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Časopis pro pěstování matematiky, Vol. 104 (1979), No. 2, 180--184

Persistent URL: http://dml.cz/dmlcz/118014

Časopis pro pěstování matematiky, roč. 104 (1979), Praha

SOME REMARKS ON HAVLÍČEK-TIETZE CONFIGURATION

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(Received December 27, 1976)

It is well known that in some projective planes there exist pairs of manifold homologic triangles. A particular case is a six-fold homology, when the triangles are homologic in each permutation of their vertices. Pairs of six-fold homologic triangles exist for example in the projective plane over the field of complex numbers.

In [5] K. HAVLÍČEK and J. TIETZE have proved the following theorem:

In the finite projective plane of order 4 *there exists a set of four triangles, no two of them with a common vertex, and each two of them in six-fold homology. The centres of homologies of any two triangles are the vertices of the other two, the axes being the corresponding opposite sides.*

The configuration described in this theorem will be called the configuration of Havlíček-Tietze, shortly (H-T).

Moreover, the authors of the cited paper proved that, in the projective plane of order 4, the lines of any configuration (H-T) meet altogether in the nine points which complete the plane.

Obviously, the projective plane of order 4 is not the only plane containing (H-T). Every projective plane with a finite subplane of order 4 contains this configuration.

In this paper we consider the problem of existence of the (H-T) configuration. In detail, we present a necessary and sufficient condition of existence of the above configuration in desarguesian planes.

Definition. We say that a pair of triangles T_1, T_2 is *special six-fold homologic* if:

1. T_1, T_2 have no common vertex,
2. T_1, T_2 are in six-fold homology,
3. the centres of homologies are vertices of a new pair of triangles T_3, T_4,
4. if a centre of any arbitrary homology of triangles T_1, T_2 is one of the vertices of a triangle T_i, $i = 3, 4$, then the axis of this homology is incident with the remaining vertices of T_i.

Theorem 1. *If in an arbitrary projective plane π there exists a pair of special six-fold homologic triangles, then π contains an* (H-T) *configuration.*

Proof. Let us assume that a pair of triangles $T_1 = \{A_1, A_2, A_3\}$ and $T_2 = \{B_1, B_2, B_3\}$ satisfies the conditions of the above definition. We define successively six points $C_1, C_2, C_3, D_1, D_2, D_3$ as centres of homologies of the following triples

$$(1)\ \begin{pmatrix} A_1 & A_2 & A_3 \\ B_3 & B_2 & B_1 \end{pmatrix}, \quad (2)\ \begin{pmatrix} A_1 & A_2 & A_3 \\ B_2 & B_1 & B_3 \end{pmatrix}, \quad (3)\ \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_3 & B_2 \end{pmatrix}$$

$$(4)\ \begin{pmatrix} A_1 & A_2 & A_3 \\ B_2 & B_3 & B_1 \end{pmatrix}, \quad (5)\ \begin{pmatrix} A_1 & A_2 & A_3 \\ B_3 & B_1 & B_2 \end{pmatrix}, \quad (6)\ \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix}.$$

Let $T_3 = \{C_1, C_2, C_3\}$, $T_4 = \{D_1, D_2, D_3\}$. It is easy to show that no three points from A_1, A_2, A_3, B_1, B_2, B_3 are collinear, and hence no pair of triangles from $T_1, \ldots, T_4$ has a common vertex. The vertices of triangles $T_1, \ldots, T_4$ determine 21 lines presented below:

$$\begin{array}{llll} a_1 = A_2A_3 & b_1 = B_2B_3 & c_1 = C_2C_3 & d_1 = D_2D_3 \\ a_2 = A_1A_3 & b_2 = B_1B_3 & c_2 = C_1C_3 & d_2 = D_1D_3 \\ a_3 = A_1A_2 & b_3 = B_1B_2 & c_3 = C_1C_2 & d_3 = D_1D_2 \end{array}$$

$$\begin{array}{lll} w_1 = A_1B_1 & w_4 = A_2B_1 & w_7 = A_3B_1 \\ w_2 = A_1B_3 & w_5 = A_2B_3 & w_8 = A_3B_3 \\ w_3 = A_1B_2 & w_6 = A_2B_2 & w_9 = A_3B_2 \,. \end{array}$$

Considering successively the homologies of triples (1), (2), ..., (6) we obtain the following incidence conditions:

$$\begin{array}{lllll} C_3, D_3 \in w_1\,, & C_1, D_2 \in w_2\,, & C_2, D_1 \in w_3\,, & C_2, D_2 \in w_4\,, & C_3, D_1 \in w_5 \\ C_1, D_3 \in w_6\,, & C_1, D_1 \in w_7\,, & C_2, D_3 \in w_8\,, & C_3, D_2 \in w_9\,. & \end{array}$$

From the conditions considered above we obtain that the lines a_i, b_i, c_i, d_i, w_i, $i = 1, 2, 3$, are all different.

We define nine points:

$$\begin{array}{lll} W_1 = a_1 \cap b_1 & W_4 = a_2 \cap b_1 & W_7 = a_3 \cap b_1 \\ W_2 = a_1 \cap b_2 & W_5 = a_2 \cap b_2 & W_8 = a_3 \cap b_2 \\ W_3 = a_1 \cap b_3 & W_6 = a_2 \cap b_3 & W_9 = a_3 \cap b_3 \,. \end{array}$$

The centre of homology of the triples (1) is C_1 and $c_1 = C_2C_3$ is its axis. Thus $W_5, W_7, W_3 \in c_1$.

Similarly, from the homologies of triples (2), (3), ..., (6) we have: $W_2, W_4, W_9 \in c_2$; $W_1, W_6, W_8 \in c_3$; $W_2, W_6, W_7 \in d_1$; $W_3, W_4, W_8 \in d_2$; $W_1, W_5, W_9 \in d_3$.

Therefore, from the incidences described above we obtain the following scheme (S):

$$
\begin{array}{ll}
a_1 = \{A_2, A_3, W_1, W_2, W_3, \ldots\} & b_1 = \{B_2, B_3, W_1, W_4, W_7, \ldots\} \\
a_2 = \{A_1, A_3, W_4, W_5, W_6, \ldots\} & b_2 = \{B_1, B_3, W_2, W_5, W_8, \ldots\} \\
a_3 = \{A_1, A_2, W_7, W_8, W_9, \ldots\} & b_3 = \{B_1, B_2, W_3, W_6, W_9, \ldots\} \\
c_1 = \{C_2, C_3, W_3, W_5, W_7, \ldots\} & d_1 = \{D_2, D_3, W_2, W_6, W_7, \ldots\} \\
c_2 = \{C_1, C_3, W_2, W_4, W_9, \ldots\} & d_2 = \{D_1, D_3, W_3, W_4, W_8, \ldots\} \\
c_3 = \{C_1, C_2, W_1, W_6, W_8, \ldots\} & d_3 = \{D_1, D_2, W_1, W_5, W_9, \ldots\}
\end{array}
$$

$$
\begin{array}{lll}
w_1 = \{A_1, B_1, C_3, D_3, \ldots\} & w_4 = \{A_2, B_1, C_2, D_2, \ldots\} & w_7 = \{A_3, B_1, C_1, D_1, \ldots\} \\
w_2 = \{A_1, B_3, C_1, D_2, \ldots\} & w_5 = \{A_2, B_3, C_3, D_1, \ldots\} & w_8 = \{A_3, B_3, C_2, D_3, \ldots\} \\
w_3 = \{A_1, B_2, C_2, D_1, \ldots\} & w_6 = \{A_2, B_2, C_1, D_3, \ldots\} & w_9 = \{A_3, B_2, C_3, D_2, \ldots\} .
\end{array}
$$

All lines of (S) are different, hence all the points of (S) are also different. From (S) we can easily verify that the set of triangles $T_1, \ldots, T_4$ realizes an (H-T) configuration (see [5]).

Theorem 2. *If in a projective Fano plane π there exsits an* (H-T) *configuration, then π has a finite subplane of order* 4.

Proof. By examining quadrangles (W_4, W_5, W_7, W_9), (W_4, W_5, W_8, W_9), (W_4, W_5, W_7, W_8), (W_1, W_2, W_7, W_9), (W_1, W_2, W_8, W_9), (W_1, W_2, W_7, W_8), (W_1, W_2, W_4, W_6), (W_1, W_2, W_5, W_6), (W_1, W_2, W_4, W_5), we obtain successively $W_1 \in w_1$; $W_2 \in w_3$; $W_3 \in w_2$; $W_4 \in w_4$; $W_5 \in w_6$; $W_6 \in w_5$; $W_7 \in w_7$; $W_8 \in w_9$; $W_9 \in w_8$. It follows from (S) that the vertices of triangles $T_1, \ldots, T_4$ with the points $W_1, W_2, \ldots W_9$ and appropriate 5-point lines form the set of all points and lines of the projective plane of order 4 (cf. [4], [5]).

Remark. In the projective Fano plane, the points $W_1, W_2, \ldots, W_9$ and the respective three-point lines determined by an (H-T) configuration from the affine plane of order 3.

Now, we shall consider the problem of existence of an (H-T) configuration on desarguesian projective planes.

It is well known that an arbitrary projective plane is desarguesian iff it is isomorphic with a projective plane over a field F, not necessarily commutative.

Theorem 3. *Let π be an arbitrary desarguesian projective plane. Then in π there exist a pair of triangles with no common vertex, and in six-fold homology if and only if π is isomorphic with a plane over a field F containing a root of the polynomial $x^2 + x + 1$ different from* 1.[1])

---

[1]) This implies that in a projective plane over any field with the characteristic 3 there exist no pairs of six-fold homologic triangles.

Proof. Let π be a projective plane over a field F containing a root of the polynomial $x^2 + x + 1$ different from 1. By means of an element $a \in F$ such that $a \neq 0, 1$ and $a^2 + a + 1 = 0$ we define the pair of triangles $T_1 = \{A_1, A_2, A_3\}$, $T_2 = \{B_1, B_2, B_3\}$ where

$$A_1 = (1, 0, 0) \quad A_2 = (0, 1, 0) \quad A_3 = (0, 0, 1)$$
$$B_1 = (a, 1, 1) \quad B_2 = (1, a, 1) \quad B_3 = (1, 1, a).$$

One can easily show that T_1 and T_2 have no common vertex and are in six-fold homology. The centres and axes of these six homologies are the points

$$C_1 = (a, 1, a) \quad D_1 = (1, a^2, a)$$
$$C_2 = (a, a, 1) \quad D_2 = (a^2, 1, a)$$
$$C_3 = (1, a, a) \quad D_3 = (1, 1, 1)$$

and lines

$$c_1 : x_1 + ax_2 + x_3 = 0 \quad d_1 : a^2x_1 + x_2 + ax_3 = 0$$
$$c_2 : x_1 + x_2 + ax_3 = 0 \quad d_2 : a^2x_1 + ax_2 + x_3 = 0$$
$$c_3 : ax_1 + x_2 + x_3 = 0 \quad d_3 : x_1 + x_2 + x_3 = 0.$$

Conversely, let $T_1' = \{A_1', A_2', A_3'\}$, $T_2' = \{B_1', B_2', B_3'\}$ be two triangles with no common vertex in the projective desarguesian plane over a field F. Assume that T_1', T_2' are in six-fold homology, and let P denote the centre of homology of the triples (A_1', A_2', A_3') and (B_1', B_2', B_3'). No three of the points A_1', A_2', A_3', P are collinear and thus there exists an automorphism of the plane, which transforms these points onto the points $A_1 = (1, 0, 0)$, $A_2 = (0, 1, 0)$, $A_3 = (0, 0, 1)$, $D_3 = (1, 1, 1)$, respectively (cf. [3]).

Let B_1^*, B_2^*, B_3^* be the images of the points B_1', B_2', B_3'. Obviously the triangles $T_1 = \{A_1, A_2, A_3\}$ and $T_2^* = \{B_1^*, B_2^*, B_3^*\}$ are also in six-fold homology and have no common vertex.

A necessary condition for the homology of the triples (A_1, A_2, A_3) and (B_1^*, B_2^*, B_3^*) with the point D_3 as its centre can be expressed in the following form of homogeneous coordinates of the points

$$B_1 = (a, 1, 1) \quad B_2 = (1, b, 1) \quad B_3 = (1, 1, c)$$

where $a, b, c \neq 0, 1$. The remaining homologies hold if the following systems of equations have non-trivial solutions:

$$\left.\begin{aligned} x_2 - bx_3 &= 0 \\ cx_1 - x_3 &= 0 \\ x_1 - ax_2 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} cx_2 - x_3 &= 0 \\ x_1 - ax_3 &= 0 \\ bx_1 - x_2 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} x_2 - bx_3 &= 0 \\ x_1 - ax_3 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} x_2 - x_3 &= 0 \\ cx_1 - x_3 &= 0 \\ bx_1 - x_2 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} cx_2 - x_3 &= 0 \\ x_1 - x_3 &= 0 \\ x_1 - ax_2 &= 0 \end{aligned}\right\}.$$

One can easily show that this is equivalent to the conditions $a = b = c$ and $a^3 = 1$. Thus we conclude that the element $a \in F$ must be a root of the polynomial $x^2 + x + 1$ different from 1.

Theorem 4. *In an arbitrary desarguesian projective plane every six-fold homology of two triangles with no common vertex is equivalent to the special six-fold homology.*

Proof. According to the proof of the preceding theorem each pair of six-fold homologic triangles can be transformed by a certain automorphism of the plane onto triangles T_1, T_2 with the homogeneous coordinates described above. It is very easy to verify that this pair of triangles has the required property (cf. [4], [5]).

Remark. By an analogous argument we can obtain that in the desarguesian plane a six-fold perspectivity of two triangles with not common vertex implies their six-fold homology. Theorems 1–4 imply immediately:

Theorem 5. *A configuration* (H-T) *exists in a desarguesian projective plane over a field F if and only if in F there exists a root of the polynomial* $x^2 + x + 1$ *different from* 1.

Theorem 6. *If an arbitrary desarguesian-Fano plane contains an* (H-T) *configuration, then this plane has a finite subplane of order* 4. *In the case of finite Fano planes, they are exactly the projective planes over the Galois field of order* $n = 2^{2m}$. (cf. [1], [2].)

References

[1] *P. Dembowski:* Finite Geometries, Berlin—Heidelberg—New York, 1968.

[2] *M. Hall:* Combinatorial Theory, Walthamm—Toronto—London, 1967.

[3] *R. Hartshorne:* Foundations of Projective Geometry, New York, 1967.

[4] *K. Havlíček:* Zur Konstruktion einer endlichen Ebene, Čas. pěst. mat., *95* (1970), 71—75.

[5] *K. Havlíček, J. Tietze:* Zur Geometrie der endlichen Ebene der Ordnung $n = 4$, Czech. Mat. Journal *21* (*96*), 1971, 157—164.

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