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SUFFICIENT CONDITIONS FOR LOCALLY CONNECTED GRAPHS

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Let G be a graph without isolated vertices, and let v be a vertex of G. The *neighborhood* of v, denoted by $\langle N(v) \rangle$, is the subgraph of G induced by the set $N(v)$ of vertices of G adjacent with v. The graph G is called *locally connected* if the neighborhood of every vertex of G is connected.

In [1] CHARTRAND and PIPPERT showed that if the minimum degree $\delta(G)$ of a graph G of order p exceeds $\frac{2}{3}(p - 1)$, then G is locally connected. More generally, it was proved in [1] that if G is a graph of order p such that for every pair u, v of vertices, $\deg u + \deg v > \frac{4}{3}(p - 1)$, then G is locally connected. Hence, it is possible for some vertex of a graph G to have degree at most $\frac{2}{3}(p - 1)$ (with the degrees of all other vertices exceeding $\frac{2}{3}(p - 1)$) and still be assured that G is locally connected.

It is the object of this article to determine the number of vertices of specified degrees not exceeding $\frac{2}{3}(p - 1)$ which insures that a given graph be locally connected.

The results we present are reminiscent of work on hamiltonian graphs. DIRAC [2] proved that for a graph G of order $p \geqq 3$, if $\delta(G) \geqq p/2$, then G is hamiltonian. ORE [4] extended this result by showing that if $\deg u + \deg v \geqq p \geqq 3$ for every pair u, v of nonadjacent vertices, then G is hamiltonian. PÓSA [5] then proceeded to provide a sufficient condition for hamiltonian graphs which allows even more vertices of degree less than $p/2$, including some of quite small degree.

First we show that no vertex of a graph G of order p can have degree much less than $\frac{2}{3}(p - 1)$ to assure local connectedness. In this respect, it is convenient to employ the *join* $G_1 + G_2$ of two disjoint graphs G_1 and G_2, defined as that graph whose vertex set is $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and whose edge set is

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{v_1 v_2 \mid v_1 \in V(G_1), \quad v_2 \in V(G_2)\}.$$

The *union* of graphs G_1 and G_2, denoted $G_1 \cup G_2$, is the graph for which

$$V(G_1 \cup G_2) = V(G_1) \cup V(G_2), \quad \text{and} \quad E(G_1 \cup G_2) = E(G_1) \cup E(G_2).$$

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The union of n graphs, each of which is isomorphic to G, is denoted by nG; if G is connected, the graph nG has n components, each of which is isomorphic to G.

As usual, $\{\ \}$ denotes the least integer function in what follows. All definitions and notation not given here may be found in [3].

Proposition. *Let G be a graph of order $p \geqq 5$. If G has one vertex of degree $2\{\frac{1}{3}(p-1)\} - 2$ and all others have degree exceeding $\frac{2}{3}(p-1)$, then G need not be locally connected.*

Proof. Let $k = \{\frac{1}{3}(p-1)\}$ and consider the graph $G = 2K_{k-1} + (\{v\} \cup \cup K_{p+1-2k})$. Then $\deg v = 2\{\frac{1}{3}(p-1)\} - 2$, and all other vertices have degree exceeding $\frac{2}{3}(p-1)$. Since $\langle N(v) \rangle$ is disconnected, G is not locally connected.

Thus, by the preceding proposition, we may not allow even a single vertex to have degree as small as $2\{\frac{1}{3}(p-1)\} - 2$ (with all other degrees exceeding $\frac{2}{3}(p-1)$) and be assured that the graph is locally connected. In the case of vertices of degree $2\{\frac{1}{3}(p-1)\} - 1$, we have the following result.

Theorem 1. *Let G be a graph of order p which has up to $2\{\frac{1}{3}(p-1)\} - p - 1$ vertices of degree $2\{\frac{1}{3}(p-1)\} - 1$ and all others of degree greater than $\frac{2}{3}(p-1)$. Then G is locally connected.*

Proof. If $p \equiv 2 \pmod 3$, then $2\{\frac{1}{3}(p-1)\} - 1 > \frac{2}{3}(p-1)$, so $\delta(G) > \frac{2}{3}(p-1)$. Thus, G is locally connected.

For $p \equiv 0 \pmod 3$ or $p \equiv 1 \pmod 3$, suppose G is not locally connected. Let v be a vertex of G such that $\langle N(v) \rangle$ is not connected.

Case 1. *Suppose* $\deg v = 2\{(\frac{1}{3}p - 1)\} - 1$.

Let G_1 be a component of $\langle N(v) \rangle$ of minimum order, say $|V(G_1)| = r$. Then $r \leqq \frac{1}{2}(2\{\frac{1}{3}(p-1)\} - 1)$, so $r \leqq \{\frac{1}{3}(p-1)\} - 1$. If $u \in V(G_1)$, then $\deg u \leqq \leqq r + p - 2\{\frac{1}{3}(p-1)\} \leqq p - 1 - \{\frac{1}{3}(p-1)\} \leqq \frac{2}{3}(p-1)$. Thus each vertex of G_1 has degree at most $\frac{2}{3}(p-1)$, so the degree of each vertex of G_1 must be $2\{\frac{1}{3}(p-1)\} - - 1$. Therefore, $r \leqq 2\{\frac{2}{3}(p-1)\} - p - 2$ since there are at most $2\{\frac{2}{3}(p-1)\} - p - 1$ vertices of degree $2\{\frac{1}{3}(p-1)\} - 1$, one of which is v. Hence $\deg u \leqq r + p - - 2\{\frac{1}{3}(p-1)\} \leqq 2(\{\frac{2}{3}(p-1)\} - \{\frac{1}{3}(p-1)\}) - 2 = 2\{\frac{1}{3}(p-1)\} - 2$, since $p \not\equiv \not\equiv 2 \pmod 3$. By hypothesis this is impossible so Case 1 cannot happen.

Case 2. *Suppose* $\deg v = k > \frac{2}{3}(p-1)$.

Select G_1 as in Case 1 so that $r \leqq k/2$. If $u \in V(G_1)$, then $\deg u \leqq r + p - 1 - k < < \frac{2}{3}(p-1)$. Thus $r \leqq 2\{\frac{2}{3}(p-1)\} - p - 1$, so $\deg u \leqq r + p - 1 - k < 2\{\frac{2}{3}(p - - 1)\} - \frac{2}{3}(p-1) - 2 < 2\{\frac{1}{3}(p-1)\} - 1$. But, by hypothesis, this is impossible, so Case 2 cannot happen.

The following example shows that the result in Theorem 1 is sharp.

Example 1. Let $G = (K_{2k-p-1} \cup K_{p-k}) + (\{v\} \cup K_{p-k})$, where $p \geqq 7$ and $k = \{\frac{2}{3}(p-1)\}$. Then G has $2\{\frac{2}{3}(p-1)\} - p$ vertices of degree $\{\frac{2}{3}(p-1)\} - 1$ and all other vertices have degrees exceeding $\frac{2}{3}(p-1)$. Since $\langle N(v)\rangle$ is disconnected, G is not locally connected.

As we noted at the beginning of the proof of Theorem 1, when $p \equiv 2 \pmod 3$, then $\delta(G) > \frac{2}{3}(p-1)$. However, if $p \equiv 2 \pmod 3$, then $2\{\frac{1}{3}(p-1)\} - 2 < \frac{2}{3}(p-1) < 2\{\frac{1}{3}(p-1)\} - 1$. Thus, by the Proposition, when $p \equiv 2 \pmod 3$, if G has as few as one vertex of degree not exceeding $\frac{2}{3}(p-1)$, then G need not be locally connected.

If $p \equiv 0 \pmod 3$, then by Theorem 1, G may have as many as $2\{\frac{2}{3}(p-1)\} - p - 1$ vertices of degree $2\{\frac{1}{3}(p-1)\} - 1$ and all others of degree greater than $\frac{2}{3}(p-1)$, and necessarily G is locally connected. Now when $p \equiv 0 \pmod 3$, we have $2\{\frac{1}{3}(p-1)\} - 1 = \{\frac{2}{3}(p-1)\} - 1$, so Theorem 1 is best possible.

The remaining case to consider is $p \equiv 1 \pmod 3$. In this case, Theorem 1 states that if G has a certain number of vertices of degree $2\{\frac{1}{3}(p-1)\} - 1 = \frac{2}{3}(p-1) - 1$ and all others have degree exceeding $\frac{2}{3}(p-1)$, then G must be locally connected. We next determine what combination of vertices of degrees $\frac{2}{3}(p-1) - 1$ and $\frac{2}{3}(p-1)$, with all other vertices having degree exceeding $\frac{2}{3}(p-1)$, insures that G is locally connected.

Theorem 2. *Let* $p \equiv 1 \pmod 3$ *and let* k *be such that* $0 < k < \frac{1}{3}(p-1) - 1$. *If a graph* G *has* k *vertices of degree* $\frac{2}{3}(p-1)$ *and* $\frac{1}{3}(p-1) - 1 - k$ *vertices of degree* $\frac{2}{3}(p-1) - 1$, *with all other vertices of degree exceeding* $\frac{2}{3}(p-1)$, *then* G *is locally connected.*

Proof. Assume G is not locally connected and let v be a vertex of G for which $\langle N(v)\rangle$ is not connected. We consider three cases determined by the degree of v.

Case 1. *Suppose* $\deg v = \frac{2}{3}(p-1) - 1$.

Let G_1 be a component of $\langle N(v)\rangle$ of smallest order, say $|V(G_1)| = r$. Thus, $r \leqq \frac{1}{3}(p-1) - 1$, since r is an integer and $p \equiv 1 \pmod 3$. Let $u \in V(G_1)$. Then $\deg u \leqq r + p - \frac{2}{3}(p-1) \leqq \frac{2}{3}(p-1)$. Thus each vertex in G_1 has degree $\frac{2}{3}(p-1) - 1$ or $\frac{2}{3}(p-1)$, and since there are $\frac{1}{3}(p-1) - 1$ such vertices, one of which is v, necessarily $r \leqq \frac{1}{3}(p-1) - 2$. Thus $\deg u \leqq \frac{2}{3}(p-1) - 1$. But G contains $\frac{1}{3}(p-1) - 1 - k$ vertices of degree $\frac{2}{3}(p-1) - 1$, one of which is v, so $r \leqq \frac{1}{3}(p-1) - 2 - k$. Therefore, $\deg u \leqq \frac{2}{3}(p-1) - 1 - k$. But $k > 0$, so $\deg u \leqq \frac{2}{3}(p-1) - 2$, which by hypothesis is impossible. Thus Case 1 cannot happen.

Case 2. *Suppose* $\deg v = \frac{2}{3}(p-1)$.

Let G_1 and r be as in Case 1. Then $r \leqq \frac{1}{3}(p-1)$. If $u \in V(G_1)$, then $\deg u \leqq r + \frac{1}{3}(p-1) \leqq \frac{2}{3}(p-1)$. But G has $\frac{1}{3}(p-1)$ such vertices, one of which is v, so $r \leqq \frac{1}{3}(p-1) - 2$. Hence $\deg u \leqq \frac{2}{3}(p-1) - 2$, which by hypothesis is impossible. Thus Case 2 cannot occur.

Case 3. *Suppose* $\deg v = t > \frac{2}{3}(p-1)$.

Let G_1 and r be as in Case 1, so $r \leqq t/2$. For $u \in V(G_1)$, $\deg u \leqq r + p - 1 - t < \frac{2}{3}(p-1)$. Since G has $\frac{1}{3}(p-1) - 1 - k$ such vertices, we must have $r \leqq \frac{1}{3}(p-1) - 1 - k$. But then $\deg u \leqq \frac{1}{3}(p-1) - 1 - k + (p-1) - t < \frac{2}{3}(p-1) - 1 - k$. Since $k > 0$, necessarily $\deg u < \frac{2}{3}(p-1) - 2$, which is impossible. Thus Case 3 is also impossible, so the assumed graph G cannot exist; that is, the theorem is valid.

An example will illustrate the sharpness of Theorem 2.

Example 2. Let $p \equiv 1 \pmod 3$, and let k satisfy $0 < k < \frac{1}{3}(p-1) - 1$. Then let $G' = (G_1' \cup G_2') + (\{v\} \cup G_3')$, where G_1', G_2', and G_3' are complete graphs of orders $\frac{1}{3}(p-1) - 1$, $\frac{1}{3}(p-1)$, and $\frac{1}{3}(p-1) + 1$, respectively. A graph G is now defined. Select $\frac{1}{3}(p-1) - 1 - k$ vertices from G_1', and for each such vertex, we decrease its degree by one by deleting an incident edge which is also incident with a vertex in G_3'. These deletions are performed so that no vertex in G_3' has degree decreased by more than one. This is possible since $|V(G_3')| > |V(G_1')|$. Then G is the graph obtained from G' by removing the edges so described. Let G_i $(i = 1, 2, 3)$ denote the subgraph of G corresponding to G_i'. The subgraph G_1 has k vertices of degree $\frac{2}{3}(p-1)$ and $\frac{1}{3}(p-1) - 1 - k$ vertices of degree $\frac{2}{3}(p-1) - 1$. All other vertices of G have degree at least $\frac{2}{3}(p-1) + 1$, except that $\deg v = \frac{2}{3}(p-1) - 1$. Since $\langle N(v) \rangle$ is disconnected, G is not locally connected.

The only situation which has not been considered is when $p \equiv 1 \pmod 3$ and the only vertices whose degrees do not exceed $\frac{2}{3}(p-1)$ have degree $\frac{2}{3}(p-1)$.

Theorem 3. *Let* $p \equiv 1 \pmod 3$. *If a graph* G *has no more than* $\frac{2}{3}(p-1)$ *vertices of degree* $\frac{2}{3}(p-1)$, *and all other vertices have degree greater than* $\frac{2}{3}(p-1)$, *then* G *is locally connected.*

Proof. Suppose there is a graph G satisfying the hypothesis which is not locally connected. Then there is a vertex v of G such that $\langle N(v) \rangle$ is not connected.

Case 1. *Suppose* $\deg v = \frac{2}{3}(p-1)$.

Let $\langle N(v) \rangle = G_1 \cup G_2$ where G_1 is a component of $\langle N(v) \rangle$ of minimum order, say $|V(G_1)| = r$. Then $r \leqq \frac{1}{3}(p-1)$. If $u \in V(G_1)$, then $\deg u \leqq r + \frac{1}{3}(p-1) \leqq \frac{2}{3}(p-1)$. Thus each vertex of G_1 has degree $\frac{2}{3}(p-1)$ since no vertex of G has smaller degree. But then $r = \frac{1}{3}(p-1)$ and consequently $|V(G_2)| = \frac{1}{3}(p-1)$. Thus if $y \in V(G_2)$, then $\deg y \leqq \frac{2}{3}(p-1)$, so $\deg y = \frac{2}{3}(p-1)$. Therefore, all vertices of G_2 have degree $\frac{2}{3}(p-1)$. Also, $\deg v = \frac{2}{3}(p-1)$, so G contains at least $\frac{2}{3}(p-1) + 1$ vertices of degree $\frac{2}{3}(p-1)$, which by hypothesis is impossible.

Case 2. *Suppose* $\deg v = t > \frac{2}{3}(p-1)$.

Let G_1 and r be as in Case 1, so $r \leqq t/2$. If $u \in V(G_1)$, then $\deg u \leqq r + p - 1 - t$

$< \frac{2}{3}(p - 1)$. But no vertex of G has degree less than $\frac{2}{3}(p - 1)$, so Case 2 cannot happen.

Theorem 3, too, is best possible.

Example 3. Let $G = 2K_r + (\{v\} \cup K_r)$, where $r = (p - 1)/3$. Then G has $\frac{2}{3}(p - 1) + 1$ vertices of degree $\frac{2}{3}(p - 1)$, and all other vertices have degree exceeding $\frac{2}{3}(p - 1)$. Since $\langle N(v) \rangle$ is disconnected, G is not locally connected.

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