Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 96 (1971), No. 4, 391--397

Persistent URL: `http://dml.cz/dmlcz/117737`

Časopis pro pěstování matematiky, roč. 96 (1971), Praha

REMARKS ON DENJOY PROPERTY AND $\mathcal{M}_2'$ PROPERTY OF REAL FUNCTIONS

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(Received March 31, 1970)

In the whole paper the interval means a normal non-degenerate interval on the real line E_1 and the measure means the Lebesgue measure on the real line. In what follows $|M|$ denotes the measure of the set M.

The real function $f : \langle 0, 1 \rangle \to E_1$ is said to have the property $\mathcal{M}_2'$ if for each $a \in E_1$ and each closed interval $I \subset \langle 0, 1 \rangle$ each of sets $I \cap E_a(f)$, $I \cap E^a(f)$,

$$E_a(f) = \{x \in \langle 0, 1 \rangle;\ f(x) > a\}\ ,\quad E^a(f) = \{x \in \langle 0, 1 \rangle;\ f(x) < a\}$$

is either void or it has a positive measure (cf. [6]).

Further, the function $f : \langle 0, 1 \rangle \to E_1$ is said to have the Denjoy property if for each two numbers $a, b \in E_1$ and each closed interval $I \subset \langle 0, 1 \rangle$ the set $I \cap E_a^b(f)$, $E_a^b(f) = \{x \in \langle 0, 1 \rangle;\ a < f(x) < b\}$ is either void or it has a positive measure (cf. [1]).

It is obvious from the previous definitions that each function $f : \langle 0, 1 \rangle \to E_1$ with the $\mathcal{M}_2'$ or Denjoy property is Lebesgue measurable.

It is easy to see that each function with the Denjoy property has the $\mathcal{M}_2'$ property, too. L. MIŠÍK has shown the equivalence of these two properties for functions of the first Baire class (cf. [2]).

The function $f : \langle 0, 1 \rangle \to E_1$ is said to have the Darboux property if f maps each interval $I \subset \langle 0, 1 \rangle$ onto an interval or a one-point set.

Denote by F the set of all functions $f : \langle 0, 1 \rangle \to E_1$. For $S \subset F$ we put $CS = F - S$. Denote by M_2', D^*, D the set of all $f \in F$ with the $\mathcal{M}_2'$, Denjoy, Darboux property, respectively. Further B_α $(\alpha \geqq 0)$ denotes the set of all functions $f \in F$ of the Baire class α.

We have already remarked that $D^* \subset M_2'$ and $D^* \cap B_1 = M_2' \cap B_1$. L. Mišík has shown (cf. [2]) that the set

$$S_2 = B_2 \cap [M_2' - (D \cup D^*)] = B_2 \cap M_2' \cap CD \cap CD^*$$

is non-void and he asked whether the set $T_2 = B_2 \cap M'_2 \cap D \cap CD^*$ is non-void, too. J. Lipiński has given an affirmative answer to this question (cf. [1]). He showed by using some properties of Köpcke derivatives that each of the sets S_2, T_2 is non-void.

In this paper we shall give new proofs for the non-voidness of each of the sets S_2, T_2, the proof of the non-voidness of T_2 being based on some properties of certain functions which were defined in the paper [5] by means of subseries of divergent series. Further we shall study some properties of the sets S_2, T_2 as subsets of the metric space $M(0, 1)$ of all bounded functions $f \in F$ (see Theorem 7 below).

At first we show a simple construction of functions $f \in S_2$. Let $A \subset \langle 0, 1 \rangle$ be an F_σ set with the following property: For each interval $I \subset \langle 0, 1 \rangle$ each of the sets $A \cap I$, $A' \cap I$ ($A' = \langle 0, 1 \rangle - A$) has a positive measure (cf. [3], p. 244). R denotes the set of all rational numbers $r \in \langle 0, 1 \rangle$. Put $B = A - R$, $B' = A' - R$. Let t be an arbitrary positive real number. Put $g_t(x) = t$ for $x \in B$, $g_t(x) = -t$ for $x \in B'$ and $g_t(x) = 0$ for $x \in R$.

Theorem 1. *The function g_t belongs to S_2.*

Proof. 1. We shall show at first that $g_t \in B_2$. Let $a \in E_1$, $E^a(g_t) = \{x \in \langle 0, 1 \rangle; g_t(x) < a\}$. Then we have

$$E^a(g_t) = \begin{cases} \emptyset & \text{for} \quad a \leqq -t\,, \\ B' & \text{for} \quad -t < a \leqq 0\,, \\ B' \cup R & \text{for} \quad 0 < a \leqq t\,, \\ \langle 0, 1 \rangle & \text{for} \quad t < a\,. \end{cases}$$

Since A is an F_σ set, B' is a $G_{\delta\sigma}$ set and we see at once that the set $E^a(g_t)$ is a $G_{\delta\sigma}$ set for each a. It can be shown analogously that for each $a \in E_1$ the set $E_a(g_t) = \{x \in \in \langle 0, 1 \rangle; g_t(x) > a\}$ is a $G_{\delta\sigma}$ set.

2. The function g_t has not the Darboux property since $g_t(\langle 0, 1 \rangle) = \{0, t, -t\}$.

3. The function g_t has not the Denjoy property since the set $E^t_{-t}(g_t) = \{x \in \in \langle 0, 1 \rangle; -t < g_t(x) < t\}$ is non-void and its measure is 0.

4. The function g_t has the $\mathcal{M}'_2$ property. Indeed, let $a \in E_1$ and let $I \subset \langle 0, 1 \rangle$ be a closed interval. If $I \cap E^a(g_t) \neq \emptyset$, then $a > -t$ and therefore the set $I \cap B'$ is contained in the set $I \cap E^a(g_t)$. In view of the properties of the set A we have $|I \cap B'| > > 0$ and so $|I \cap E^a(g_t)| > 0$. It can be shown analogously that if $I \cap E_a(g_t) \neq \emptyset$ then $|I \cap E_a(g_t)| > 0$. This completes the proof.

Remark. From the previous theorem we obtain a set of the power c (c is the power of the continuum) of functions from S_2. Since $S_2 \subset B_2$ and the power of the set B_2 is c, we see that the set S_2 has the power c.

In what follows we shall use some functions defined in [5] by subseries of divergent series. Let $\sum_{k=1}^{\infty} |a_k| = +\infty$, $x \in (0, 1\rangle$, $x = \sum_{k=1}^{\infty} \varepsilon_k(x) \cdot 2^{-k}$ (non-terminating dyadic expansion of x, $\varepsilon_k(x) = 0$ or 1 and for an infinite number of k's we have $\varepsilon_k(x) = 1$). Denote by $f = f(\sum_1^\infty a_n)$ the function defined on $(0, 1\rangle$ in the following way: If the series

$$(1) \qquad \sum_{k=1}^\infty \varepsilon_k(x)\, a_k$$

converges and has the sum $S(x)$, then we put $f(x) = S(x)/(1 + |S(x)|)$. If $\sum_{k=1}^\infty \varepsilon_k(x)\, a_k = +\infty$ ($\sum_{k=1}^\infty \varepsilon_k(x)\, a_k = -\infty$), then we put $f(x) = 1$ ($f(x) = -1$). If (1) oscillates, then $f(x) = 0$.

It is well-known (cf. [6] Theorem 1, p. 6) that each function $f \in F$ of the first Baire class with the $\mathcal{M}'_2$ property has the Darboux property. So the inclusion $B_1 \cap M'_2 \subset B_1 \cap D$ holds. In the connection with this fact we shall show that for the functions of the second Baire class the inclusion $B_2 \cap D \subset B_2 \cap M'_2$ is not true.

Theorem 2. *Let $a_k \to 0$ and let the series $\sum_{k=1}^\infty a_k$ fulfil one of the following conditions:*

1) $\sum_{k; a_k \geqq 0} a_k = +\infty, \quad \sum_{k; a_k < 0} |a_k| < +\infty;$
2) $\sum_{k; a_k < 0} a_k = -\infty, \quad \sum_{k; a_k \geqq 0} a_k < +\infty.$

Define the function $g : \langle 0, 1\rangle \to E_1$ in the following way: $g(0) = 1$ in the case 1) and $g(0) = -1$ in the case 2). Further we put $g(x) = f(\sum_1^\infty a_n)(x)$ for $x \in (0, 1\rangle$ (in both cases).

Then $g \in B_2 \cap (D - M'_2)$.

Proof. Let $\sum_{k=1}^\infty a_k$ fulfil the condition 1) (in the case 2) the theorem can be proved in an analogous way). We know that $f(\sum_1^\infty a_n)$ is a function from the second Baire class (see [5], Theorem 2,6). From this it follows easily that $g \in B_2$.

Further it is known that $f(\sum_1^\infty a_n)$ has the Darboux property and the set $\{x \in (0, 1\rangle; f(\sum_1^\infty a_n)(x) = 1\}$ is dense in $(0, 1\rangle$ (see [5], Theorem 2,4 and 1,10). From this it can be easily deduced that $g \in D$.

Since $a_k \to 0$ there exists a sequence $k_1 < k_2 < \ldots$ of natural numbers such that $\sum_{n=1}^\infty |a_{k_n}| < +\infty$. Put $x_0 = \sum_{n=1}^\infty 2^{-k_n} = \sum_{k=1}^\infty \varepsilon_k(x_0)\, 2^{-k}$ ($\varepsilon_k(x_0) = 0$ for $k \neq k_n$ and $\varepsilon_{k_n}(x_0) = 1$, $n = 1, 2, \ldots$). Then it follows from the definition of g that $g(x_0) < 1$ and so $E^1(g) = \{x \in \langle 0, 1\rangle; g(x) < 1\} \neq \emptyset$. From the theorem 1,10 of the paper [5] we get $|\{x \in (0, 1\rangle; \sum_{k=1}^\infty \varepsilon_k(x)\, a_k = +\infty\}| = 1$ and so we have $|E^1(g)| = 0$. Hence $g \notin M'_2$ and so finally $g \in B_2 \cap (D - M'_2)$. This completes the proof.

Remark. In view of Theorem 2 there exists a function $f_1 \in U_2$, $U_2 = B_2 \cap \cap (D - M'_2)$. It is easy to check that each of the functions $f_1 + a$ $(a \in E_1)$ belongs to U_2, too. From this we see at once that the set U_2 has the power c.

We shall prove now the non-voidness of the set T_2. The proof of this fact will be based on some properties of functions $f(\sum_1^\infty a_n)$.

Theorem 3. $T_2 = B_2 \cap M'_2 \cap D \cap CD^* \neq \emptyset$.

Proof. Let C_0 denote the Cantor set in $\langle 0, 1\rangle$. In the closure of the longest component interval J_1 of the set $\langle 0, 1\rangle - C_0$ we construct again a Cantor-like set C_1. Thus the only common points of C_1, C_0 are the end-points of the interval J_1. In the closure of the longest component interval J_2 of the set $\langle 0, 1\rangle - (C_0 \cup C_1)$ we construct again a Cantor-like set C_2. Thus the only common points of C_2, $C_0 \cup C_1$ are the end-points of the interval J_2. We continue this construction by induction. Hence we obtain the set $C = \bigcup_{n=0}^{\infty} C_n$. Obviously $|C| = 0$ and the sets $C_n (n > 0)$, $C_0 \cup C_1 \cup \ldots \ldots \cup C_{n-1}$ have only two common points ($\inf C_n$ and $\sup C_n$). If $I \subset \langle 0, 1\rangle$ is an arbitrary interval, then there exists an m such that $C_m \subset I$. Let $\varphi_n : C_n \to \langle -1, 1\rangle$ denote the function which maps C_n onto $\langle -1, 1\rangle$, φ_n being continuous and non-decreasing on C_n (this function is analogous to the well-known Cantor function - see [3] p. 410).

Further we construct an F_σ set $A \subset \langle 0, 1\rangle$ such that for each interval $P \subset \langle 0, 1\rangle$ we have

$$(*) \qquad |A \cap P| > 0\,, \quad |A' \cap P| > 0 \;\; (A' = \langle 0, 1\rangle - A)$$

(cf. [3], p. 244). Put $G = \langle 0, 1\rangle - C$. Then $\langle 0, 1\rangle = C \cup GA \cup GA'$, the summands on the right-hand side being pairwise disjoint. Let

$$a_k > 0\,, \quad a_k \to 0\,, \quad \sum_{k=1}^{\infty} a_k = +\infty\,; \quad b_k < 0\,, \quad b_k \to 0\,, \quad \sum_{k=1}^{\infty} b_k = -\infty\,.$$

Define the function g in the following way: $g(x) = \varphi_0(x)$ for $x \in C_0 = C_0^*$, $g(x) = = \varphi_1(x)$ for $x \in C_1 - C_0 = C_1^*, \ldots, g(x) = \varphi_n(x)$ for $x \in C_n - (C_0 \cup C_1 \cup \ldots \ldots \cup C_{n-1}) = C_n^*, \ldots$ Further we put $g(x) = f(\sum_1^\infty a_n)(x)$ for $x \in GA$ and $g(x) = = f(\sum_1^\infty b_n)(x)$ for $x \in GA'$.

1) We show that $g \in B_2$. For $a \in E_1$ we have $E^a(g) = M_1 \cup M_2 \cup M_3$, where

$$M_1 = \bigcup_{n=0}^{\infty} \{x \in C_n^*;\ \varphi_n(x) < a\}\,, \quad M_2 = \{x \in GA;\ f(\sum_1^\infty a_n)(x) < a\}\,,$$

$$M_3 = \{x \in GA';\ f(\sum_1^\infty b_n)(x) < a\}\,.$$

Owing to the continuity of φ_n on C_n^* the set $\{x \in C_n^*; \varphi_n(x) < a\}$ is open in C_n^* and therefore it is a $G_{\delta\sigma}$ set. So the set M_1 is a $G_{\delta\sigma}$ set, too. Further $M_2 = GA \cap \{x \in \in (0, 1\rangle; f(\sum_1^\infty a_n)(x) < a\}$. Since GA is a $G_{\delta\sigma}$ set and $\{x \in (0, 1\rangle; f(\sum_1^\infty a_n)(x) < a\}$ is a $G_{\delta\sigma}$ set, too (see [5], Theorem 2,6), the set M_2 is a $G_{\delta\sigma}$ set. In an analogous way we can verify that M_3 is also a $G_{\delta\sigma}$ set. So $E^a(g)$ is a $G_{\delta\sigma}$ set. Analogously it can be shown that $E_a(g)$ is a $G_{\delta\sigma}$ set.

2) We shall show that g has the property $\mathscr{M}_2'$. Let $a \in E_1$ and let $I \subset \langle 0, 1\rangle$ be a closed interval. If

$$I \cap E^a(g) \neq \emptyset, \tag{2}$$

then $a > -1$ and the set $I \cap E^a(g)$ contains the set $I \cap \{x \in \langle 0, 1\rangle; g(x) = -1\}$. According to the theorem 1,10 from [5] we have $g(x) = -1$ for almost all $x \in GA'$ and so owing to the property (*) of the set A we obtain $|I \cap E^a(g)| > 0$. In an analogous way we can show that also the set $I \cap E_a(g)$ is either void or it has a positive measure.

3) We shall show that g has the Darboux property. If $I \subset \langle 0, 1\rangle$ is an interval then there exists an m such that $C_m \subset I$ and so

$$g(I) \supset g(C_m) \supset \varphi_m(C_m^*) = (-1, 1). \tag{3}$$

In view of (*) and $|C| = 0$ we have $|(GA) \cap I| > 0$, $|(GA') \cap I| > 0$. But for almost all $x \in GA (x \in GA')$ we have $g(x) = 1$ $(g(x) = -1)$ (see[5], Theorem 1,10). Owing to this fact there exist two points $x_1, x_2 \in I$ such that $g(x_1) = 1$, $g(x_2) = -1$. This together with (3) gives $g(I) \supset \langle -1, 1\rangle$. But $g(\langle 0, 1\rangle) \subset \langle -1, 1\rangle$, therefore $g(I) = \langle -1, 1\rangle$.

4) We shall prove that g has not the Denjoy property.

Let us choose $a = -1$, $b = 1$, $I = \langle 0, 1\rangle$. Then $I \cap E_a^b(g) = \{x \in \langle 0, 1\rangle; -1 < g(x) < 1\} \neq \emptyset$ and $I \cap E_a^b(g) \subset C \cup M$ where M denotes the set of all such $x \in GA \cup GA'$ for which at least one of the series $\sum_{k=1}^\infty \varepsilon_k(x)\, a_k$, $\sum_{k=1}^\infty \varepsilon_k(x)\, b_k$ converges. It follows from the theorem 1,10 of the paper [5] that $|M| = 0$ and since $|C| = 0$, we have $|I \cap E_a^b(g)| = 0$. This completes the proof.

Remark. It is easy to verify that T_2 has the power c.

It is easy to check that if $f \in M_2'$ or $f \in D^*$, then for each $k \in E_1$ also the function kf belongs to M_2', D^* respectively. In connection with this fact the question arises whether the sum of two functions from M_2' or D^* is again a function belonging to M_2' or D^*, respectively (i.e. whether M_2' or D^* is a linear function space). The following example gives a negative answer to this question.

Example. Let $C \subset \langle 0, 1\rangle$ be the Cantor set, $C' = \langle 0, 1\rangle - C$. Let $A \subset \langle 0, 1\rangle$ be such an F_σ set that for each interval $P \subset \langle 0, 1\rangle$ we have $|A \cap P| > 0, |A' \cap P| > 0$ $(A' = \langle 0, 1\rangle - A)$. Then $\langle 0, 1\rangle = C \cup C'A \cup C'A'$, the summands on the right-hand side being pairwise disjoint. Put $h_1(x) = 1$ for $x \in C \cup C'A$ and $h_1(x) = -1$ for $x \in C'A'$. Further put $h_2(x) = 1$ for $x \in C \cup C'A'$ and $h_2(x) = -1$ for $x \in C'A$. If we put $h = h_1 + h_2$, then $h(x) = 2$ for $x \in C$ and $h(x) = 0$ for $x \in C'$. It is easy to verify that $h_1, h_2 \in D^*$. Since $\{x \in \langle 0, 1\rangle; h(x) > 0\} = C$, the function h does not belong to M_2'.

In what follows we shall study the structure of the space $M(0, 1)$ (with the metric $\varrho(f, g) = \sup_{0 \leq t \leq 1} |f(t) - g(t)|$) from the point of view of the Denjoy and Zahorski's property $\mathscr{M}_2'$. Let $D^*(0, 1)$ and $M_2'(0, 1)$ denote the set $D^* \cap M(0, 1)$, $M_2' \cap M(0, 1)$, respectively. Let us remark that if (X, ϱ) is a metric space, then the symbol $S(p, \delta)$ $(p \in X, \delta > 0)$ denotes the spherical neighbourhood of the point p in the space X, i.e. $S(p, \delta) = \{x \in X; \varrho(p, x) < \delta\}$.

Theorem 4. *Each of the sets $D^*(0, 1)$, $M_2'(0, 1)$ is a perfect non-dense set in $M(0, 1)$.*

Proof. We shall prove the theorem for $D^*(0, 1)$ (the proof for $M_2'(0, 1)$ being analogous). It suffices to prove the following assertions:

1) $D^*(0, 1)$ is a closed subset of the space $M(0, 1)$;
2) $D^*(0, 1)$ has no isolated point;
3) $D^*(0, 1)$ is non-dense in $M(0, 1)$.

1) Let $f_n \in D^*(0, 1)$ $(n = 1, 2, \ldots)$ and let $\{f_n\}_{n=1}^{\infty}$ uniformly converge to f. Then it is known that $f \in D^*(0, 1)$ (cf. [7], Theorem 15).

2) Let $f \in D^*(0, 1)$ and $\delta > 0$. It is easy to check that each of the functions $f + t$, $|t| < \delta$ belongs to $S(f, \delta)$ and $f + t \in D^*(0, 1)$.

3) Since each of the functions $f \in D^*(0, 1)$ is measurable, we have $D^*(0, 1) \subset L(0, 1)$, $L(0, 1)$ being the set of all Lebesgue measurable functions from $M(0, 1)$. But $L(0, 1)$ is a non-dense set in $M(0, 1)$ (see [4]) and therefore $D^*(0, 1)$ is non-dense, too. The proof is complete.

In an analogous way we can prove the following

Theorem 5. *Each of the sets $Z \cap M(0, 1)$, $Z = S_2, T_2, U_2$ is a perfect non-dense set in $M(0, 1)$.*

Proof. It follows from the inclusions $S_2 \subset M_2'$, $T_2 \subset M_2'$ that $S_2 \cap M(0, 1)$, $T_2 \cap M(0, 1)$ are non-dense. Further $U_2 \subset D$ and $D \cap M(0, 1)$ is non-dense in $M(0, 1)$ (see [4]), so that $U_2 \cap M(0, 1)$ is non-dense, too. The perfectness of the sets $Z \cap M(0, 1)$, $Z = S_2, T_2, U_2$ can be proved in an analogous way as the perfectness of $D^*(0, 1)$ was proved in Theorem 4.

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