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Časopis pro pěstování matematiky, Vol. 104 (1979), No. 2, 185--187

Persistent URL: `http://dml.cz/dmlcz/118015`

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A GEOMETRICAL CHARACTERIZATION OF THE PROJECTIVE PLANE OF ORDER 4

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(Received December 27, 1976)

By a hexagon in an arbitrary projective plane we mean a sextuple $(P_1, P_2, \ldots, P_6)$ of disctinct points such that the lines $P_iP_{i+1}, P_{i+2}P_{i+3}$ subscripts mod 6 – are distinct. The lines P_iP_{i+1} and $P_{i+3}P_{i+4}$ are pairs of opposite sides. A hexagon is pascalian if each two opposite sides intersect at a point collinear with the vertices of the hexagon which do not belong to these sides.

A k-arc is a set of k points no three of which are collinear (cf. [1]).

Definition. We call a 6-arc *fully pascalian* if each hexagon determined by its points is pascalian.

One of the possible formulations of the Pappus-Pascal proposition has the following form:

(P) *An arbitrary hexagon with two pairs of opposite sides intersecting at a point collinear with the vertices not belonging to these sides is pascalian.*

We shall formulate the following stronger proposition:

(P*) *Each 6-arc is fully pascalian.*

The aim of this paper is to prove that the only projective plane satisfying non-trivially the proposition (P*) is the plane of order 4.

Theorem 1. *If in a projective plane π the proposition* (P*) *is non-trivially satisfied, then π is a Fano plane.*

Proof. If in a projective plane π there exists a 6-arc, then the order of π is greater than 3.

Let $(A_1, \ldots, A_4)$ be an arbitrary quadrangle in π and let D_1, D_2, D_3 be its diagonal points. The set of vertices of this quadrangle is a 4-arc and can be completed to a 6-arc $(A_1, A_2, \ldots, A_6)$ (see [1]). Since this 6-arc is fully pascalian we have that the hexagons

$(A_1, A_2, A_5, A_3, A_4, A_6)$; $(A_1, A_3, A_5, A_2, A_4, A_6)$; $(A_1, A_4, A_5, A_2, A_3, A_6)$ are pascalian. Therefore we obtain successively that the points D_3, D_2, D_1 are collinear with A_5, A_6. This implies the collinearity of the points D_1, D_2, D_3.

Theorem 2. *A 6-arc is fully pascalian if and only if any its partition onto two 3-arcs results in two 3-arcs in six-fold perspective.*

Proof. If a 6-arc A is fully pascalian, then for any its partition onto three disjoint pairs of points, the joins of these pairs are concurrent. This property is equivalent to the fact that by an arbitrary partition of A onto two disjoint 3-arcs one obtains two 3-arcs in six-fold perspective, i.e. each pair of triples determined by the two 3-arcs is in perspective. This equivalence implies the desired conclusion.

Theorem 3. *If in a projective plane π the proposition* (P*) *is satisfied then π is a finite plane of order $n \leqq 4$.*

Proof. Let d be any line in the plane π satisfying (P*) and let D_1, D_2 be two different points on d.

There exist four different lines d_i $(i = 1, \ldots, 4)$ satisfying the conditions $d_1 \cap d_2 = D_1$, $d_3 \cap d_4 = D_2$, $d_i \neq d$, $i = 1, \ldots, 4$. Let us denote $A_1 = d_1 \cap d_4$, $A_2 = d_3 \cap d_1$, $A_3 = d_2 \cap d_4$, $A_4 = d_2 \cap d_3$, $D_3 = A_1A_2 \cap A_3A_4$.

It is easy to prove that the points $A_1, \ldots, A_4$ are vertices of a certain quadrangle with D_1, D_2, D_2 as its diagonal points.

Let A_5, A_6, A_7 be points of the line d different from D_i $(i = 1, 2, 3)$. Assume that $A_5 \neq A_6, A_7$. Then the sets of points $\{A_1, A_2, \ldots, A_6\}$ and $\{A_1, A_2, \ldots, A_5, A_7\}$ are fully pascalian 6-arcs. From the definition we obtain that the triples of lines A_1A_5, A_2A_4, A_3A_6; A_1A_5, A_2A_4, A_3A_7; A_1A_2, A_4A_5, A_3A_6; A_1A_2, A_4A_5, A_3A_7 are concurrent.

Let $B = A_1A_5 \cap A_2A_4$, $C = A_1A_2 \cap A_4A_5$. Then $B \neq C$ and $BC = A_3A_6 = A_3A_7$ i.e. $A_6 = A_7 = BC \cap d$.

Thus we obtain that the line d has at most 5 points.

Theorem 4. *The proposition* (P*) *is non-trivially satisfied in the projective plane of order* 4.

Proof. The desired conclusion follows from Theorem 2 of the present paper and from Theorem 3 in [3] which implies that every partition of a 6-arc on to two 3-arcs gives a pair of triangles in six-fold homology.

Theorem 5. *The proposition* (P*) *is non-trivially satisfied in the projective plane π if and only if π is a finite plane of order* 4.

Proof. The conclusion is implied by Theorems 3 and 4 and by the fact that there is no six-arc in the plane of order less than 4.

Using the results of the present paper and of [3], [4] we can easily obtain:

Corollary 1. *If in the desarguesian projective plane there exists a fully pascalian 6-arc then this plane has a finite subplane of order* 4.

Corollary 2. *In every desarguesian projective plane, a fully pascalian* 6*-arc can be completed to at least one* (H-T)*-configuration.*

In [4] and in the present paper we have examined certain configurations mentioned in [2] or [3] as satisfied in the finite plane of order 4. We have analysed the classes of projective planes corresponding to these configurations. The configuration (H-T) is correlated with a large class of projective planes; the proposition (P*) characterizes completely the finite plane of order 4.

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