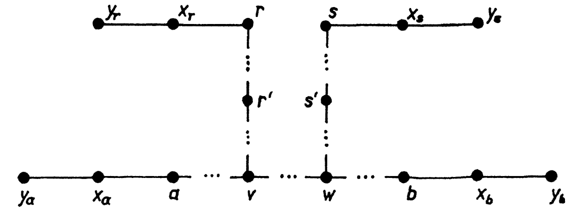

Fig. 1.

$d(v, w, S) \leqq i - 2$. Since $\mathscr{P}_0$ is $(i - 1)$-good for S, we have that $vw \in E(P_1)$. Let r and s be the end vertices of P_1. There exist vertices r' and s' such that rr' and ss' are edges of P_1. Without loss of generality we assume that if $s \in W$, then $r \in W$. We denote by $\bar{P}_0$ the path

$$(P_0 \cup P(a)) + ay_a + x_a b$$

and by $\bar{P}_1$ the path

$$(P_1 \cup P(b)) + vx_b + y_b w\,, \quad \text{if} \quad r, s \notin W,$$

$$(P_1 \cup P(b) \cup P(r)) + vx_b + y_b w + ry_r + x_r r'\,, \quad \text{if} \quad r \in W, \quad s \notin W,$$

$$(P_1 \cup P(b) \cup P(r) \cup P(s)) + vx_b + y_b w + x_r r' + ry_r + sy_s + x_s s'\,, \quad \text{if} \quad r, s \in W.$$

It is easy to see that both $\bar{P}_0$ and $\bar{P}_1$ are paths in T^{i+1}. If we continue for the paths in $\mathscr{P}' - \{P_0, P_1\}$ as in Case 1, we can extend $\mathscr{P}'$ to a (U_1, U_2)-path system, say $\bar{\mathscr{P}}$, on T^{i+1} such that $\bar{P}_0, \bar{P}_1 \in \bar{\mathscr{P}}$.

Subcase 2.2.2. Assume that for every $P \in \mathscr{P}' - \{P_0\}$ at most one vertex of P belongs to the $a - b$ path in S. Since $d(a, b, S) = i$, we have that for every $P \in \mathscr{P}' - \{P_0\}$ exactly one vertex of P belongs to the $a - b$ path in S. Since $|V(S)| \geqq 2i$, there exists $v \in V(S)$ which is adjacent to a vertex on the $a - b$ path in S, say a vertex z. Since $a, b \in V_1(S)$, we have that $a \neq z \neq b$. There exists $P_1 \in \mathscr{P}' - \{P_0\}$ such that $v \in V(P_1)$. Obviously, there exists exactly one vertex $w \in V(P_1)$ which belongs to the $a - b$ path in S. Without loss of generality we assume that $d(a, z, S) \leqq d(a, w, S)$. We have that $d(v, w, S) \leqq i - 1$. Since $\mathscr{P}_0$ is $(i - 1)$-good for S, we have that $vw \in E(P_1)$. Obviously, $2 \leqq d(a, v, S) \leqq i$ and $d(y_a, w, S) \leqq i + 1$. Assume that $v \in W$ (see Fig. 2).

If $d(a, v, S) < i$, then $d(x_a, x_v, S) \leqq i + 1$.

If $d(a, v, S) = i$, then $z = w$, and therefore $d(x_v, x_b, S) = 4 \leqq i + 1$ and $d(w, y_b, S) = 3$.

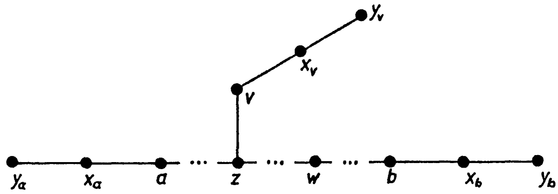

Fig. 2.

Let r and s be the end vertices of P_1. The above observation shows that there exist an $a - b$ path P_0^* in T^{i+1} and a $r - s$ path P_1^* in T^{i+1} such that $V(P_0^*) \cap V(P_1^*) = \emptyset$ and

$$V(P_0^*) \cup V(P_1^*) = V(P_1) \cup V(T(a)) \cup V(T(b)) \cup V(T)r)) \cup V(T(s)).$$

If we continue for the paths in $\mathscr{P}' - \{P_0, P_1\}$ as in Case 1, we can extend $\mathscr{P}'$ to a (U_1, U_2)-path system, say $\mathscr{P}^*$, on T^{i+1} such that $P_0^*, P_1^* \in \mathscr{P}^*$.

If T is a tree with at least six vertices, then we denote

$$\mathscr{D}^*(T) = \bigcup_{i=3}^{\infty} \mathscr{D}_i(T).$$

Lemma 5. *Let T be a tree with at least six vertices, and let $(U_1, U_2) \in \mathscr{D}^*(T)$. Assume that T is (U_1, U_2)-primitive. Then there exists a (U_1, U_2)-path system on T^{i+1}, where $i = |U_1|$.*

Proof. If $m(T, U_1, U_2) = 0$, then the result follows immediately from Lemma 4. Let $m(T, U_1, U_2) \geqq 1$. We shall assume that for every tree T' with at least six vertices and for every $(U_1', U_2') \in \mathscr{D}^*(T')$ such that T' is (U_1', U_2')-primitive and that $m(T', U_1', U_2') < m(T, U_1, U_2)$ there exists a (U_1', U_2')-path system on $(T')^{i'+1}$, where $i' = |U_1'|$.

Since $m(T, U_1, U_2) \geqq 1$, there exists $u \in V(T(U_1, U_2)) - V_1(T(U_1, U_2))$ with the property that $T(u, U_1, U_2)$ is nontrivial. We denote by S the graph $T - V(T(u, U_1, U_2) - u)$. Obviously, S is a tree, $(U_1, U_2) \in \mathscr{D}^*(S)$, and S is (U_1, U_2)-primitive. Since $m(S, U_1, U_2) = m'T, U_1, U_2) - 1$, the induction assumption implies that there exists a (U_1, U_2)-path system, say $\mathscr{Q}$, on S^{i+1}. Let Q_0 be the path in $\mathscr{Q}$ with the property that u belongs to Q_0. We distinguish the following two cases:

1. There exists $Q \in \mathscr{Q} - \{Q_0\}$ with the property that there exist distinct $v, w \in V(Q)$ such that $vw \in E(Q)$ and u belongs to the $v - w$ path in S.

2. There exists no $Q \in \mathscr{Q} - Q_0$ with the above property.

Case 1. Corollary 2 implies that there exist $x_u, y_u \in V(T(u, U_1, U_2) - u)$ such that $x_u \in T(u, 1)$, $y_u \in T(u, 2)$, and that there exists a hamiltonian $x_u - y_u$ path, say P, in $(T(u, U_1, U_2) - u)^3$. Since $d(v, w, S) \leqq i + 1$ and $Q \neq Q_0$ we have that $d(v, u, S) \leqq i$ and $d(w, x_u, S) \leqq i + 1$.

If $d(v, u, S) < i$, then $d(v, y_u, S) \leqq i + 1$, and we denote by Q' the path $((Q - vw) \cup P) + vy_u + wx_u$. If $d(v, u, S) = i$, then $uw \in E(S)$, $d(v, x_u, S) = i + 1$ and $d(y_u, w, S) \leqq 3 \leqq i$, and we denote by Q' the path $((Q - vw) \cup P) + vx_u + wy_u$.

It is clear that Q' is a path in T^{i+1}. Therefore, $(\mathcal{Q} - \{Q\}) \cup \{Q'\}$ is a (U_1, U_2)-path system on T^{i+1}.

Case 2. We denote by u_1 and u_2 the end vertices of Q_0 such that $u_1 \in U_1$ and $u_2 \in U_2$. Divide the tree S into two nontrivial trees S_1 and S_2 such that

(i) S is identical with $S_1 \cup S_2$,

(ii) $V(S_1) \cap V(S_2) = \{u\}$,

(iii) $u \in V_1(S_1)$, and

(iv) $u_1 \in V(S_1)$ and $u_2 \in V(S_2)$.

We denote by T_1 the tree $S_1 \cup T(u, U_1, U_2)$. Clearly, T is identical with $T_1 \cup S_2$. Since there exists no path $Q \in \mathcal{Q} - \{Q_0\}$ with the property defined in Case 1, we conclude that for every $Q \in \mathcal{Q} - \{Q_0\}$ either $V(Q) \subseteq V(T_1)$ or $V(Q) \subseteq V(S_2)$. Denote:

$$U_{11} = U_1 \cap V(T_1)\,,$$
$$U_{12} = (U_2 \cap V(T_1)) \cup \{u\}\,,$$
$$U_{21} = (U_1 \cap V(S_2)) \cup \{u\}\,,$$
$$U_{22} = U_2 \cap V(S_2)\,.$$

Obviously, $(U_{11}, U_{12}) \in \mathcal{D}(T_1)$ and $(U_{21}, U_{22}) \in \mathcal{D}(S_2)$. It is easy to see that T_1 is (U_{11}, U_{12})-primitive, S_2 is (U_{21}, U_{22})-primitive.

Since $u \in V(T_1(U_{11}, U_{12})) \cap V(S_2(U_{21}, U_{22}))$, we have that $m(T_1, U_{11}, U_{12}) < m(T, U_1, U_2)$ and $m(S_2, U_{21}, U_{22}) < m(T, U_1, U_2)$.

Obviously, $\max(4, |U_{11}| + 1, |U_{21}| + 1) \leqq i + 1$. Combining the induction assumption and Corollary 2, we get that there exists a (U_{11}, U_{12})-path system $\mathcal{P}_1$ on $(T_1)^{i+1}$ and a (U_{21}, U_{22})-path system $\mathcal{P}_2$ on $(S_2)^{i+1}$. Let $P_1 \in \mathcal{P}_1$ and $P_2 \in \mathcal{P}_2$ be the paths with the property that $u \in V(P_1) \cap V)P_2)$. Since T is identical with $T_1 \cup S_2$ and $V(T_1) \cap V(S_2) = \{u\}$, we have that

$$(\mathcal{P}_1 - \{P_1\}) \cup (\mathcal{P}_2 - \{P_2\}) \cup \{(P_1 \cup P_2)\}$$

is a (U_1, U_2)-path system on T^{i+1}, which completes the proof.

Now, we can state the main result of the present paper.

Theorem 1. *Let $i \geqq 3$ and let G be a connected graph with at least $2i$ vertices. Then G^{i+1} is i-traceable.*

Proof. Since G is connected, it is spanned a tree T. Let $(U_1, U_2) \in \mathscr{D}_i(T)$. It is sufficient to prove that there exists a (U_1, U_2)-path system on T^{i+1}.

It is easy to see that there exist vertex-disjoint subtrees $T_1, \ldots, T_k$ of T, where $k \geqq 1$, such that $V(T) = V(T_1) \cup \ldots \cup V(T_k)$ and, for every $j = 1, \ldots, k$,

$$(V(T_j) \cap U_1,\ V(T_j) \cap U_2) \in \mathscr{D}(T_j) \quad \text{and}$$

$$T_j \quad \text{is} \quad (V(T_j) \cap U_1,\ V(T_j) \cap U_2)\text{-primitive}\,.$$

Since $i \geqq 3$, we have that

$$\max(4, |V(T_1) \cap U_1| + 1, \ldots, |V(T_k) \cap U_1| + 1) \leqq i + 1\,.$$

Combining Corollary 2 and Lemma 5, we get that for every $j = 1, \ldots, k$ there exists a $(V(T_j) \cap U_1, V(T_j) \cap U_2)$-path system, say $\mathscr{P}_j$, on $(T_j)^{i+1}$. This means that $\mathscr{P}_1 \cup \ldots \cup \mathscr{P}_k$ is a (U_1, U_2)-path system on T^{i+1}. Hence the theorem follows.

Remark 1. G^{i+1} in Theorem 1 cannot be replaced by G^i. For example, if G is the graph in Fig. 3 and U_1 and U_2 are the sets of vertices denoted by 1 and 2, respectively, then there exists no (U_1, U_2)-path system on G^i.

Fig. 3.

Remark 2. According to Corollary 2, if G is a connected graph with at least four vertices, then G^4 is 2-traceable. This power cannot be decreased. For example, if G is the graph in Fig. 4 and U_1 and U_2 are the sets of vertices denoted by 1 and 2, respectively, then there exists no (U_1, U_2)-path system on G^3.

In the end of the present paper we shall prove two results concerning 2-traceable graphs.

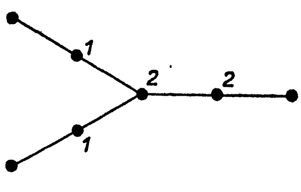

Fig. 4.

Theorem 2. *Let G be a 2-traceable graph with at least five vertices. Then G is 3-connected.*

Proof. On the contrary, we assume that G is not 3-connected. Since $|V(G)| > 3$, there exists a set U_1 of two vertices of G such that $G - U_1$ is disconnected. Let G'

be a component of $G - U_1$ with the minimum number of vertices. Since $|V(G)| \geqq 5$, we have that $|V(G) - U_1 - V(G')| \geqq 2$. Consider an arbitrary two-element subset U_2 of $V(G) - U_1 - V(G')$. Let $v \in V(G')$. It is obvious that in G the vertex v is separated from U_2 by U_1. This implies that there exists no (U_1, U_2)-path system on G, which is a contradiction. Hence the theorem follows.

Theorem 3. *Let G be a 2-traceable graph with at least five vertices. Then G is hamiltonian-connected.*

Proof. According to Theorem 2, G is 3-connected. Let u and v be distinct vertices of G. Since $G - u - v$ is connected, there exist distinct vertices a and b of $G - u - v$ such that $ab \in E(G)$. Since G is 2-traceable, there exists a $(\{u, v\}, \{a, b\})$-path system on G. Without loss of generality we assume that there exist a $u - a$ path P_1 and a $v - b$ path P_2 such that $V(P_1) \cap V(P_2) = \emptyset$ and $V(P_1) \cup V(P_2) = V(G)$. This means that $(P_1 \cup P_2) + ab$ is a hamiltonian $u - v$ path in G. Hence the theorem follows.

Remark 3. The cycle with exactly four vertices is 2-traceable but not hamiltonian-connected.

Acknowledgement. The author wishes to express her gratitude to M. SEKANINA and to L. NEBESKÝ for their helpful comments during the preparation of this paper.

References

[1] *F. Harary:* Graph Theory. Addison-Wesley, Reading (Mass.) 1969.
[2] *M. Behzad* and *G. Chartrand:* Introduction to the Theory of Graphs. Allyn and Bacon, Boston 1971.
[3] *Jean-Loup Jolivet:* Sur les puissances des graphes connexes. C.R. Acad. Sc. Paris, t. *272*, p. 107—109 (1971), Serie A.
[4] *G. Chartrand* and *S. F. Kapoor:* The cube of every connected graph is 1-hamiltonian. J. Res. Nat. Bur. Stand. *73* B (1969) 47—48.
[5] *M. Sekanina:* On an ordering of the of vertices of a connected graph. Publ. Fac. Sci. Univ. Brno *412* (1960), 137—142.
[6] *M. Sekanina:* Private communication.
[7] *L. Lesniak-Foster:* Some recent results in hamiltonian graphs. Journal of Graph Theory, vol. *1*, 27—36 (1977).

Author's address: 010 88 Žilina, Marxa-Engelsa 25 (Fakulta SET, Vysoká škola dopravná).