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# A NOTE ON WEAKLY BOREL MEASURES

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In [1] S. K. BERBERIAN compared several of the commonly used definitions of "regular measure". In Theorem 3 he proved that

if $\varrho$ is a finite measure on the weakly Borel sets of a locally compact Hausdorff space $X$, the following conditions are equivalent:

(A) $\varrho$ is inner regular,

(B) $\varrho$ is biregular,

(C) $\varrho$ is sesquiregular,

(D) $\varrho$ is outer regular, and there exists a Borel set $E$ such that $\varrho(X - E) = 0$.

In the present paper we show: 1. the assumption of the local compactness of $X$ can be dropped, 2. the conditions (A) and (D) can be replaced by weaker ones, 3. the finiteness of $\varrho$ can be replaced by $(\mathbf{U}, \sigma)$-finiteness.

Let $X$ be an arbitrary nonvoid set of elements. Let $\mathbf{S}$ be the $\sigma$-ring of subsets of $X$, and $\mathbf{C}$ and $\mathbf{U}$ nonempty subfamilies of $\mathbf{S}$. Let $\mu$ be a measure defined on $\mathbf{S}$. Measure $\mu$ is said to be *inner $\mathbf{C}$-regular* on $\mathbf{S}$ if

$$\mu(A) = \sup \{\mu(C) : A \supset C \in \mathbf{C}\} \quad \text{for all sets} \quad A \in \mathbf{S},$$

*outer $\mathbf{U}$-regular* on $\mathbf{S}$ if

$$\mu(A) = \inf \{\mu(U) : A \subset U \in \mathbf{U}\} \quad \text{for all sets} \quad A \in \mathbf{S},$$

and *$(\mathbf{C}, \mathbf{U})$-regular* on $\mathbf{S}$ if it is both inner $\mathbf{C}$-regular and outer $\mathbf{U}$-regular on $\mathbf{S}$.

Troughout the paper $X$ denotes an arbitrary Hausdorff space, $\mathbf{C}$ the family of all compact subsets of $X$, $\mathbf{D}$ the family of all closed subsets of $X$ and $\mathbf{U}$ denotes the family of all open subsets of $X$. By $\mathbf{S}(\mathbf{C})$ and $\mathbf{S}(\mathbf{D})$ we denote the $\sigma$-rings generated by $\mathbf{C}$ and $\mathbf{D}$ respectively.

A measure $\mu$ on $\mathbf{S}(\mathbf{D})$ is said to be *$(\mathbf{U}, \sigma)$-finite* if $X = \bigcup_{n=1}^{\infty} U_n$, $U_n \in \mathbf{U}$, $\mu(U_n) < \infty$ $(n = 1, 2, \ldots)$.

Remark 1. If $\mu$ is a $\sigma$-finite and outer $\mathbf{U}$-regular measure on $\mathbf{S}(\mathbf{D})$ then $\mu$ is $(\mathbf{U}, \sigma)$-finite. In fact, if $E \in \mathbf{S}(\mathbf{D})$ and $\mu(E) < \infty$ then there exists a set $U \in \mathbf{U}$ such that $U \supset E$ and $\mu(U) < \infty$.

We compare the following conditions:

(a) $\mu(U) = \sup\{\mu(D) : U \supset D \in \mathbf{D}\}$ for all sets $U \in \mathbf{U}$ and there exists a set $Y \in \mathbf{S}(\mathbf{C})$ such that $\mu(X - Y) = 0$,

(b) $\mu(U) = \sup\{\mu(C) : U \supset C \in \mathbf{C}\}$ for all sets $U \in \mathbf{U}$,

(c) $\mu$ is inner $\mathbf{C}$-regular on $\mathbf{S}(\mathbf{D})$,

(d) $\mu$ is sesquiregular on $\mathbf{S}(\mathbf{D})$ (i.e. $\mu$ is outer $\mathbf{U}$-regular on $\mathbf{S}(\mathbf{D})$ and satisfies the condition (b)),

(e) $\mu$ is $(\mathbf{C}, \mathbf{U})$-regular on $\mathbf{S}(\mathbf{D})$,

(f) $\mu$ is $(\mathbf{D}, \mathbf{U})$-regular on $\mathbf{S}(\mathbf{D})$ and there exists a set $Y \in \mathbf{S}(\mathbf{C})$ such that $\mu(X - Y) = 0$,

(g) $\mu$ is outer $\mathbf{U}$-regular on $\mathbf{S}(\mathbf{D})$ and there exists a set $Y \in \mathbf{S}(\mathbf{C})$ such that $\mu(X - Y) = 0$,

(h) $\mu(D) = \inf\{\mu(U) : D \subset U \in \mathbf{U}\}$ for all sets $D \in \mathbf{D}$ and there exists a set $Y \in \mathbf{S}(\mathbf{C})$ such that $\mu(X - Y) = 0$.

**Theorem 1.** *If $X$ is an arbitrary Hausdorff topological space and $\mu$ is a $(\mathbf{U}, \sigma)$-finite measure on $\mathbf{S}(\mathbf{D})$, the conditions* (a)–(f) *are equivalent.*

Proof. (a) $\Rightarrow$ (f): Let $E \in \mathbf{S}(\mathbf{D})$ such that $E \subset U_0 \in \mathbf{U}$, $\mu(U_0) < \infty$. The formula $\mu^0(A) = \mu(A \cap U_0)$ defines a finite measure on $\mathbf{S}(\mathbf{D})$. If $U \in \mathbf{U}$ then

$$\mu^0(U) = \mu(U \cap U_0) = \sup\{\mu(D) : U \cap U_0 \supset D \in \mathbf{D}\} = \\ = \sup\{\mu^0(D) : U \cap U_0 \supset D \in \mathbf{D}\} \leqq \sup\{\mu^0(D) : U \supset D \in \mathbf{D}\} \leqq \mu^0(U).$$

By ([2], Theorem 8, p. 43, or example 3, p. 45) $\mu^0$ is $(\mathbf{D}, \mathbf{U})$-regular on $\mathbf{S}(\mathbf{D})$. Hence

$$\mu(E) = \mu^0(E) = \sup\{\mu^0(D) : E \supset D \in \mathbf{D}\} = \sup\{\mu(D) : E \supset D \in \mathbf{D}\}$$

and

$$\mu(E) = \mu^0(E) = \inf\{\mu^0(U) : E \subset U \in \mathbf{U}\} = \inf\{\mu(U \cap U_0) : E \subset U \in \mathbf{U}\} \geqq \\ \geqq \inf\{\mu(U) : E \subset U \in \mathbf{U}\} \geqq \mu(E).$$

Let $A$ be an arbitrary set of $\mathbf{S}(\mathbf{D})$. From the $(\mathbf{U}, \sigma)$-finiteness of $\mu$ it follows that $A = \bigcup_{n=1}^{\infty} (A \cap U_n)$, where $U_n \in \mathbf{U}$, $U_n \subset U_{n+1}$ and $\mu(U_n) < \infty$, $n = 1, 2, \ldots$ According to what was said above, $A \cap U_n$ and hence also $A$ (see the proof of Theorem 3, [5], p. 220) are $(\mathbf{D}, \mathbf{U})$-regular sets according to $\mu$. Hence $\mu$ is $(\mathbf{D}, \mathbf{U})$-regular on $\mathbf{S}(\mathbf{D})$.

(f) $\Rightarrow$ (e): Let $E_0 \in \mathbf{S}(\mathbf{C})$ such that $E_0 \subset C \in \mathbf{C}$. Then

$$\mu(E_0) = \sup\{\mu(D) : E_0 \supset D \in \mathbf{D}\} = \sup\{\mu(C) : E_0 \supset C \in \mathbf{C}\},$$

since $D \in \mathbf{D}$, $D \subset E_0$ implies $D \in \mathbf{C}$.

Let $E \in \mathbf{S}(\mathbf{C})$ be an arbitrary set. Then $E = \bigcup_{n=1}^{\infty} E_n$, where $E_n \in \mathbf{S}(\mathbf{C})$, $E_n \subset E_{n+1}$, $E_n \subset C_n \in \mathbf{C}$ $(n = 1, 2, \ldots)$. Hence $\mu$ is inner $\mathbf{C}$-regular on $\mathbf{S}(\mathbf{C})$. By ([3], Theorem 1, p. 135) $\mu$ is $(\mathbf{C}, \mathbf{U})$-regular on $\mathbf{S}(\mathbf{D})$.

It is trivial that (e) $\Rightarrow$ (d) $\Rightarrow$ (b) and (e) $\Rightarrow$ (c) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a): Since $\mathbf{C} \subset \mathbf{D}$, it is

$$\mu(U) = \sup \{\mu(D) : U \supset D \in \mathbf{D}\} \quad \text{for all} \quad U \in \mathbf{U}\,.$$

From the $(\mathbf{U}, \sigma)$-finiteness of $\mu$ it follows that $X = \bigcup_{n=1}^{\infty} U_n$, $U_n \in \mathbf{U}$, $\mu(U_n) < \infty$ $(n = 1, 2, \ldots)$. By ([3], Lemma 1, p. 136) there exist sets $Y_n \in \mathbf{S}(\mathbf{C})$ such that $\mu(U_n - Y_n) = 0$. Let $Y = \bigcup_{n=1}^{\infty} Y_n$. Then $Y \in \mathbf{S}(\mathbf{C})$ and $\mu(X - Y) \leqq \sum_{n=1}^{\infty} \mu(U_n - Y_n) = 0$.

**Theorem 2.** *If $X$ is a locally compact Hausdorff space and $\mu$ is a $(\mathbf{U}, \sigma)$-finite measure on $\mathbf{S}(\mathbf{D})$, the conditions* (a)–(h) *are equivalent.*

Proof. It is trivial that (f) $\Rightarrow$ (g) $\Rightarrow$ (h).

(h) $\Rightarrow$ (e): From the $(\mathbf{U}, \sigma)$-finiteness of $\mu$ it follows that $\mu(C) < \infty$ for all $C \in \mathbf{C}$. If $C \in \mathbf{C}$ and $C \subset U \in \mathbf{U}$, there exists an open Baire set $O$ such that $C \subset O \subset U$. Hence

$$\mu(C) = \inf \{\mu(U) : C \subset U,\ U \text{ open Baire set}\}\,.$$

This proves the $(\mathbf{C}, \mathbf{U})$-regularity of $\mu$ on $\mathbf{S}(\mathbf{C})$. By ([3], Theorem 1 p. 135) $\mu$ is $(\mathbf{C}, \mathbf{U})$-regular on $\mathbf{S}(\mathbf{D})$.

The other implications follow from Theorem 1.

**Theorem 3.** *If $X$ is an arbitrary Hausdorff topological space and $\mu$ is a finite measure on $\mathbf{S}(\mathbf{D})$, the conditions* (a)–(h) *are equivalent.*

Proof. It is trivial that (f) $\Rightarrow$ (g) $\Rightarrow$ (h).

(h) $\Rightarrow$ (f): By ([2], Theorem 8, p. 43, or example 3 p. 45). The other implications follow from Theorem 1.

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