Časopis pro pěstování matematiky, roč. 101 (1976), Praha

ON THE CONDITIONS FOR THE OSCILLATION OF SOLUTIONS OF NON-LINEAR THIRD ORDER DIFFERENTIAL EQUATIONS

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(Received August 8, 1974)

In this article we study the problems of oscillation of the solutions of the differential equation

$$x''' + f(t, x) = 0 . \tag{1}$$

We shall assume that the function $f(t, x)$ satisfies the Carathéodry conditions in every bounded subregion of the rectangular region $0 \leqq t < \infty$, $|x| < \infty$. Here

$$x f(t, x) \geqq 0 , \tag{2}$$

$$|f(t, x_1)| \leqq |f(t, x_2)| , \quad \text{if} \quad |x_1| \leqq |x_2| , \quad x_1 x_2 \geqq 0 . \tag{3}$$

A solution $x(t)$ of (1) which exists in the future is said to be oscillatory if for every $T > 0$, there is a $t_0 > T$ such that $x(t_0) = 0$.

Theorem 1. *For all solutions of Equation* (1) *to be oscillatory it is necessary that conditions*

$$\int_{t_0}^{\infty} t^2 |f(t, C)| \, dt = \infty , \quad \int_{t_0}^{\infty} |f(t, Ct^2)| \, dt = \infty \tag{4}$$

be satisfied for any number $C \neq 0$.

Proof. We have to prove that if either the condition

$$\int_{t_0}^{\infty} t^2 |f(t, C)| \, dt < \infty \tag{5}$$

or the condition

$$\int_{t_0}^{\infty} |f(t, Ct^2)| \, dt < \infty \tag{6}$$

is satisfied for some constant C, then Equation (1) has at least one nonoscillatory solution.

We first assume that condition (5) is satisfied. We consider the integral equation

$$x(t) = \frac{C}{2} + \frac{1}{2!}\int_t^\infty (s-t)^2 f(s, x(s))\,\mathrm{d}s \tag{7}$$

where $t > t_1 > 1$ is so large that

$$\int_{t_1}^\infty s^2 |f(s, C)|\,\mathrm{d}s \leqq |C|\,.$$

Now consider the sequence $\{x_n(t)\}$, defined in the following manner

$$x_0(t) = \frac{C}{2}, \tag{9}$$

$$x_n(t) = \frac{C}{2} + \frac{1}{2!}\int_t^\infty (s-t)^2 f(s, x_{n-1}(s))\,\mathrm{d}s\,.$$

In accordance with conditions (2), (3) and (9) we easily find from (9) that

$$\frac{|C|}{2} < x_n(t)\operatorname{sign} C < |C|\,; \quad n = 1, 2, \ldots \tag{10}$$

from (9), we find

$$x_n'(t) = -\int_t^\infty (s-t) f(s, x_{n-1}(s))\,\mathrm{d}s\,.$$

hence

$$|x_n'(t)| \leqq \int_t^\infty s f(s, C)|\,\mathrm{d}s < \int_t^\infty s^2 |f(s, C)|\,\mathrm{d}s \leqq |C| \tag{11}$$

for $t > t_1 > 1$. It follows form (10) and (11) that the sequence $\{x_n(t)\}$ defines a uniformly bounded and equicontinuous family on (t_1, ∞), hence it follows from the Arzela-Ascoli theorem there exists a subsequence $\{x_{n_K}(t)\}$ uniformly convergent on every subinterval of (t_1, ∞). Now a standard argument, see for example [2], yields a function $x(t)$ which is a solution to (7), as easily checked, a solution of the differential equation (1). But on the other hand according to (10), $x(t)$ is nonoscillatory.

Now let condition (6) be satisfied. We consider the integral equation

$$x(t) = \frac{C}{2}t^2 + \int_{t_0}^t s\left(t - \frac{s}{2}\right) f(s, x(s))\,\mathrm{d}s + \frac{t^2}{2}\int_t^\infty f(s, x(s))\,\mathrm{d}s \tag{12}$$

where t_0 is chosen so that

(13) $$\int_{t_0}^{\infty} |f(t, Ct^2)| \, dt < \frac{|C|}{2}.$$

Consider, the sequence $\{x_n(t)\}$, defined in the following manner

(14) $$x_0(t) = \frac{C}{2} t^2,$$

$$x_n(t) = \frac{C}{2} t^2 + \int_{0t}^{t} s\left(t - \frac{s}{2}\right) f(s, x_{n-1}(s)) \, ds + \frac{t^2}{2} \int_t^{\infty} f(s, x_{n-1}(s)) \, ds$$

in accordance with conditions (2), (3) and (13), we easily find from (14), that

(15) $$\frac{|C|}{2} t^2 < x_n(t) \operatorname{sign} C < |C| \, t^2, \quad n = 1, 2, 3, \ldots$$

and

(16) $$|x_{n-1}(t)| < |x_n(t)|, \quad n = 1, 2, \ldots.$$

It is obvious from (15) and (16) that the sequence $\{x_n(t)\}$ converges to some function $x(t)$. Furthermore

(17) $$\frac{|C|}{2} t^2 < x(t) \operatorname{sign} C \leqq |C| \, t^2$$

we show that $x(t)$ is a solution of the integral equation (12). For any preassigned $\varepsilon > 0$, we choose T in such a way that

$$\int_T^{\infty} |f(t, Ct^2)| \, dt < \varepsilon.$$

Then, according to (14), (15) and (17) we obtain

$$x_n(t) - \frac{Ct^2}{2} - \frac{t^2}{2} \int_t^{\infty} f(s, x(s)) \, ds - \int_{t_0}^{t} s\left(t - \frac{s}{2}\right) f(s, x(s)) \, ds \leqq$$

$$\leqq \frac{t^2}{2} \int_t^{T} |f(s, x(s)) - f(s, x_{n-1}(s))| \, ds + \frac{t^2}{2} \int_T^{\infty} |f(s, x_{n-1}(s)) -$$

$$- |f(s, x(s))| \, ds + \int_{t_0}^{t} s\left(t - \frac{s}{2}\right) |f(s, x_{n-1}(s)) - f(s, x(s))| \, ds \leqq$$

$$\leqq t^2 \int_{t_0}^{T} |f(s, x_{n-1}(s)) - f(s, x(s))| \, ds + \frac{t^2}{2} \int_T^{\infty} f(s, x_{n-1}(s) - f(s, x(s))| \, ds \leqq$$

$$\leqq t^2 \int_{t_0}^{T} |f(s, x_{n-1}(s)) - f(s, x(s))| \, ds + \varepsilon t^2.$$

If in the latter inequality, we pass to the limit as $n \to \infty$ we obtain

$$\left| x(t) - \frac{Ct^2}{2} - \frac{t^2}{2} \int_t^\infty f(s, x(s))\, ds - \int_{t_0}^t s\left(t - \frac{s}{2}\right) f(s, x(s))\, ds \right| \leqq \varepsilon t^2 .$$

But since ε is arbitrary, it follows from the last inequality that $x(t)$ is a solution of the integral equation (12), as well, as easily checked, a solution of the differential equation (1). But on the other hand, according to (17) $x(t)$ is nonoscillatory.

Theorem 2. *If the condition*

$$\int_{t_0}^\infty |f(t, C)|\, dt = \infty \tag{18}$$

is satisfied for every constant $C \neq 0$, then any solution of (1) which exists for $t > t_0$ is oscillatory.

Proof. Assume it is not oscillatory, without loss of generality, we assume $x(t) > 0$ for $t \geqq t_0$. Then $x'''(t) < 0$ which implies $x''(t) > 0$ for $t \geqq t_0$. Therefore $x'(t) > 0$ for $t \geqq t_1 > t_0$, or $x'(t) < 0$ for $t \geqq t_0$. Replacing t by t_0, when necessary we may consider both cases.

Assume, $x'(t) > 0$ i.e. $x(t)\, x'(t) > 0$ which implies $|x(t)| > |x(t_0)|$ and

$$x''(t) = x''(t_0) - \int_{t_0}^t f(s, x(s))\, ds$$

or

$$|x''(t)| = |x''(t_0)| - \int_{t_0}^t |f(s, x(s))|\, ds \leqq |x''(t_0)| - \int_{t_0}^t |f(s, x(t_0))|\, ds ,$$

this implies $|x''(t)| \to -\infty$, which is a contradiction.

Assume $x'(t) < 0$ i.e. $x(t)\, x'(t) < 0$, or

$$|x(t)| < |x(t_0)| .$$

From the identity

$$tx''(t_0) - x'(t_0) = tx''(t) - x'(t) + \int_{t_0}^t s f(s, x(s))\, ds$$

it follows, that

$$A > \int_{t_0}^t s f(s, x(s))\, ds > \int_{t_0}^t f(s, x(t_0))\, ds ,$$

which is again a contradiction.

Theorem 3. *If for any nonzero constant C we can find constants $\lambda \neq 0$ and $M > 0$, depending on C, such that the inequality*

$$|f(t, C)| \geqq M|f(t, \lambda t^2)| \tag{19}$$

is satisfied for t sufficiently large, then for every solution of Equation (1) to be oscillatory condition (18) is necessary and sufficient.

Proof. The sufficiency of the condition follows from Theorem 2, we prove the necessity of the condition. For this we show that if $\int_{t_0}^{\infty} |f(t, C)| \, dt < \infty$ then (1) has at least one nonoscillatory solution. Indeed, according to condition (19) we have

$$\int_{t_0}^{\infty} |f(t, \lambda t^2)| \, dt \leqq \frac{1}{M} \int_{t_0}^{\infty} |f(t, C)| \, dt < \infty .$$

But, then by Theorem 1 Equation (1) has at least one nonoscillatory solution. This proves the theorem.

Corollary. *Let $a(t) \geqq 0$, $f(x)$ be continuous function satisfying the condition*

$$x f(x) > 0 , \quad \textit{when} \quad x \neq 0 ,$$

$$|f(x_1)| < |f(x_2)| \quad \text{when} \quad |x_1| < |x_2| , \quad x_1 x_2 \geqq 0 \tag{20}$$

and

$$\sup |f(x)| < \infty . \tag{21}$$

Then for all the solutions of the equation

$$x''' + a(t) f(x) = 0 \tag{22}$$

to be oscillatory, the condition

$$\int_{t_0}^{\infty} a(t) \, dt = \infty \tag{23}$$

is necessary and sufficient.

Proof. It is clear according to (20) and (23) that conditions (2), (3) are observed for Equation (22). On the other hand (21) and (23) imply that condition (18) is fulfilled. Therefore, all the hypotheses of Theorem 3 are satisfied, hence follows the validity of our assertion.

References

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