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Antonín Otáhal

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MEASURE OF NONCOMPACTNESS OF SUBSETS OF LEBESGUE SPACES

ANTONÍN OTÁHAL, Praha

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0. INTRODUCTION

The purpose of this paper is to derive a characterization of bounded noncompact subsets of spaces L_p and ℓ_p. Similarly as in [4] where analogous considerations are carried out for the space of continuous functions defined on a compact metric space, our basic notion will be that of measure of noncompactness. See also [5, 6, 7, 8, 9] for related results.

0.1. Definition. Let M be a metric space. The number

$$\chi(\Omega) = \inf\{\varepsilon > 0;\ \Omega \text{ has a finite } \varepsilon\text{-net in } M\}$$

is called the *Hausdorff measure* of noncompactness of the set Ω.

Clearly, $\chi(\Omega) = 0$ iff Ω is relatively compact.

0.1.1. Remark. The notion of the Kuratowski measure of noncompactness is also often used. It is defined, for a metric space M, by

$$\alpha(M) = \inf\{d > 0;\ M \text{ can be divided into a finite number of sets with diameters less than } d\}.$$

All the following considerations based on the Hausdorff measure of noncompactness would be analogous if the Kuratowski measure were considered instead.

The Hausdorff and Kuratowski measures of noncompactness were introduced respectively in [2] and [3].

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1.1. Definition. Let $L_p(X, \mathscr{S}, \mu)$ be a Lebesgue space with the usual norm denoted by $\|\cdot\|_p$ (in the following we assume $1 \leqq p < \infty$). Let us denote by W the set of all finite sequences $w = \{E_i\}_{i=1}^m$ of pairwise disjoint sets from $\mathscr{S}$ of finite strictly positive measure.

We define: two elements w_1, w_2 of W are in the relation $w_1 \prec w_2$ iff for every set $E \in w_1$ there exist sets $E_1, \ldots, E_n \in w_2$ such that

$$\mu(E \,\Delta (\bigcup_{i=1}^{n} E_i)) = 0 .$$

It is easy to see that W with this ordering is a directed set.

For $f \in L_p$; $w \in W$, $w = \{E_i\}_{i=1}^n$ we define

$$f_w(x) = \begin{cases} 0 , & x \notin \bigcup_{i=1}^{n} E_i , \\ \dfrac{1}{\mu(E_i)} \displaystyle\int_{E_i} f \, d\mu , & x \in E_i . \end{cases}$$

By U_w we denote the mapping from L_p into L_p defined as

$$U_w(f) = f_w .$$

1.2. Lemma. *Let $f \in L_p$, W, U_w as above. Then*

(a) *the mapping U_w is linear and continuous;*
(b) $\|U_w\| = 1$;
(c) $\lim_{w \in W} U_w f = f$.

Proof cf. [1], IV.8.18.

1.3. Definition. Let K be a bounded subset of L_p. Let us denote

$$a(K, w_0) = \sup \{\|f - U_w f\|_p ;\ w_0 \prec w,\ f \in K\} ,$$

$$a(K) = \inf_{w \in W} a(K, w) \equiv \lim_{w \in W} a(K, w) .$$

1.4. Theorem. *Let K, $a(K)$ be as above. Then the following inequalities hold:*

$$\tfrac{1}{2}\, a(K) \leqq \chi(K) \leqq a(K) .$$

Proof. 1) In order to prove the first inequality it is sufficient to show that for every $\varepsilon > 0$ there exists $w \in W$ such that

$$a(K, w) \leqq 2\, \chi(K) + \varepsilon .$$

Let $\varepsilon > 0$. Then there exists a finite $(\chi(K) + \varepsilon/3)$-net A of K. According to 1.2(c) there exists $w_0 \in W$ such that for every $w \succ w_0$ and every $g \in A$ we have

$$\|U_w g - g\|_p < \varepsilon/3 .$$

For any $f \in K$ there exists $g \in A$ such that

$$\|f - g\|_p < \chi(K) + \varepsilon/3 .$$

$$\|f - U_w f\|_p \leqq \|f - g\|_p + \|g - U_w g\|_p + \|U_w g - U_w f\|_p$$

and by 1.2(b),

$$\|U_w g - U_w f\|_p \leqq \|f - g\|_p ,$$

we obtain that for every $w \succ w_0$

$$\|f - U_w f\|_p \leqq 2\,\chi(K) + \varepsilon ,$$

that is,

$$a(K, w_0) \leqq 2\,\chi(K) + \varepsilon .$$

2) We shall prove that $\chi(K) \leqq a(K) + \varepsilon$ for every $\varepsilon > 0$. Let $\varepsilon > 0$. There exists $w \in W$ such that $a(K, w) \leqq a(K) + \varepsilon/2$. $U_w(K)$ is a precompact $(a(K) + \varepsilon/2)$-net of K. It is clear that there exists a finite $(a(K) + \varepsilon)$-net of K. Hence the result.

1.5. Remark. As an immediate corollary we obtain the well known result (see e.g. [1], IV.8.18): A *bounded subset* K *of the space* $L_p(X, \mathscr{S}, \mu)$ *is relatively compact iff* $\lim_{w \in W} f_w = f$ *uniformly on* K.

We can give more useful conditions in the case of $L_p(\Omega)$, where Ω is a bounded measurable subset of the Euclidean N-space.

1.6. Definition. Let us denote by E_N the N-dimensional Euclidean space, $|\cdot|_N$ a norm on E_N, $\mathfrak{M}_N$ the σ-algebra of Lebesgue measurable subsets of E_N, and Ω an element of $\mathfrak{M}_N$. By $\mathbb{N}$ we denote the set of all positive integers.

For $f \in L_p(\Omega)$ and $M \in \mathfrak{M}_N$ we define

$$f_M(t) = \begin{cases} 0 & t \in E_N \setminus (M \cap \Omega) \\ f(t) & t \in M \cap \Omega ; \end{cases}$$

by f_0 we denote the function f_Ω.

For $\delta > 0$, $M \in \mathfrak{M}_N$ and a bounded set K in $L_p(\Omega)$, we define $\omega(K, M, \delta) = \sup \{(\int_M |f_M(x + h) - f(x)|^p \, dx)^{1/p};\ 0 < |h|_N < \delta, f \in K\}$, $\omega(K) = \inf_{\delta > 0} \omega(K, \Omega, \delta) = \lim_{\delta \to 0+} \omega(K, \Omega, \delta)$.

1.7. Theorem. *Let* K, $\omega(K)$ *be as above. Then*

$$\tfrac{1}{2}\,\omega(K) \leqq \chi(K) .$$

Proof. The proof is quite analogous to part 1) of the proof of 1.4, where we take f_0 instead of $U_w f$.

1.8. Remark. The previous theorem generalizes the well-known result: *A bounded subset K of $L_p(\Omega)$ is relatively compact iff*

$$\lim_{h\to 0+} \left(\int_\Omega |f_0(x+h) - f(x)|^p \, dx \right)^{1/p} = 0$$

uniformly on K.

2. SUBSETS OF SPACES $\ell_p(X)$

2.1. Definition. Let X be a normed linear space with a norm $\|\cdot\|_X$, p a real number, $1 \leqq p < \infty$. Let us denote by $\ell_p(X)$ the set of all sequences $\{x_i\}_{i=1}^\infty$ with elements in X such that

$$\sum_{i=1}^{8} \|x_i\|_X^p < \infty .$$

It is evident that the set $\ell_p(X)$ may be considered as a normed linear space with the norm

$$\|\{x_i\}\|_p = \left(\sum_{i=1}^\infty \|x_i\|_X^p\right)^{1/p} .$$

2.2. Definition. Let $x \in \ell_p(X)$, $x = \{x_i\}_{i=1}^\infty$, $n \in \mathbb{N}$. We define the mappings P_n, R_n from $\ell_p(X)$ into $\ell_p(X)$ by

$$P_n(x) = (0, \dots, 0, x_n, x_{n+1}, \dots), \quad R_n(x) = (x_1, x_2, \dots, x_{n-1}, 0, 0, \dots) .$$

Clearly, P_n, R_n are linear continuous mappings and $\|P_n\| = \|R_n\| = 1$.

2.3. Definition. Let K be a bounded subset of $\ell_p(X)$. We denote

$$a(K) = \inf_{n\in N} \chi(P_n(K)) = \lim_{n\to\infty} \chi(P_n(K)) ,$$

$$b(K) = \sup_{n\in N} \chi(R_n(K)) ,$$

$$\omega(K) = \limsup_{n\to\infty} \left(\sup_{x\in K} \|P_n x\|_p\right) .$$

2.4. Lemma. *Let K, $a(K)$, $b(K)$ be as above. Then the following inequalities hold:*

$$\max(a(K), b(K)) \leqq \chi(K) \leqq a(K) + b(K) .$$

Proof. 1) Clearly $a(K) \leqq \chi(K)$ and from $\|P_n\| = 1$ it follows $b(K) \leqq \chi(K)$.

2) Let $\varepsilon > 0$ be arbitrary. From 2.3 it follows that there exist sets S, T such that S is a finite $(a(K) + \varepsilon/2)$-net of $P_n(K)$ and T is a finite $(b(K) + \varepsilon/2)$-net of $R_n(K)$. As $V = S + T$ is a finite $(a(K) + b(K) + \varepsilon)$-net of K and ε was arbitrary,

$$\chi(K) \leqq a(K) + b(K).$$

2.5. Theorem. *Let K, $b(K)$, $\omega(K)$ be the same as in* 2.3. *Then*

$$(*) \qquad \max(\omega(K), b(K)) \leqq \chi(K) \quad \text{and} \quad a(K) \leqq \omega(K).$$

If the following additional condition is fulfilled: $(+)$ *there exists a compact subset N of the space X such that every member of every sequence from K is an element of N, then*

$$(**) \qquad \chi(K) = \omega(K) = a(K).$$

Proof. 1) We have proved $b(K) \leqq \chi(K)$ till now. In order to prove $\omega(K) \leqq \chi(K)$ it is sufficient, for any $\varepsilon > 0$, to find $n \in \mathbb{N}$ such that for every $m \geqq n$ and every $x \in K$, $\|P_m x\|_p \leqq \chi(K) + \varepsilon$ holds. So let $\varepsilon > 0$. There exists a finite $(\chi(K) + \varepsilon/2)$-net of K, say A. Then there exists $n \in \mathbb{N}$ such that for every $m \geqq n$ and for every $a \in A$, $\|P_m a\|_p \leqq \varepsilon/2$. Hence it follows

$$\|P_m x\|_p \leqq \|P_m x - P_m a\|_p + \|P_m a\|_p,$$

where a is that element of A for which

$$\|x - a\|_p < \chi(K) + \varepsilon/2.$$

$\|P_m\| = 1$ now implies

$$\|P_m x\|_p < \chi(K) + \varepsilon.$$

2) Clearly, for every $\varepsilon > 0$ the zero sequence in $\ell_p(X)$ is a finite $(\sup_{x \in K} \|P_n x\|_p + \varepsilon)$-net of $P_n(K)$. That is why for every $n \in \mathbb{N}$ the inequality $\chi(P_n(K)) \leqq \sup_{x \in K} \|P_n x\|_p$ holds. This implies $a(K) \leqq \omega(K)$.

3) If the condition $(+)$ is fulfilled then clearly $b(K) = 0$ and the relation $(*)$ changes into $(**)$.

2.6. Remark. The evident consequence of the preceding theorem is the well-known result: *Let X be a real line. Then a bounded subset K of $\ell_p(X)$ is relatively compact iff*

$$\lim_{n \to \infty} \Big(\sum_{i=n}^{\infty} |x_i|_1^p \Big)^{1/p} = 0$$

uniformly on K.

Added in proof. It is possible to prove the upper estimate for $\chi(K)$ that is analogous to the lower one proved in the theorem 1.7. The remark 1.8. is based on both those estimates.

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Author's address: 186 00 Praha 8 - Karlín, Sokolovská 83 (Matematicko-fyzikální fakulta UK).