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ON ALGEBRAIC PROPERTIES OF DISPERSIONS OF THE 3RD AND 4TH KIND OF THE DIFFERENTIAL EQUATION $y'' = q(t)\,y$

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Academician O. BORŮVKA introduced in [1] the definitions and established properties of general dispersions, giving a characterization of dispersions of the 1st, 2nd, 3rd and 4th kind as well as of central dispersions. Further he studied the sets of dispersions of the 1st and 2nd kind.

The subject of the present paper was suggested by Professor M. LAITOCH who directed my attention to the possibility of a parallel study of the 3rd and 4th kind dispersion sets.

The opening part establishes a representation of the 3rd kind dispersions by means of unimodular matrices.

In the second part we define equivalence relations $\sim$ and $\approx$ in the 3rd kind dispersion set $\boldsymbol{D}_3$:

$X_3 \sim Y_3$ if and only if there exists $\varphi_\nu \in \boldsymbol{C}_1$ such that $X_3\varphi_\nu = Y_3$, where $\boldsymbol{C}_1$ is the group of central dispersions of the 1st kind, $X_3, Y_3 \in \boldsymbol{D}_3$;

$X_3 \approx Y_3$ if and only if there exists $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ such that $X_3 \in \boldsymbol{C}_3\mathscr{X}_1$ and at the same time $Y_3 \in \boldsymbol{C}_3\mathscr{X}_1$.

The relations turn out to be the same and hence the decompositions $\boldsymbol{D}_3/\sim$ and $\boldsymbol{D}_3/\approx$ coincide. Hence, for any coset $\mathscr{X}_3 \in \boldsymbol{D}_3/\sim$ we can uniquely determine a coset $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ by $\mathscr{X}_3 = \mathscr{X}_3\boldsymbol{C}_1 = \boldsymbol{C}_3\mathscr{X}_1$. Moreover, any dispersions $X_1 \in \mathscr{X}_1$ and $X_3 \in \mathscr{X}_3$ satisfy $\mathscr{X}_3 = X_3\boldsymbol{C}_1 = \boldsymbol{C}_3X_1$.

In the next part we show the existence of a $1-1$ mapping of the set $\boldsymbol{D}_3/\sim$ onto the factor group $\boldsymbol{L}/\{\mathbf{E}, -\mathbf{E}\}$. (Any coset $\mathscr{X}_3 \in \boldsymbol{D}_3/\sim$ is associated with a couple of unimodular matrices $\{\mathbf{C}, -\mathbf{C}\}$). Further, $\mathscr{X}_3\mathfrak{B}_1 = \mathfrak{B}_3\mathscr{X}_1 = \mathfrak{B}_3$, where $\mathfrak{B}_3(\mathfrak{B}_1)$ is the set (the group) of the 3rd kind (the 1st kind) direct dispersions and $\mathscr{X}_3$ is an arbitrary element in $\mathfrak{B}_3$, $\mathscr{X}_1 \in \mathfrak{B}_1$.

The concluding part of the paper is devoted to transfering the results proved for the dispersions of the 3rd kind to the case of the dispersions of the 4th kind.

Basic concepts and relations. (q) will always denote an ordinary linear differential equation of the 2^{nd} order in the real domain $y'' = q(t)\,y$, where $q(t) \in C_j^2$ ($j = (a, b)$ is an open definition interval) and $q(t) < 0$ for every $t \in j$; the differential equation (q) will be always assumed oscillatory in (a, b), that is, the integrals of this equation vanish infinitely many times in both directions towards the endpoints a, b of the interval (a, b). (q_1) will always denote the associated equation of (q). (See [1].) The integral space (i.e., the space of all solutions) of the differential equation (q), (q_1) will be denoted by R, R_1, respectively. The concepts not defined in this paper were adopted from [1].

1. DISPERSIONS OF THE 3^{RD} KIND

Representation by means of unimodular matrices. Let $X_3 \in \boldsymbol{D}_3$ be an arbitrary dispersion of the 3^{rd} kind, $\boldsymbol{D}_3$ the set of all dispersions of the 3^{rd} kind. Choose a basis (U_1, V_1) of the integral space R_1 and denote its Wronskian by W_1; let $u(t)$, $v(t)$ be the functions

$$u(t) = \frac{U_1[X_3(t)]}{\sqrt{|X_3'(t)|}}, \quad v(t) = \frac{V_1[X_3(t)]}{\sqrt{|X_3'(t)|}}. \tag{1}$$

By [1, § 20, 6.3], the functions $u(t)$, $v(t)$ are linearly independent integrals of (q) and thus they form a basis of the integral space R. Their Wronskian w satisfies

$$w = W_1 \,.\, \operatorname{sgn} X_3'\,. \tag{2}$$

By [1, § 1, 9] there exists exactly one integral y of (q) for each integral y_1 of differential equation (q_1) such that

$$y_1(t) = \frac{y'(t)}{\sqrt{|q(t)|}}\,. \tag{3}$$

Consequently, it is possible to determine exactly one basis (U, V) of R for the basis (U_1, V_1) of R_1 such that the corresponding functions U, U_1 and V, V_1 satisfy (3). The bases (u, v), (U, V) of the same space R are connected in the following way

$$u(t) = c_{11}\,U(t) + c_{12}\,V(t)\,, \quad v(t) = c_{21}\,U(t) + c_{22}\,V(t) \tag{4}$$

and hence

$$w = W \,.\, \det \mathbf{C}\,, \tag{5}$$

where w and W are the Wronskians of the bases (u, v) and (U, V), respectively. Further,

$$W_1 = \begin{vmatrix} U_1 & V_1 \\ U_1' & V_1' \end{vmatrix} = (U'V - UV')\,.\, \operatorname{sgn} q = W\,.$$

Now by (5),

$$w = W_1 \,.\, \det \mathbf{C} \tag{6}$$

and (2) and (6) imply

$$\det \mathbf{C} = \operatorname{sgn} X_3' \,. \tag{7}$$

Therefore the matrix $\mathbf{C}$ is unimodular.

Theorem 1.1. *For any dispersion $X_3 \in \boldsymbol{D}_3$, the unimodular matrix $\mathbf{C}$ is uniquely determined by* (4).

The theorem results from the above consideration.

Theorem 1.2. *For any unimodular matrix, there exists at least one dispersion of the 3^{rd} kind associated with it through the relations* (4) *and* (1).

Proof. Let $\mathbf{C} = \|c_{ik}\|$ be an arbitrary unimodular matrix. Let us consider the integral $c_{21}\, U + c_{22}\, V$ and let t_0 be its arbitrary zero point. Let T_0 be a zero point of the integral V_1, such that

$$\operatorname{sgn} U_1(T_0) = \operatorname{sgn}\left(c_{11}\, U(t_0) + c_{12}\, V(t_0)\right), \tag{8}$$

where $U(t), V(t)$ is a basis of R, $U_1(t), V_1(t)$ is the basis of R_1 such that

$$U_1(t) = \frac{U'(t)}{\sqrt{|q|}}, \quad V_1(t) = \frac{V'(t)}{\sqrt{|q|}} \,.$$

Let us consider the linear mapping p

$$p = \left[u(t) \to U_1(t),\ v(t) \to V_1(t)\right],$$

where $u(t) = c_{11}\, U(t) + c_{12}\, V(t)$, $v(t) = c_{21}\, U(t) + c_{22}\, V(t)$. This mapping is normalized with respect to t_0, T_0 and thus uniquely determines the dispersion $X_3 \in \boldsymbol{D}_3$. (See [1, § 20,2].) Further,

$$\chi_p = \frac{\det \mathbf{C} \,.\, W}{W_1} = \frac{\det \mathbf{C} \,.\, W}{W} = \det \mathbf{C} = \pm 1 \,,$$

where χ_p is the characteristic of the linear mapping p. Hence by [1, § 20,6 (17)] we have for any $Y_1 \in R_1$,

$$\frac{Y_1[X_3(t)]}{\sqrt{|X_3'(t)|}} = \pm y$$

where $y \in R$ and $Y_1 = py$. The sign $+$ or $-$ does not depend on the choice of the integral Y_1. Therefore by (7)

$$\frac{U_1[X_3(t)]}{\sqrt{|X_3'(t)|}} = +u(t) = c_{11}\,U(t) + c_{12}\,V(t)\,,$$
$$\frac{V_1[X_3(t)]}{\sqrt{|X_3'(t)|}} = +v(t) = c_{21}\,U(t) + c_{22}\,V(t)\,.$$

The dispersion $X_3(t)$ is associated with the matrix $\mathbf{C}$ in the required manner.

The decomposition of the set D_3 determined by the equivalence relation $\sim$ or $\approx$. Now we shall introduce the relation $\sim$ in the dispersion set D_3 as follows:

Let C_1 be the group of central dispersions of the first kind, X_3, Y_3 arbitrary dispersions of the 3rd kind of D_3.

(9) $\qquad X_3 \sim Y_3$ iff there exists $\varphi_\nu \in C_1$ such that $X_3\varphi_\nu = Y_3$.

Theorem 1.3. *The relation* (9) *is an equivalence relation on the set* D_3.

Proof. Let X_3, Y_3, Z_3 be arbitrary dispersions in D_3. Since there exists a dispersion $\varphi_0(t) = t \in C_1$ such that $X_3\varphi_0 = X_3$, it holds $X_3 \sim X_3$ for each $X_3 \in D_3$. Let $X_3 \sim Y_3$. Then $X_3\varphi_\nu = Y_3$, $X_3\varphi_\nu\varphi_{-\nu} = Y_3\varphi_{-\nu}$ and hence $Y_3\varphi_{-\nu} = X_3$. Thus $Y_3 \sim X_3$. Let $X_3 \sim Y_3$ and $Y_3 \sim Z_3$, $X_3\varphi_\nu = Y_3$ and $Y_3\varphi_\mu = Z_3$. Therefore $X_3\varphi_\nu\varphi_\mu = Y_3\varphi_\mu = Z_3$, $X_3\varphi_\sigma = Z_3$ and thus $X_3 \sim Z_3$.

Theorem 1.4. *The relation* (9) *forms a decomposition* $D_3/\sim$. *The set* C_3 *of all central dispersions of the* 3rd *kind forms exactly one coset of* $D_3/\sim$.

Proof. a) Any central dispersion $\chi_\nu \in C_3$ ($\nu = \pm 1, \pm 2, \ldots$) can be expressed (see [1, § 12,4 (7)]) in the following manner:

$$\chi_n = \chi_1\varphi_{n-1}\,, \quad \chi_{-n} = \chi_1\varphi_{-n}\,, \quad n = 1, 2, \ldots\,.$$

This implies that any central dispersion χ_ν ($\nu = \pm 1, \pm 2, \ldots$) and the dispersion χ_1 are equivalent (in equivalence $\sim$) and thus all dispersions χ_ν belong to the same coset of $D_3/\sim$.

b) If a dispersion X_3 and an arbitrary central dispersion χ_ϱ are equivalent, then X_3 is also a central dispersion. Indeed, in this case $X_3\varphi_\nu = \chi_\varrho$, $X_3\varphi_\nu\varphi_{-\nu} = \chi_\varrho\varphi_{-\nu}$ and therefore $X_3 = \chi_\sigma \in C_3$.

Corollary 1.1. *Let* $\mathcal{X}_3$ *be an arbitrary equivalence coset of* $D_3/\sim$. *Then* $\mathcal{X}_3 = X_3C_1$, *where* X_3 *is a dispersion in the coset* $\mathcal{X}_3$. *Also* $X_3C_1 = \mathcal{X}_3C_1$.

Let us now consider the group $\boldsymbol{D}_1$ of the first kind dispersions and its cyclic subgroup $\boldsymbol{S}_1$ of central dispersions with an even index. The latter one is a normal subgroup of $\boldsymbol{D}_1$. The factor group $\boldsymbol{D}_1/\boldsymbol{S}_1$ and the group $\boldsymbol{L}$ of all unimodular matrices of the 2nd order are isomorphic. (See [1, § 21,6].) Let

$$\varphi : \boldsymbol{D}_1/\boldsymbol{S}_1 \to \boldsymbol{L}$$

be the isomorphism considered in [1]. In this isomorphism the group $\boldsymbol{S}_1$ (the set $\bar{\boldsymbol{S}}_1$ of the first kind central dispersions with an odd index) and the unity matrix $\mathbf{E}$ (the matrix $-\mathbf{E}$) correspond to each other. $\boldsymbol{S}_1 \cup \bar{\boldsymbol{S}}_1 = \boldsymbol{C}_1$ is the group of the first kind central dispersions and it is also a cyclic subgroup of the group $\boldsymbol{D}_1$. (See [1, § 21,6].)

Let us now consider the induced isomorphism

$$\{\boldsymbol{S}_1, \bar{\boldsymbol{S}}_1\} \to \{\mathbf{E}, -\mathbf{E}\}$$

between two-element subgroups $\{\boldsymbol{S}_1, \bar{\boldsymbol{S}}_1\}$ and $\{\mathbf{E}, -\mathbf{E}\}$ of the groups $\boldsymbol{D}_1/\boldsymbol{S}_1$ and $\boldsymbol{L}$, respectively. Since $\{\mathbf{E}, -\mathbf{E}\}$ is a normal subgroup of $\boldsymbol{L}$, $\{\boldsymbol{S}_1, \bar{\boldsymbol{S}}_1\}$ is a normal subgroup of $\boldsymbol{D}_1/\boldsymbol{S}_1$ and the relative factor groups are isomorphic:

$$\varphi' : (\boldsymbol{D}_1/\boldsymbol{S}_1)/\{\boldsymbol{S}_1, \bar{\boldsymbol{S}}_1\} \to \boldsymbol{L}/\{\mathbf{E}, -\mathbf{E}\}\,.$$

Furthermore, for arbitrary $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{S}_1$, $\mathscr{X}_1 \,.\, \{\boldsymbol{S}_1, \bar{\boldsymbol{S}}_1\} = \{\boldsymbol{S}_1, \bar{\boldsymbol{S}}_1\} \,.\, \mathscr{X}_1$ and hence $\mathscr{X}_1\bar{\boldsymbol{S}}_1 = \boldsymbol{S}_i\mathscr{X}_1$, where $\boldsymbol{S}_i \in \{\boldsymbol{S}_1, \bar{\boldsymbol{S}}_1\}$. Thus, for arbitrary $X_1 \in \mathscr{X}_1$ and $\varphi_\nu \in \bar{\boldsymbol{S}}_1$, there exists $\varphi_\mu \in \boldsymbol{S}_i$ and $\bar{X}_1 \in \mathscr{X}_1$ such that $X_1\varphi_\nu = \varphi_\mu \bar{X}_1$. Since X_1, $\bar{X}_1 \in \mathscr{X}_1$, there exists $\varphi_\varrho \in \boldsymbol{S}_1$ such that $\bar{X}_1 = \varphi_\varrho X_1$. Hence $X_1\varphi_\nu = \varphi_\mu\varphi_\varrho X_1$ and $X_1\boldsymbol{C}_1 \subseteq \boldsymbol{C}_1X_1$. The converse relation $X_1\boldsymbol{C}_1 \supseteq \boldsymbol{C}_1X_1$ can be proved by analogy. Consequently, it holds $X_1\boldsymbol{C}_1 = \boldsymbol{C}_1X_1$. Since $\boldsymbol{C}_1$ is a normal subgroup of $\boldsymbol{D}_1$, we can form the factor group $\boldsymbol{D}_1/\boldsymbol{C}_1$. Any two elements $X_1, Y_1 \in \boldsymbol{D}_1$ belong to the same coset of $\boldsymbol{D}_1/\boldsymbol{C}_1$ if and only if $X_1 \sim Y_1$, i.e., if there exists $\varphi_\nu \in \boldsymbol{C}_1$ such that $X_1 = Y_1\varphi_\nu$. Let

$$\alpha : \boldsymbol{D}_1/\boldsymbol{S}_1 \to \boldsymbol{D}_1/\boldsymbol{C}_1$$

be a mapping of the factor group $\boldsymbol{D}_1/\boldsymbol{S}_1$ onto the factor group $\boldsymbol{D}_1/\boldsymbol{C}_1$ such that each coset $X_1\boldsymbol{C}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ is mapped onto the coset $X_1\boldsymbol{S}_1 \in \boldsymbol{D}_1/\boldsymbol{S}_1$. Thus $\alpha(X_1\boldsymbol{S}_1) = X_1\boldsymbol{C}_1$.

Lemma 1.1. *The mapping $\alpha : \boldsymbol{D}_1/\boldsymbol{S}_1 \to \boldsymbol{D}_1/\boldsymbol{C}_1$ is a homomorphism. The kernel of this homomorphism is the two-element subgroup $\{\boldsymbol{S}_1, \bar{\boldsymbol{S}}_1\}$ to which the element $\boldsymbol{C}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ corresponds.*

Proof. Let $X_1\boldsymbol{S}_1, Y_1\boldsymbol{S}_1$ be arbitrary elements in the group $\boldsymbol{D}_1/\boldsymbol{S}_1$. Then $\alpha(X_1\boldsymbol{S}_1 . Y_1\boldsymbol{S}_1) = \alpha(X_1Y_1\boldsymbol{S}_1\boldsymbol{S}_1) = \alpha(X_1Y_1\boldsymbol{S}_1) = X_1Y_1\boldsymbol{C}_1 = X_1Y_1\boldsymbol{C}_1\boldsymbol{C}_1 = X_1\boldsymbol{C}_1 . Y_1\boldsymbol{C}_1 = \alpha(X_1\boldsymbol{S}_1) \,.\, \alpha(Y_1\boldsymbol{S}_1)$. Thus α is a homomorphism. Further $\alpha(\boldsymbol{S}_1) = \boldsymbol{C}_1$, $\alpha(\bar{\boldsymbol{S}}_1) = \alpha(\varphi_\sigma\boldsymbol{S}_1) = \varphi_\sigma \,.\, \boldsymbol{C}_1 = \boldsymbol{C}_1$ where σ is an odd integer. If $\alpha(\mathscr{X}_1) = \boldsymbol{C}_1$ for a coset $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{S}_1$, then $\mathscr{X}_1 = X_1\boldsymbol{S}_1$ where $X_1 \in \mathscr{X}_1$ implies $\boldsymbol{C}_1 = \alpha(\mathscr{X}_1) = \alpha(X_1\boldsymbol{S}_1) = X_1\boldsymbol{C}_1$. Thus $X_1 \in \boldsymbol{C}_1$ and hence either $X_1\boldsymbol{S}_1 = \boldsymbol{S}_1$ or $X_1\boldsymbol{S}_1 = \bar{\boldsymbol{S}}_1$.

Remark 1.1. According to Lemma 1.1, there exists an isomorphism

$$\tau : D_1/C_1 \to (D_1/S_1)/\{S_1, \bar{S}_1\} .$$

If we now compose the isomorphisms τ and φ' we obtain an isomorphism

$$\varphi'\tau : D_1/C_1 \to L/\{\mathbf{E}, -\mathbf{E}\} .$$

Clearly $\varphi'\tau(C_1) = \{\mathbf{E}, -\mathbf{E}\}$. Also for any element $X_1C_1 \in D_1/C_1$, $\varphi'\tau(X_1C_1) = \{\mathbf{C}, -\mathbf{C}\}$ where $\mathbf{C}$ is a matrix in L such that $\varphi(X_1S_1) = \mathbf{C}$. ($\varphi$ is the above described isomorphism $D_1/S_1 \to L$).

Now, let us return to the set D_3 and the equivalence relation (9).

Lemma 1.2. *For each dispersion of the* 3^{rd} *kind* $X_3 \in D_3$ *and for each central dispersion* $\chi_\varrho \in C_3$ *there exists a dispersion of the first kind* $X_1 \in D_1$ *such that*

$$X_3 = \chi_\varrho X_1 . \tag{10}$$

Proof. For each central dispersion of the 3^{rd} kind $\chi_\varrho(\varrho = \pm 1, \pm 2, \ldots)$ there exists a central dispersion of the 4^{th} kind $\omega_{-\varrho} \in C_4$ such that $\omega_{-\varrho}\chi_\varrho = \chi_\varrho\omega_{-\varrho} = \varphi_0(t) = t$. (See [1, § 12,4 (6)].) Consider now the function $\omega_{-\varrho}X_3$, where $\omega_{-\varrho} \in C_4$ and $X_3 \in D_3$. Then by [1, § 21,8] $\omega_{-\varrho}X_3 \in D_1$ and there exists $X_1 \in D_1$ such that $X_1 = \omega_{-\varrho}X_3$. Also $\chi_\varrho X_1 = \chi_\varrho\omega_{-\varrho}X_3 = \varphi_0 X_3 = X_3$ and hence $X_1 \in D_1$ satisfies the equality (10).

Lemma 1.3. *If two different central dispersions of the* 3^{rd} *kind* $\chi_\varrho, \chi_\sigma \in C_3$ *fulfil* $X_3 = \chi_\varrho X_1$, $X_3 = x_\sigma Y_1$, *then* $X_1 = Y_1\varphi_\nu$ *and thus* X_1, Y_1 *lie in the same coset of the factor group* D_1/C_1.

Proof. It holds $\chi_\varrho \sim \chi_\sigma$, $\chi_\varrho = \chi_\sigma\varphi_\mu$, where $\varphi_\mu \in C_1$. From $\chi_\sigma Y_1 = \chi_\varrho . X_1$ it follows that $\chi_\sigma Y_1 = \chi_\sigma\varphi_\mu X_1$ and hence $\omega_{-\sigma}x_\sigma Y_1 = \omega_{-\sigma}\chi_\sigma\varphi_\mu X_1$. Therefore $Y_1 = \varphi_\mu X_1$ and also $X_1 = Y_1\varphi_\nu$.

Corollary 1.2. *For each dispersion of the* 3^{rd} *kind* $X_3 \in D_3$ *there exists a dispersion of the* 1^{st} *kind* $X_1 \in D_1$ *such that* $X_3 \in C_3X_1$. *Thus, for each dispersion* X_3 *there exists exactly one coset* $\mathscr{X}_1 = X_1C_1$ *of the factor group* D_1/C_1 *such that* $X_3 \in C_3\mathscr{X}_1$. *Consequently,* $C_3X_1 = C_3\mathscr{X}_1$ *holds.*

We shall now introduce a binary relation $\approx$ in the set D_3 as follows:

(11) $X_3 \approx Y_3$ iff there exists $\mathscr{X}_1 \in D_1/C_1$ such that $X_3 \in C_3\mathscr{X}_1$ and at the same time $Y_3 \in C_3\mathscr{X}_1$.

Theorem 1.5. *The relation* (11) *is an equivalence relation in the set* D_3.

Proof. By Corollary 1.2, $X_3 \approx X_3$ holds for each $X_3 \in D_3$. Now let $X_3 \approx Y_3$. Then there exists a coset $\mathscr{X}_1 \in D_1/C_1$ such that $X_3 \in C_3\mathscr{X}_1$ and $Y_3 \in C_3\mathscr{X}_1$. Therefore

$Y_3 \approx X_3$. Let $X_3 \approx Y_3$. Then there exists $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ such that $X_3 \in \boldsymbol{C}_3\mathscr{X}_1$ and $Y_3 \in \boldsymbol{C}_3\mathscr{X}_1$. Let also $Y_3 \approx Z_3$. Then there exists $\mathscr{Y}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ such that $Y_3 \in \boldsymbol{C}_3\mathscr{Y}_1$ and $Z_3 \in \boldsymbol{C}_3\mathscr{Y}_1$. From $Y_3 \in \boldsymbol{C}_3\mathscr{X}_1$ and $Y_3 \in \boldsymbol{C}_3\mathscr{Y}_1$ it follows (by Lemma 1.3) that there exists $\varphi_\nu \in \boldsymbol{C}_1$ such that $X_1 = Y_1\varphi_\nu$ and hence $\mathscr{X}_1 = \mathscr{Y}_1$. Herefrom $X_3 \approx Z_3$.

Theorem 1.6. *Two arbitrary dispersions of the* 3rd *kind* $X_3, Y_3 \in \boldsymbol{D}_3$ *fulfil* $X_3 \sim Y_3$ *if and only if* $X_3 \approx Y_3$.

Proof. a) Let $X_3 \approx Y_3$. Then there exists $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ such that $X_3 \in \boldsymbol{C}_3\mathscr{X}_1$ and $Y_3 \in \boldsymbol{C}_3\mathscr{X}_1$. Thus $X_3 = \chi_\varrho X_1$, $Y_3 = \chi_\sigma Y_1$, where $\chi_\varrho, \chi_\sigma \in \boldsymbol{C}_3$, $X_1, Y_1 \in \mathscr{X}_1$. Since it holds $X_1 = Y_1\varphi_\nu$ and $\chi_\varrho = \chi_\sigma\varphi_\mu$ where φ_ν, φ_μ are proper dispersions of $\boldsymbol{C}_1$ we have $X_3 = \chi_\sigma\varphi_\mu Y_1\varphi_\nu = \chi_\sigma Y_1\varphi_{\mu_1}\varphi_\nu = Y_3\varphi_{\nu_1}$, thus $X_3 \sim Y_3$.

b) Let $X_3 \sim Y_3$. Then there exists φ_ν such that $X_3\varphi_\nu = Y_3$. By Corollary 1.2 there exists a coset $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ such that $X_3 \in \boldsymbol{C}_3\mathscr{X}_1$. So $X_3 = \chi_\varrho X_1$ where $\chi_\varrho \in \boldsymbol{C}_3$ and $X_1 \in \mathscr{X}_1$. Then $Y_3 = X_3\varphi_\nu = \chi_\varrho \,.\, X_1\varphi_\nu$. Thus $Y_3 \in \boldsymbol{C}_3\mathscr{X}_1$ and therefore $X_3 \approx Y_3$.

Hence the decompositions $\boldsymbol{D}_3/\sim$ and $\boldsymbol{D}_3/\approx$ coincide.

Let us recall that if we consider an arbitrary dispersion $X_3 \in \boldsymbol{D}_3$ and compose it with all central dispersions of the 1st kind (i.e., with dispersions from $\boldsymbol{C}_1$) we obtain exactly one coset $\mathscr{X}_3 \in \boldsymbol{D}_3/\sim$. Thus $\mathscr{X}_3 = X_3\boldsymbol{C}_1$, where $X_3 \in \mathscr{X}_3$. (See Corollary 1.1.)

Theorem 1.7. *Let* X_3 *be an arbitrary dispersion in* $\boldsymbol{D}_3$ *and let* $\mathscr{X}_3 = X_3\boldsymbol{C}_1$ *be a coset of* $\boldsymbol{D}_3/\sim$. *If we compose a dispersion* $X_1 \in \boldsymbol{D}_1$ *associated with* X_3 *by* (10) *with all central dispersions of the* 3rd *kind (i.e., with dispersions in* $\boldsymbol{C}_3$) *we obtain exactly one coset* $\mathscr{X}_3 \in \boldsymbol{D}_3/\sim$. *Thus* $\mathscr{X}_3 = \boldsymbol{C}_3X_1$ *and*

$$\boldsymbol{C}_3X_1 = X_3\boldsymbol{C}_1 = \mathscr{X}_3\,. \tag{12}$$

Proof. This theorem is an immediate consequence of those above.

Theorem 1.8. *For each coset* $\mathscr{X}_3 \in \boldsymbol{D}_3/\sim$ *there exists exactly one coset* $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ *such that*

$$\mathscr{X}_3 = \mathscr{X}_3\boldsymbol{C}_1 = \boldsymbol{C}_3\mathscr{X}_1\,. \tag{13}$$

Proof. By Corollaries 1.1 and 1.2 it holds $X_3\boldsymbol{C}_1 = \mathscr{X}_3\boldsymbol{C}_1$ and $\boldsymbol{C}_3X_1 = \boldsymbol{C}_3\mathscr{X}_1$. Herefrom and by (12) we have (13). Let us now consider a coset $\mathscr{X}_3 \in \boldsymbol{D}_3/\sim$. Let $\mathscr{X}_3 = \boldsymbol{C}_3Y_1$, where $Y_1 \in \boldsymbol{D}_1$. Then $Y_1 \sim X_1$ and therefore $Y_1 \in \mathscr{X}_1$ and for a coset $\mathscr{X}_3$, the coset $\mathscr{X}_1$ is uniquely determined. The converse is evident.

The properties of the factor set $\mathfrak{D}_3$ and the factor groups $\mathfrak{L}$ and $\mathfrak{D}_1$. Let us denote the factor set $\boldsymbol{D}_3/\sim$ by $\mathfrak{D}_3$, the group $\boldsymbol{D}_1/\boldsymbol{C}_1$ by $\mathfrak{D}_1$ and the group $\boldsymbol{L}/\{\mathbf{E}, -\mathbf{E}\}$ by $\mathfrak{L}$. On the basis of the results contained in the preceding part we can express the following

Theorem 1.9. *There exists a* $1-1$ *mapping*

$$\beta : \mathfrak{D}_3 \to \mathfrak{L}$$

given in the following way: For each $\mathscr{X}_3 \in \mathfrak{D}_3$, $\beta(\mathscr{X}_3) = \{\mathbf{C}, -\mathbf{C}\}$ *where* $\{\mathbf{C}, -\mathbf{C}\} = \varphi'\tau(\mathscr{X}_1)$ *for* $C_3\mathscr{X}_1 = \mathscr{X}_3$.

Lemma 1.4. *If we compose an arbitrary dispersion* $X_3 \in \mathscr{X}_3$ *and another one* $X_1 \in \mathscr{X}_1$ *we always obtain a dispersion from the same coset* $\mathscr{Y}_3 \in \mathfrak{D}_3$.

Proof. Let X_3 and X_1 be arbitrary dispersions in $\mathscr{X}_3$ and $\mathscr{X}_1$, respectively. Then $X_3X_1 \in \mathscr{Y}_3$. Now let $X_3 \sim \overline{X}_3$, $X_1 \sim \overline{X}_1$, that is $X_3 = \varphi_\nu \overline{X}_3$, $X_1 = \varphi_\mu \overline{X}_1$. Then $\overline{X}_3\overline{X}_1 = X_3\varphi_\nu X_1\varphi_\mu = X_3X_1\varphi_{\nu_1}\varphi_\mu = X_3X_1\varphi_\sigma$ and therefore $\overline{X}_3\overline{X}_1 \sim X_3X_1$. Consequently $\overline{X}_3\overline{X}_1 \in \mathscr{Y}_3$.

Now we can introduce a multiplication of cosets from $\mathfrak{D}_3$ and $\mathfrak{D}_1$ by means of Lemma 1.4 as follows:

$$\mathscr{X}_3\mathscr{X}_1 = \mathscr{Y}_3,$$

where $\mathscr{Y}_3$ is the coset from $\mathfrak{D}_3$ containing the product X_3X_1, where X_3, X_1 are arbitrary elements of $\mathscr{X}_3$ and $\mathscr{X}_1$, respectively.

Lemma 1.5. *Let* β *be the mapping from Theorem* 1.9 *and* $\varphi'\tau$ *the isomorphism from Remark* 1.1. *If* $\beta(\mathscr{X}_3) = \{\mathbf{C}, -\mathbf{C}\}$ *and* $\varphi'\tau(\mathscr{Y}_1) = \{\mathbf{G}, -\mathbf{G}\}$, *where* $\mathscr{X}_3 \in \mathfrak{D}_3$, $\mathscr{Y}_1 \in \mathfrak{D}_1$ *and* $\{\mathbf{C}, -\mathbf{C}\}, \{\mathbf{G}, -\mathbf{G}\} \in \mathfrak{L}$, *then* $\beta(\mathscr{X}_3\mathscr{Y}_1) = \{\mathbf{CG}, -\mathbf{CG}\}, \{\mathbf{CG}, -\mathbf{CG}\} \in \mathfrak{L}$.

The proof is evident.

$\mathfrak{L}$ is decomposed into two equivalent subsets:

the subset of unimodular matrix cosets whose determinant is equal to $+1$ and that one whose matrices have determinant equal to -1. A consequence of this is that $\mathfrak{D}_3$ (and also $\mathfrak{D}_1$, see [1, § 21]) decomposes into equivalent subsets as well:

the set $\mathfrak{B}_3(\mathfrak{B}_1)$ of direct, i.e., increasing dispersion cosets the corresponding matrices of which have determinant equal to $+1$ (compare with (7) in the first part of this paper);

the set of indirect (decreasing) dispersion cosets the corresponding matrices of which have determinant -1.

Theorem 1.10. *Choosing an arbitrary coset* $\mathscr{X}_3 \in \mathfrak{B}_3$ *and composing it with all* $\mathscr{X}_1 \in \mathfrak{B}_1$, *we obtain again the whole set* $\mathfrak{B}_3$. *That is,* $\mathscr{X}_3\mathfrak{B}_1 = \mathfrak{B}_3$ *for any* $\mathscr{X}_3 \in \mathfrak{B}_3$.

Proof. Let $\beta(\mathscr{X}_3) = \{\mathbf{C}, -\mathbf{C}\}$. Clearly $\det \mathbf{C} = \det(-\mathbf{C}) = +1$. Let $\varphi'\tau(\mathscr{X}_1) = \{\mathbf{G}, -\mathbf{G}\}$. Clearly $\det \mathbf{G} = \det(-\mathbf{G}) = +1$. By Lemma 1.5 $\beta(\mathscr{X}_3\mathscr{X}_1) =$

$= \{\mathbf{CG}, -\mathbf{CG}\}$ and since $\det \mathbf{CG} = +1$, $\mathscr{X}_3\mathscr{X}_1 \in \mathfrak{B}_3$ for each $\mathscr{X}_1 \in \mathfrak{B}_1$. Let now $\overline{\mathscr{X}}_3$ be an arbitrary element of $\mathfrak{B}_3$. Then for $\mathscr{X}_3$ there always exists $\mathscr{Y}_1 \in \mathfrak{D}_1$ such that $\overline{\mathscr{X}}_3 = \mathscr{X}_3\mathscr{Y}_1$. We now prove the relation $\mathscr{Y}_1 \in \mathfrak{B}_1$ by means of the matrix representation:

Let $\beta(\overline{\mathscr{X}}_3) = \{\mathbf{A}, -\mathbf{A}\}$ and $\varphi'\tau(\mathscr{Y}_1) = \{\mathbf{B}, -\mathbf{B}\}$. From $\overline{\mathscr{X}}_3 = \mathscr{X}_3\mathscr{Y}_1$ we obtain by Lemma 1.5 for the corresponding cosets of matrices $\{\mathbf{A}, -\mathbf{A}\} = \{\mathbf{CB}, -\mathbf{CB}\}$.

Suppose first that $\mathbf{A} = \mathbf{CB}$. Hence the elements of the matrices satisfy

$$\begin{aligned} c_{11}b_{11} + c_{12}b_{21} &= a_{11}, \\ c_{11}b_{12} + c_{12}b_{22} &= a_{12}, \\ c_{21}b_{11} + c_{22}b_{21} &= a_{21}, \\ c_{21}b_{12} + c_{22}b_{22} &= a_{22} \end{aligned}$$

and hence $\det \mathbf{B} = \det \mathbf{A} \,.\, \det \mathbf{C} = +1$.

Similarly, if $-\mathbf{A} = \mathbf{CB}$ then $\det \mathbf{B} = \det \mathbf{A} \,.\, \det \mathbf{C} = +1$. Evidently also $\det(-\mathbf{B}) = +1$. Matrices corresponding to the coset $\mathscr{Y}_1$ have the determinant equal to $+1$, that is, $\mathscr{Y}_1 \in \mathfrak{B}_1$.

Completely analogously we could prove the following

Theorem 1.11. *Choosing an arbitrary coset $\mathscr{X}_1 \in \mathfrak{B}_1$ and composing it with all $\mathscr{X}_3 \in \mathfrak{B}_3$ we obtain again the whole set $\mathfrak{B}_3$. Further it holds*

$$\mathscr{X}_3\mathfrak{B}_1 = \mathfrak{B}_3\mathscr{X}_1 = \mathfrak{B}_3,$$

where $\mathscr{X}_3$ is an arbitrary element of $\mathfrak{B}_3$ and $\mathscr{X}_1$ is an arbitrary element of $\mathfrak{B}_1$.

2. DISPERSIONS OF THE 4TH KIND

Representation by means of unimodular matrices. The representation will be realized analogously to that of the dispersions of the 3rd kind. Let $X_4 \in \boldsymbol{D}_4$ be an arbitrary dispersion of the 4th kind and $\boldsymbol{D}_4$ the set of all dispersions of the 4th kind. Now choose a basis (U, V) of the integral space R and denote its Wronskian by W; let $u_1(t)$, $v_1(t)$ be the functions

$$u_1(t) = \frac{U[X_4(t)]}{\sqrt{|X_4'(t)|}}, \quad v_1(t) = \frac{V[X_4(t)]}{\sqrt{|X_4'(t)|}}. \tag{14}$$

The functions $u_1(t)$, $v_1(t)$ form a basis of the integral space R_1. Their Wronskian w_1 fulfils

$$w_1 = W \,.\, \operatorname{sgn} X_4'. \tag{15}$$

Following (3) we can uniquely determine a basis (u, v) of R for the basis (u_1, v_1) of R_1. Two bases (u, v) and (U, V) are connected by (4). Thus the Wronskians satisfy (5). Since $w_1 = w$ holds, we get

$$w_1 = \det \mathbf{C} \,.\, W \tag{16}$$

and therefore $\det \mathbf{C} = \operatorname{sgn} X_4'$.

We now present a number of theorems concerning the properties of the 4[th] kind dispersions without giving their proofs since they are analogous to those of the theorems for the 3[rd] kind dispersions.

Theorem 2.1. *For any dispersion $X_4 \in \boldsymbol{D}_4$, the unimodular matrix $\mathbf{C}$ is uniquely determined by* (4).

Theorem 2.2. *For any unimodular matrix there exists at least one* 4[th] *kind dispersion associated with it through the relations* (4) *and* (14).

The decomposition of the set D_4 determined by the equivalence relation $\sim$ or $\approx$. We now introduce a relation $\sim$ in the dispersion set $\boldsymbol{D}_4$ as follows:

Let C_1 be the group of central dispersions of the 1[st] kind and let X_4, Y_4 be arbitrary dispersions from $\boldsymbol{D}_4$.

$$X_4 \sim Y_4 \text{ iff there exists } \varphi_\nu \in C_1 \text{ such that } \varphi_\nu X_4 = Y_4\,. \tag{17}$$

Theorem 2.3. *The relation* (17) *is an equivalence relation on the set $\boldsymbol{D}_4$.*

Theorem 2.5. *The relation* (17) *forms a decomposition $\boldsymbol{D}_4/\sim$. The set C_4 of all central dispersions of the* 4[th] *kind forms exactly one coset of $\boldsymbol{D}_4/\sim$.*

Corollary 2.1. *Let $\mathscr{X}_4$ be an arbitrary coset of $\boldsymbol{D}_4/\sim$. Then $\mathscr{X}_4 = C_1X_4$ where X_4 is an arbitrary dispersion in the coset $\mathscr{X}_4$. Consequently $C_1X_4 = C_1\mathscr{X}_4$.*

Lemma 2.1. *For each dispersion of the* 4[th] *kind $X_4 \in \boldsymbol{D}_4$ and for each central dispersion of the* 4[th] *kind $\omega_\varrho \in C_4$ there exists a dispersion of the* 1[st] *kind $X_1 \in \boldsymbol{D}_1$ such that*

$$X_4 = X_1\omega_\varrho\,. \tag{18}$$

Lemma 2.2. *If two different central dispersions of the* 4[th] *kind ω_ϱ, $\omega_\sigma \in C_4$ satisfy $X_4 = X_1\omega_\varrho$, $X_4 = Y_1\omega_\sigma$, then $X_1 = Y_1\varphi_\nu$ and thus X_1, Y_1 belong to the same coset of the factor group $\boldsymbol{D}_1/C_1$.*

Corollary 2.2. *For each dispersion of the* 4[th] *kind $X_4 \in \boldsymbol{D}_4$ there exists a dispersion $X_1 \in \boldsymbol{D}_1$ such that $X_4 \in X_1C_4$. Thus, for each dispersion X_4 there exists exactly* *one coset* $\mathscr{X}_1 = X_1 C_1$ *of the factor group* $\boldsymbol{D}_1/\boldsymbol{C}_1$ *such that* $X_4 \in \mathscr{X}_1 \boldsymbol{C}_4$. *Hence* $X_1 \boldsymbol{C}_4 \doteq \mathscr{X}_1 \boldsymbol{C}_4$ *holds.*

We now introduce a relation $\approx$ in the set $\boldsymbol{D}_4$ as follows:

(19) $X_4 \approx Y_4$ iff there exists $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ such that $X_4 \in \mathscr{X}_1 \boldsymbol{C}_4$ and at the same time $Y_4 \in \mathscr{X}_1 \boldsymbol{C}_4$.

Theorem 2.5. *The relation* (19) *is an equivalence relation in the set* $\boldsymbol{D}_4$.

Theorem 2.6. *Two arbitrary dispersions of the* 4th *kind fulfil* $X_4 \sim Y_4$ *if and only if* $X_4 \approx Y_4$.

Theorem 2.7. *Let* X_4 *be an arbitrary dispersion from* $\boldsymbol{D}_4$ *and let* $\mathscr{X}_4 = \boldsymbol{C}_1 X_4$ *be a coset of* $\boldsymbol{D}_4/\sim$. *Composing the dispersion* $X_1 \in \boldsymbol{D}_1$ *associated with the dispersion* X_4 *through* (18) *with all central dispersions of the* 4th *kind we obtain exactly one coset* $\mathscr{X}_4 \in \boldsymbol{D}_4/\sim$. *Thus* $\mathscr{X}_4 = X_1 \boldsymbol{C}_4$ *and*

$$X_1 \boldsymbol{C}_4 = \boldsymbol{C}_1 X_4 = \mathscr{X}_4 . \tag{20}$$

Theorem 2.8. *For each coset* $\mathscr{X}_4 \in \boldsymbol{D}_4/\sim$ *there exists exactly one coset* $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$ *such that*

$$\mathscr{X}_4 = \boldsymbol{C}_1 \mathscr{X}_4 = \mathscr{X}_1 \boldsymbol{C}_4 . \tag{21}$$

The properties of the factor set $\mathfrak{D}_4$. Let us denote the factor set $\boldsymbol{D}_4/\sim$ by $\mathfrak{D}_4$.

Theorem 2.9. *Between the elements of the set* $\mathfrak{D}_4$ *and those of the group* $\mathfrak{L}$ *there exists a* $1-1$ *correspondence*

$$\gamma : \mathfrak{D}_4 \to \mathfrak{L}$$

determined as follows: For each $\mathscr{X}_4 \in \mathfrak{D}_4$, $\gamma(\mathscr{X}_4) = \{\mathbf{C}, -\mathbf{C}\}$ *where* $\{\mathbf{C}, -\mathbf{C}\} = \varphi'\tau(\mathscr{X}_1)$ *for* $\mathscr{X}_1 \boldsymbol{C}_4 = \mathscr{X}_4$.

Lemma 2.3. *If we compose an arbitrary dispersion* $X_4 \in \mathscr{X}_4$ *with an arbitrary dispersion* $X_1 \in \mathscr{X}_1$ *we always obtain a dispersion from the same coset* $\mathscr{Y}_4 \in \mathfrak{D}_4$.

By means of Lemma 2.3 we can now introduce a multiplication of cosets from $\mathfrak{D}_1$ and $\mathfrak{D}_4$ as follows:

$$\mathscr{X}_1 \mathscr{X}_4 = \mathscr{Y}_4 ,$$

where $\mathscr{Y}_4$ is the coset from $\mathfrak{D}_4$ containing the product $X_1 X_4$, where X_1 is an element of $\mathscr{X}_1$ and X_4 is an element of $\mathscr{X}_4$.

Lemma 2.4. *If* $\gamma(\mathscr{X}_4) = \{\mathbf{C}, -\mathbf{C}\}$ *and* $\varphi'\tau(\mathscr{Y}_1) = \{\mathbf{G}, -\mathbf{G}\}$, *where* $\mathscr{X}_4 \in \mathfrak{D}_4$,

$\mathscr{Y}_1 \in \mathfrak{D}_1$ *and* $\{\mathbf{C}, -\mathbf{C}\}, \{\mathbf{G}, -\mathbf{G}\} \in \mathfrak{L}$, *then* $\gamma(\mathscr{Y}_1\mathscr{X}_4) = \{\mathbf{CG}, -\mathbf{CG}\}, \{\mathbf{CG}, -\mathbf{CG}\} \in \in \mathfrak{L}$.

Let us denote the set of all direct dispersions of the 4^{th} kind by $\mathfrak{B}_4$.

Theorem 2.10. *Choosing an arbitrary coset* $\mathscr{X}_4 \in \mathfrak{B}_4$ *and composing it with all* $\mathscr{X}_1 \in \mathfrak{B}_1$ *we obtain again the whole set* $\mathfrak{B}_4$. *That is,* $\mathfrak{B}_1\mathscr{X}_4 = \mathfrak{B}_4$ *for any* $\mathscr{X}_4 \in \mathfrak{B}_4$.

Theorem 2.11. *Choosing an arbitrary coset* $\mathscr{X}_1 \in \mathfrak{B}_1$ *and composing it with all* $\mathscr{X}_4 \in \mathfrak{B}_4$ *we obtain again the whole set* $\mathfrak{B}_4$. *Further it holds* $\mathscr{X}_1\mathfrak{B}_4 = \mathfrak{B}_1\mathscr{X}_4 = \mathfrak{B}_4$, *where* $\mathscr{X}_4$ *and* $\mathscr{X}_1$ *are elements of* $\mathfrak{B}_4$ *and* $\mathfrak{B}_1$, *respectively.*

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