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SOME GENERALIZATIONS OF THE NOTION OF CONTINUITY AND DENJOY PROPERTY OF FUNCTIONS

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1. INTRODUCTION

Let X, Y be two topological spaces. The function $f : X \to Y$ is said to be quasi-continuous at the point $x_0 \in X$ if for each neighbourhood $U(x_0)$ of the point x_0 (in X) and each neighbourhood $V(f(x_0))$ of the point $f(x_0)$ (in Y) there exists an open set $U \subset U(x_0)$, $U \neq \emptyset$ such that $f(U) \subset V(f(x_0))$. The property of the quasi-continuity is equivalent to the property of the neighbourliness (cf. [1], [6], [8]).

The function $f : X \to Y$ is said to be somewhat continuous if for each set $V \subset Y$ open in Y such that $f^{-1}(V) \neq \emptyset$ there exists an open set $U \subset X$, $U \neq \emptyset$ so that $U \subset f^{-1}(V)$ (cf. [4]).

Let X be a topological and Y a metric space (with the metric ϱ). The function $f : X \to Y$ is said to be cliquish at the point $x_0 \in X$ if for each neighbourhood $U(x_0)$ of the point x_0 and each $\varepsilon > 0$ there exists an open set $U \subset U(x_0)$, $U \neq \emptyset$ such that $\varrho(f(x'), f(x'')) < \varepsilon$ holds for each two points $x', x'' \in U$ (cf. [6], [11]).

The function f defined on the topological space X is said to be quasi-continuous or cliquish on X if it is quasi-continuous or cliquish, respectively, at each point $x \in X$.

In the sequel, I_0 denotes an arbitrary interval (it may be $I_0 = (-\infty, +\infty) = R$). The function $f : I_0 \to R$ is said to have the Denjoy property $\mathcal{D}_0$ if for each $a, b \in R$, $a < b$, the set $\{x \in I_0; a < f(x) < b\}$ is either void or has a positive Lebesgue measure (cf. [7]).

We introduce further the following notation: We shall say that $f : I_0 \to R$ has the Denjoy property $\mathcal{D}_1(\mathcal{D}_2)$ if it has the property $\mathcal{D}_0$ for each interval $I \subset I_0$ which is a closed (an open) set in I_0, i.e., if for each such interval I and each $a, b \in R$, $a < b$, the set $\{x \in I;\ a < f(x) < b\}$ is either void or has a positive Lebesgue measure. Evidently the property $\mathcal{D}_0$ follows from any of the properties $\mathcal{D}_1$, $\mathcal{D}_2$. Further, if a function has the property $\mathcal{D}_1$, then it has $\mathcal{D}_2$ as well. The converse is not true (see Example II).

Example I. Let $I_0 = \langle 0, 1\rangle$. We put $f(x) = 1$ for $0 \leqq x \leqq \frac{1}{2}$, $f(x) = 1$ for x rational, $\frac{1}{2} \leqq x \leqq 1$ and $f(x) = 0$ for x irrational, $\frac{1}{2} \leqq x \leqq 1$. Then f has the property $\mathscr{D}_0$ but not the property $\mathscr{D}_k$ $(k = 1, 2)$.

Example II. Let $g(x) = x$ for $0 \leqq x \leqq \frac{1}{2}$ and $g(x) = 0$ for $\frac{1}{2} < x \leqq 1$. Then g has the property $\mathscr{D}_2$ but not the property $\mathscr{D}_1$ since for $I = \langle \frac{1}{2}, 1\rangle$ we have $H = \{x \in I; 0 < g(x) < 1\} = \{\frac{1}{2}\} \neq \emptyset$ and $|H| = 0$ ($|M|$ denotes the Lebesgue measure of the set M).

2. QUASI-CONTINUITY, SOMEWHAT CONTINUITY, CLIQUISHNESS AND DENJOY PROPERTY

Let X, Y be two topological spaces. Denote by $Q(X, Y)$ and $S(X, Y)$ the set of all quasi-continuous and somewhat continuous functions $f : X \to Y$, respectively. If Y is a metric space, then $Q^*(X, Y)$ denotes the set of all cliquish functions $f : X \to Y$.

It is easy to see that $Q(X, Y) \subset S(X, Y)$ (see [4]) and if Y is a metric space, then

$$(*) \qquad Q(X, Y) \subset Q^*(X, Y)$$

(see [8]).

First we prove the following simple

Theorem 1. *Let I be an interval. Then*

$$S(I, R) - Q^*(I, R) \neq \emptyset \neq Q^*(I, R) - S(I, R).$$

Proof. We shall give the proof of Theorem for $I = \langle 0, 1\rangle$. For an arbitrary interval the proof is quite analogous.

Define $h(x) = 0$ for $x \in \langle 0, 1)$ and $h(1) = 1$. Then obviously $h \in Q^*(I, R) - S(I, R)$.

Define $f(x) = 1$ for x rational, $x \in (\frac{1}{2}, 1\rangle$, $f(x) = 0$ for x irrational, $x \in (\frac{1}{2}, 1\rangle$. Further put $f(x) = 0$ for $x \in (\frac{1}{3}, \frac{1}{2}\rangle$ and $f(x) = 1$ for $x \in \langle 0, \frac{1}{3}\rangle$. Then $f \in S(I, R) - Q^*(I, R)$.

It is obvious that any continuous function $f : I_0 \to R$ has each of the properties $\mathscr{D}_k$ $(k = 0, 1, 2)$. Further it is well-known that also any derivative has these properties (cf. [2], [3]). There exist such quasi-continuous functions which are not derivatives. Such a function is e.g. the function g from Example II. This function is evidently quasi-continuous on $\langle 0, 1\rangle$ but it is not a derivative since it has not the Darboux property. In connection with these facts we prove the following theorem. In the sequel we consider I_0 a metric space with the usual Euclidean metric.

Theorem 2. a) *Let $f : I_0 \to R$ be a Lebesgue measurable somewhat continuous function. Then f has the property $\mathscr{D}_2$.*

b) *There exists a function $f : I_0 \to R$ in the first Baire class which is quasi-continuous on I_0 but has not the property $\mathscr{D}_1$.*

Proof. a) Let $I \subset I_0$ be an interval which is an open set in I_0. Let

$$a, b \in R\,, \quad a < b\,, \quad E_a^b(f) = \{x \in I;\ a < f(x) < b\}\,.$$

Let $x_1 \in E_a^b(f)$, i.e. $x_1 \in f^{-1}((a, b))$. Then $f^{-1}((a, b)) \neq \emptyset$. In virtue of the somewhat continuity of the function f there exists an interval $I_1 \subset I$ such that $I_1 \subset E_a^b(f)$. This yields $|E_a^b(f)| > 0$.

b) Such a function for $I_0 = \langle 0, 1 \rangle$ is e.g. the function g from Example II. For an arbitrary I_0 it can be constructed in an analogous way as g in Example II.

Remark 1. a) Every function $f : I_0 \to R$ with the Denjoy property $\mathscr{D}_0$ is obviously Lebesgue measurable. Since there exist non-measurable quasi-continuous functions (cf. [6]), the assumption of the measurability of the function f in Theorem 2 is necessary.

b) Let us remark that Theorem 2a) cannot be extended to cliquish functions. E.g., let $f(x) = 0$ for $0 \leqq x < 1$ and $f(1) = 1$. Then f is measurable and cliquish on $\langle 0, 1 \rangle$, but it has not even the property $\mathscr{D}_0$ since

$$B = \{x \in \langle 0, 1 \rangle;\ 0 < f(x) < 2\} = \{1\} \neq \emptyset$$

and $|B| = 0$.

The converse of Theorem 2a) is not true. This is shown by the following*)

Theorem 2'.*) *There exists a function $f : I_0 \to R$ in the first Baire class such that f has the property $\mathscr{D}_1$ while it is not somewhat continuous on the interval I_0.*

Proof. Choose f as a derivative which is not identically equal to zero on I_0 and vanishes at each point of a dense subset of the interval I_0 (see [9]). Then f has the asserted properties.

3. QUASI-CONTINUOUS, SOMEWHAT CONTINUOUS AND CLIQUISH FUNCTIONS IN THE SPACE $M(X)$

In the sequel X denotes a topological space and $M(X)$ the linear normed space of all functions $f : X \to R$ bounded on X with the sup-norm (i.e. $\|f\| = \sup_{t \in X} |f(t)|$).

The properties of the quasi-continuity, somewhat continuity and cliquishness introduced above lead us to the study of the space $M(X)$ from the point of view of these properties.

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*) The author is thankful to the referee for his suggestion improving the original version of Theorem 2'.

In the sequel $Q(X)$, $S(X)$ and $Q^*(X)$ denote the sets of all $f \in M(X)$ which are quasi-continuous, somewhat continuous and cliquish on X, respectively.

Theorem 3. *Each of the sets $Q(X)$, $S(X)$, $Q^*(X)$ is a perfect set in $M(X)$. If $M(X) \neq Q^*(X)$, then each of the sets $Q(X)$, $Q^*(X)$ is a nowhere dense set in $M(X)$.*

First we prove the following

Lemma 1. *$Q^*(X)$ is a linear subspace of the space $M(X)$.*

Proof. Let $f_k \in Q^*(X)$ $(k = 1, 2)$ and $a \in R$. We shall prove that $f_1 + f_2 \in Q^*(X)$, $af_1 \in Q^*(X)$. Putting $f = f_1 + f_2$ we see easily that $f \in M(X)$. Let $x_0 \in X$, $\varepsilon > 0$. Let $U(x_0)$ be a neighbourhood of the point x_0 (in X). Since f_1 is cliquish at x_0, there exists an open set $U_1 \subset U(x_0)$, $U_1 \neq \emptyset$ such that

$$|f_1(y_1) - f_1(y_2)| < \frac{\varepsilon}{2} \tag{1}$$

for each two points $y_1, y_2 \in U_1$. We take a point $x_1 \in U_1$. In virtue of the cliquishness of the function f_2 at the point x_1 there exists an open set $U_2 \subset U_1$, $U_2 \neq \emptyset$ such that

$$|f_2(y_1) - f_2(y_2)| < \frac{\varepsilon}{2} \tag{2}$$

for each two points $y_1, y_2 \in U_2$. From (1), (2) we obtain for $y_1, y_2 \in U_2 \subset U_1$ the inequality $|f(y_1) - f(y_2)| < \varepsilon$. Hence f is cliquish on X. The proof for af_1 is easy.

Remark 2. In general $Q(X)$.($S(X)$) does not form a linear subspace of the space $M(X)$. Let e.g. $X = R, f_1(x) = 0$ for $x < 0$ and $f_1(x) = 1$ for $x \geqq 0$, further $f_2(x) = -1$ for $x \leqq 0$ and $f_2(x) = 0$ for $x > 0$. It is obvious that $f_k \in Q(R)$ $(f_k \in S(R))$ $(k = 1, 2)$ but $f_1 + f_2$ is not quasi-continuous at 0 (somewhat continuous on R).

Proof of Theorem 3. It can be easily verified that the limit function of any uniformly convergent sequence of functions from $Q^*(X)$ belongs again to $Q^*(X)$. Thus according to Lemma 1 $Q^*(X)$ is a closed linear subspace of the space $M(X)$. Further, if $f \in Q^*(X)$ and $a \in R$, then $f + a \in Q^*(X)$. From this and from the closedness of $Q^*(X)$ it follows that $Q^*(X)$ is a perfect set in $M(X)$.

Now if W is a closed linear subspace of a linear normed space E and $W \neq E$, then W is a nowhere dense set in E (cf. [5]). Therefore in the case $Q^*(X) \neq M(X)$ the set $Q^*(X)$ is a nowhere dense set in $M(X)$.

Let $f_n \in S(X)$ $(n = 1, 2, \ldots)$. Let

$$\{f_n\}_{n=1}^{\infty} \tag{3}$$

converge uniformly to the function f. We prove that $f \in S(X)$. Let G be an open set, $G \subset R$ and let $f^{-1}(G) \neq \emptyset$. Then there exists an $x_0 \in X$ such that $y_0 = f(x_0) \in G$.

Choose an $\varepsilon > 0$ such that $H = (y_0 - 2\varepsilon, y_0 + 2\varepsilon) \subset G$. On account of the uniform convergence of the sequence (3) to the function f there exists an m such that for each $x \in X$ the inequality

$$|f_m(x) - f(x)| < \varepsilon \tag{4}$$

holds. Particularly, $|f_m(x_0) - f(x_0)| < \varepsilon$, therefore

$$f_m(x_0) \in (y_0 - \varepsilon,\ y_0 + \varepsilon) = L, \quad x_0 \in f_m^{-1}(L).$$

Since f_m is a somewhat continuous function, there exists an open set $V_m \subset X$, $V_m \neq \emptyset$ such that $V_m \subset f_m^{-1}(L)$. For $x \in V_m$ we have

$$f_m(x) \in (y_0 - \varepsilon,\ y_0 + \varepsilon). \tag{5}$$

On account of (4), (5) we get for $x \in V_m$

$$|f(x) - f(x_0)| \leqq |f(x) - f_m(x)| + |f_m(x) - f(x_0)| < \varepsilon + \varepsilon = 2\varepsilon.$$

Hence $V_m \subset f^{-1}(H) \subset f^{-1}(G)$.

Further, the function $f + a$ $(a \in R)$ is somewhat continuous if f is somewhat continuous. Hence $S(X)$ is a perfect set in $M(X)$.

The perfectness of the set $Q(X)$ can be proved analogously as the perfectness of $S(X)$ and $Q^*(X)$. In virtue of the inclusion $Q(X) \subset Q^*(X)$ (see (*)) it follows from the previous part of the proof that in the case $Q^*(X) \neq M(X)$ the set $Q(X)$ is nowhere dense in $M(X)$. The proof is complete.

Remark 3. a) If I is any interval, then obviously $Q^*(I) \neq M(I)$. Theorem 3 implies that each of the sets $Q(I)$, $Q^*(I)$ is a perfect nowhere dense set in the space $M(I)$.

b) In the paper [10] it is proved that the set of all functions $f \in M(I_0)$, $I_0 = \langle 0, 1 \rangle$ having the property $\mathscr{D}_1$ is a perfect nowhere dense set in $M(I_0)$. Analogous statements can be obtained also for the classes of all functions $f \in M(I_0)$ having the property $\mathscr{D}_0$ or $\mathscr{D}_2$, respectively.

Problem. In connection with Theorem 1 and Theorem 3 the question arises whether an analogous statement is true for $S(X)$ as for $Q^*(X)$, i.e., whether in the case $S(X) \neq M(X)$ the set $S(X)$ is nowhere dense in $M(X)$.

Addenda in proofs: The problem was already solved by the author of this paper in an affirmative way.

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