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A NOTE TO THE CONSTRUCTION OF A LINEAR DIFFERENTIAL EQUATION WITH GIVEN SOLUTIONS

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0. Let be given n functions with continuous n-th derivative in the interval (a, b), such that their Wronskian is different from zero in this interval. It is a well known fact that then there exists a homogeneous linear differential equation of the n-th order with continuous coefficients in the form

$$x^{(n)} + a_1(t)\, x^{(n-1)} + \ldots + a_{n-1}(t)\, x' + a_n(t)\, x = 0$$

such that the given functions form its fundamental system of solutions.

We shall show that a similar result holds even when the number of given functions is $k < n$ and if we know only that the $k \times n$-matrix constructed from the given functions and their derivatives in a similar way as the Wronski matrix has the maximum rank, i.e. k, at each point of the interval (a, b).

1. Let us denote by $W(f_1, f_2, \ldots, f_k)(t)$ the value of the Wronskian of functions $f_1, f_2, \ldots, f_k$ at t.

Theorem 1. *Let a, b be real numbers, $a < b$, k positive integer. Let functions $x_1(t), x_2(t), \ldots, x_k(t)$ have continuous derivative of the k-th order in the interval (a, b) and let the matrix*

$$\begin{pmatrix} x_1(t), & x_2(t), & \ldots, & x_k(t) \\ x_1'(t), & x_2'(t), & \ldots, & x_k'(t) \\ \ldots & \ldots & \ldots & \ldots \\ x_1^{(k)}(t), & x_2^{(k)}(t), & \ldots, & x_k^{(k)}(t) \end{pmatrix}$$

be of the rank k for all $t \in (a, b)$.

Then there exists a function $x_{k+1}(t)$ with continuous k-th derivative in (a, b), such that

$$W(x_1, x_2, \ldots, x_k, x_{k+1})(t) \neq 0$$

for all $t \in (a, b)$.

The assumptions of Theorem 1 obviously do not guarantee that the Wronskian of functions $x_1(t), x_2(t), \ldots, x_k(t)$ is different from zero in (a, b). Nevertheless, the following lemma holds:

Lemma. *Let a, b be real numbers, $a < b$, k, s positive integers, $s \geqq k$. Let functions $x_1(t), x_2(t), \ldots, x_k(t)$ have continuous derivative of the s-th order and let the matrix*

$$\begin{pmatrix} x_1(t), & x_2(t), & \ldots, & x_k(t) \\ x_1'(t), & x_2'(t), & \ldots, & x_k'(t) \\ \ldots & \ldots & \ldots & \ldots \\ x_1^{(s)}(t), & x_2^{(s)}(t), & \ldots, & x_k^{(s)}(t) \end{pmatrix} \tag{1}$$

be of the rank k for all $t \in (a, b)$.

Then there exists a set of numbers a_m, $m = 0, \pm 1, \pm 2, \ldots$, $a < \ldots < a_{-n} < \ldots \ldots < a_{-1} < a_0 < a_1 < \ldots < a_n < \ldots < b$,

$$\lim_{n \to \infty} a_n = b\,, \quad \lim_{n \to \infty} a_{-n} = a$$

such that the Wronskian $W(x_1, x_2, \ldots, x_k)(t) \neq 0$ for all $t \in (a, b)$, $t \neq a_m$, m integer.

Proof of the Lemma follows from Theorem 1 [1]. Denote by N the set of all $t \in (a, b)$ such that $W(x_1, x_2, \ldots, x_k)(t) = 0$ and assume that there is an accumulation point c of the set N, $c \in (a, b)$. The continuity of $W(x_1, x_2, \ldots, x_k)(t)$ implies $c \in N$ which is a contradiction with Theorem 1 [1].

Note that the assumptions of Theorem 1 are those of lemma with $s = k$.

Proof of Theorem 1. Let us choose numbers α_0^i, $i = 0, 1, 2, \ldots, k$ such that

$$\begin{vmatrix} x_1(a_0), & x_2(a_0), & \ldots, & x_k(a_0), & \alpha_0^0 \\ x_1'(a_0), & x_2'(a_0), & \ldots, & x_k'(a_0), & \alpha_0^1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_1^{(k)}(a_0), & x_2^{(k)}(a_0), & \ldots, & x_k^{(k)}(a_0), & \alpha_0^k \end{vmatrix} \neq 0$$

and a function $u(t)$ with continuous k-th derivative in (a_{-1}, a_1), $u^{(i)}(a_0) = \alpha_0^i$ for $i = 0, 1, 2, \ldots, k$. [1]) Evidently there exists $\varepsilon_0 > 0$ such that $W(x_1, x_2, \ldots, x_k, u)(t) \neq \neq 0$ for $t \in \langle a_0 - \varepsilon_0, a_0 + \varepsilon_0 \rangle$. Put

$$x_{k+1}(t) = u(t) \quad \text{for} \quad t \in \langle a_0 - \varepsilon_0, a_0 + \varepsilon_0 \rangle\,.$$

Let us now suppose that the function $x_{k+1}(t)$ has been already defined (and satisfies Theorem 1) on $\langle a_{-j} - \varepsilon, a_j + \varepsilon \rangle$, $\varepsilon > 0$, j nonnegative integer.

[1]) We denote $u^{(0)}(t) = u(t)$.

Let us choose a function $f_j(t)$ continuous on $\langle a_j, a_{j+1}\rangle$ and a function $f_{-j}(t)$ continuous on $\langle a_{-j-1}, a_{-j}\rangle$ such that

$$f_j(t) \neq 0\,, \quad f_{-j}(t) \neq 0 \quad \text{for} \quad t \in \langle a_j, a_{j+1}\rangle\,, \quad t \in \langle a_{-j-1}, a_{-j}\rangle\,,$$

respectively. Moreover, let

$$(2) \qquad f_j(a_j + \varepsilon) = W(x_1, x_2, \dots, x_k, x_{k+1})\,(a_j + \varepsilon)\,,$$
$$f_{-j}(a_{-j} - \varepsilon) = W(x_1, x_2, \dots, x_k, x_{k+1})\,(a_{-j} - \varepsilon)\,.$$

(Since $x_{k+1}(t)$ is defined and satisfies Theorem 1 on $\langle a_{-j} - \varepsilon, a_j + \varepsilon\rangle$, the values of the Wronskian on the righthand side of the last two equations are nonzero and hence such functions f_j, f_{-j} exist.)

Let us consider differential equations

$$(3) \qquad \begin{vmatrix} x_1(t), & x_2(t), & \dots, & x_k(t), & y(t) \\ x_1'(t), & x_2'(t), & \dots, & x_k'(t), & y'(t) \\ \dots & \dots & \dots & \dots & \dots \\ x_1^{(k)}(t), & x_2^{(k)}(t), & \dots, & x_k^{(k)}(t), & y^{(k)}(t) \end{vmatrix} = f_J(t)\,,$$

$J = \pm j$. The coefficient at the highest derivative $y^{(k)}(t)$ in these equations is $W(x_1, x_2, \dots, x_k)\,(t)$. Denote by $Y_+(t), Y_-(t)$ the solution of the equation with $J = j$, $J = -j$ and with the initial condition

$$(4) \qquad Y_+^{(i)}(a_j + \varepsilon) = x_{k+1}^{(i)}(a_j + \varepsilon)\,, \quad Y_-^{(i)}(a_{-j} - \varepsilon) = x_{k+1}^{(i)}(a_{-j} - \varepsilon)$$

respectively, $i = 0, 1, \dots, k - 1$. Since $W(x_1, x_2, \dots, x_k)\,(t) \neq 0$ for $t \in (a_j, a_{j+1}) \cup \cup\, (a_{-j-1}, a_{-j})$, the functions $Y_+(t), Y_-(t)$ are continuous and have continuous k-th derivative in $\langle a_j + \varepsilon, a_{j+1})$, $(a_{-j-1}, a_{-j} - \varepsilon\rangle$, respectively. Moreover, the inequalities

$$W(x_1, x_2, \dots, x_k, Y_+)\,(t) \neq 0\,, \quad W(x_1, x_2, \dots, x_k, Y_-)\,(t) \neq 0$$

hold in the respective intervals.

Further, let us choose (analogously to the first part of the proof) the numbers $\alpha_{j+1}^i, \alpha_{-j-1}^i$, $i = 0, 1, \dots, k$ such that the determinant

$$\begin{vmatrix} x_1(a_{j+1}), & x_2(a_{j+1}), & \dots, & x_k(a_{j+1}), & \alpha_{j+1}^0 \\ x_1'(a_{j+1}), & x_2'(a_{j+1}), & \dots, & x_k'(a_{j+1}), & \alpha_{j+1}^1 \\ \dots & \dots & \dots & \dots & \dots \\ x_1^{(k)}(a_{j+1}), & x_2^{(k)}(a_{j+1}), & \dots, & x_k^{(k)}(a_{j+1}), & \alpha_{j+1}^k \end{vmatrix}$$

as well as the determinant obtained by replacing the index $j + 1$ by $-j - 1$ are different from zero. Let $u_{j+1}(t), u_{-j-1}(t)$ be functions with continuous k-th derivative in $\langle a_j, a_{j+2}\rangle$, $\langle a_{-j-2}, a_{-j}\rangle$ respectively and $u_{j+1}^{(i)}(a_{j+1}) = \alpha_{j+1}^i$, $u_{-j-1}^{(i)}(a_{-j-1}) =$

$= \alpha^i_{-j-1}$ for $i = 0, 1, \ldots, k$. (Such functions obviously exist.) From the continuity of $W(x_1, x_2, \ldots, x_k, u_{j+1})(t)$ and $W(x_1, x_2, \ldots, x_k, u_{-j-1})(t)$ there follows that

$$W(x_1, x_2, \ldots, x_k, u_{j+1})(t) \neq 0\,, \quad W(x_1, x_2, \ldots, x_k, u_{-j-1})(t) \neq 0$$

in some interval $\langle a_{j+1} - \varepsilon', a_{j+1} + \varepsilon' \rangle$, $\langle a_{-j-1} - \varepsilon', a_{-j-1} + \varepsilon' \rangle$ respectively, $\varepsilon' > 0$.

In the interval $\langle a_{j+1} - \varepsilon', a_{j+1} + \varepsilon' \rangle$ put

$$v_{j+1}(t) = \beta_0 u_{j+1}(t) + \sum_{p=1}^{k} \beta_p x_p(t)\,,$$

the constants $\beta_0, \beta_1, \ldots, \beta_k$ being the solution of the linear system

$$\beta_0 u^{(i)}_{j+1}(a_{j+1} - \varepsilon') + \sum_{p=1}^{k} \beta_p x^{(i)}_p(a_{j+1} - \varepsilon') = Y^{(i)}_+(a_{j+1} - \varepsilon')\,,$$

$i = 0, 1, \ldots, k$. Since from the choice of $u_{j+1}(t)$ and ε' there follows that the determinant of this system is nonzero, there exists a unique solution $\beta_0, \beta_1, \ldots, \beta_k$.[2])

Analogously put

$$v_{-j-1}(t) = \gamma_0 u_{-j-1}(t) + \sum_{p=1}^{k} \gamma_p x_p(t)$$

in $\langle a_{-j-1} - \varepsilon', a_{-j-1} + \varepsilon' \rangle$; we obtain the constants $\gamma_0, \gamma_1, \ldots, \gamma_k$ as the (unique) solution of the system

$$\gamma_0 u^{(i)}_{-j-1}(a_{-j-1} + \varepsilon') + \sum_{p=1}^{k} \gamma_p x^{(i)}_p(a_{-j-1} + \varepsilon') = Y^{(i)}_-(a_{-j-1} + \varepsilon')\,.$$

Let us now define

$$x_{k+1}(t) = \begin{cases} Y_+(t) & \text{in the interval } \langle a_j + \varepsilon, a_{j+1} - \varepsilon' \rangle \\ v_{j+1}(t) & \text{in the interval } \langle a_{j+1} - \varepsilon', a_{j+1} + \varepsilon' \rangle \\ Y_-(t) & \text{in the interval } \langle a_{-j-1} + \varepsilon', a_{-j} - \varepsilon \rangle \\ v_{-j-1}(t) & \text{in the interval } \langle a_{-j-1} - \varepsilon', a_{-j-1} + \varepsilon' \rangle\,. \end{cases}$$

By this way it is evidently possible to define the function $x_{k+1}(t)$ on the whole interval (a, b). It follows from the construction that $x_{k+1}(t)$ has all properties required by the assertion of Theorem 1. It is just necessary to verify that the k-th derivative $x^{(k)}_{k+1}(t)$ is continuous at the points $a_j + \varepsilon$, $a_{-j} - \varepsilon$.

There is

$$W(x_1, x_2, \ldots, x_k, Y_+)(a_j + \varepsilon) = f_j(a_j + \varepsilon) = \\ = W(x_1, x_2, \ldots, x_k, x_{k+1})(a_j + \varepsilon)$$

according to the choice of $f_j(t)$; moreover, $Y_+(t)$ fulfils the initial conditions $Y^{(i)}_+(a_j +$

[2]) Moreover, $\beta_0 \neq 0$ since otherwise $W(x_1, x_2, \ldots, x_k, Y_+)(a_{j+1} - \varepsilon') = 0$; hence $W(x_1, x_2, \ldots, x_k, v_{j+1})(t) \neq 0$ implies $W(x_1, x_2, \ldots, x_k, u_{j+1})(t) \neq 0$ and conversely.

$+ \varepsilon) = x^{(i)}_{k+1}(a_j + \varepsilon)$ for $i = 0, 1, \ldots, k - 1$, which implies immediately

$$Y^{(k)}_{+}(a_j + \varepsilon) = x^{(k)}_{k+1}(a_j + \varepsilon)$$

(the derivative of Y_+ being taken from the right, the derivative of x_{k+1} from the left). The continuity of the k-th derivative $x^{(k)}_{k+1}(t)$ at $a_{-j} - \varepsilon$ is proved quite analogously.

2. In this article we shall generalize Theorem 1 assuming that the functions $x_1, x_2, \ldots, x_k$ have continuous derivatives of the s-th order, $s \geqq k$ and that the matrix from Theorem 1 has $s + 1$ rows. (For $s = k$ we get Theorem 1.) We shall prove

Theorem 2. *Let a, b be real numbers, $a < b$, k, s positive integers, $s \geqq k$. Let functions $x_1(t), x_2(t), \ldots, x_k(t)$ have continuous derivatives of the s-th order in the interval (a, b) and let the matrix (1) be of the rank k for all $t \in (a, b)$.*

Then there exists a function $x_{k+1}(t)$ with continuous s-th derivative in (a, b) such that the matrix

$$\begin{pmatrix} x_1(t), & x_2(t), & \ldots, & x_k(t), & x_{k+1}(t) \\ x'_1(t), & x'_2(t), & \ldots, & x'_k(t), & x'_{k+1}(t) \\ \multicolumn{5}{c}{\dotfill} \\ x^{(s)}_1(t), & x^{(s)}_2(t), & \ldots, & x^{(s)}_k(t), & x^{(s)}_{k+1}(t) \end{pmatrix}$$

is of the rank $k + 1$ for all $t \in (a, b)$.

Proof will follow the same lines as that of Theorem 1. If a_m, $m = 0, \pm 1, \pm 2, \ldots$ are the points from Lemma[3]) then again $W(x_1, x_2, \ldots, x_k)(t) \neq 0$ for all $t \in (a, b)$, $t \neq a_m$, $m = 0, \pm 1, \pm 2, \ldots$. We start constructing $x_{k+1}(t)$ at a_0 again, choosing numbers α^i_0, $i = 0, 1, 2, \ldots, s$, a function $u(t)$ and $\varepsilon_0 > 0$ so that

(i) the matrix

$$\begin{pmatrix} x_1(a_0), & x_2(a_0), & \ldots, & x_k(a_0), & \alpha^0_0 \\ x'_1(a_0), & x'_2(a_0), & \ldots, & x'_k(a_0), & \alpha^1_0 \\ \multicolumn{5}{c}{\dotfill} \\ x^{(s)}_1(a_0), & x^{(s)}_2(a_0), & \ldots, & x^{(s)}_k(a_0), & \alpha^s_0 \end{pmatrix}$$

has the rank $k + 1$;

(ii) $u(t)$ has continuous s-th derivative;

(iii) $u^{(i)}(a_0) = \alpha^i_0$, $i = 0, 1, 2, \ldots, s$;

(iv) the matrix

$$\begin{pmatrix} x_1(t), & x_2(t), & \ldots, & x_k(t), & u(t) \\ x'_1(t), & x'_2(t), & \ldots, & x'_k(t), & u'(t) \\ \multicolumn{5}{c}{\dotfill} \\ x^{(s)}_1(t), & x^{(s)}_2(t), & \ldots, & x^{(s)}_k(t), & u^{(s)}(t) \end{pmatrix}$$

has the rank $k + 1$ for all $t \in \langle a_0 - \varepsilon_0, a_0 + \varepsilon_0 \rangle$.

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[3]) Actually, the assumptions of Lemma are the same as those of Theorem 2.

The functions $x_1(t), x_2(t), \ldots, x_k(t), u(t)$ satisfy the assumptions of Lemma on $(a_0 - \varepsilon_0, a_0 + \varepsilon_0)$. Hence there is $\varepsilon > 0$ such that $W(x_1, x_2, \ldots, x_k, u)(t) \neq 0$ for all $t \neq a_0$, $t \in \langle a_0 - \varepsilon, a_0 + \varepsilon \rangle$ [4]); for $t = a_0$ we have (iv).

Put $x_{k+1}(t) = u(t)$ for $t \in \langle a_0 - \varepsilon, a_0 + \varepsilon \rangle$. If $x_{k+1}(t)$ is defined (and satisfies Theorem 2) for all $t \in \langle a_{-j} - \varepsilon, a_j + \varepsilon \rangle$,

$$W(x_1, x_2, \ldots, x_k, x_{k+1})(t) \neq 0$$

for $t \neq a_m$, $m = 0, \pm 1, \pm 2, \ldots, \pm j$, let us consider again equation (3) where the function $f_J(t)$, $J = \pm j$ is defined in the following manner:

(i′) f_J is defined and has continuous s-th derivative in the intervals $\langle a_j, a_{j+1} \rangle$, $\langle a_{-j-1}, a_{-j} \rangle$ respectively;

(ii′) $f_J(t) \neq 0$ in its interval of definition;

(iii′) the values $f_j^{(i)}(a_j + \varepsilon)$, $f_{-j}^{(i)}(a_{-j} - \varepsilon)$ are given so as to satisfy equations

$$\frac{\mathrm{d}^i}{\mathrm{d}t^i} W(x_1, x_2, \ldots, x_k, x_{k+1})(a_j + \varepsilon) = f_j^{(i)}(a_j + \varepsilon)$$

$$\frac{\mathrm{d}^i}{\mathrm{d}t^i} W(x_1, x_2, \ldots, x_k, x_{k+1})(a_{-j} - \varepsilon) = f_{-j}^{(i)}(a_{-j} - \varepsilon),$$

$i = 0, 1, \ldots, s - k$ (for $i = 0$, this equations are equivalent to (2)).[5])

The solutions Y_+, Y_- with the corresponding initial condition (4) have then continuous s-th derivative (since the same holds for $x_1, x_2, \ldots, x_k, f_J$) and, moreover,

$$Y_+^{(i)}(a_j + \varepsilon) = x_{k+1}^{(i)}(a_j + \varepsilon), \quad Y_-^{(i)}(a_{-j} - \varepsilon) = x_{k+1}^{(i)}(a_{-j} - \varepsilon)$$

for $i = 0, 1, 2, \ldots, s$. In fact, for $i = 0, 1, \ldots, k - 1$ these relations coincide with the initial conditions; for $i = k, k + 1, \ldots, s$ we get them successively from (iii′).

Let us now choose numbers ${}_p\alpha^i_{j+1}$, ${}_p\alpha^i_{-j-1}$, $i = 0, 1, 2, \ldots, s$, $p = k + 1, k + 2, \ldots$ $\ldots, s + 1$ so that

$$(5) \qquad \begin{vmatrix} x_1(a_{j+1}), & \ldots, & x_k(a_{j+1}), & {}_{k+1}\alpha^0_{j+1}, & \ldots, & {}_{s+1}\alpha^0_{j+1} \\ x_1'(a_{j+1}), & \ldots, & x_k'(a_{j+1}), & {}_{k+1}\alpha^1_{j+1}, & \ldots, & {}_{s+1}\alpha^1_{j+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x_1^{(s)}(a_{j+1}), & \ldots, & x_k^{(s)}(a_{j+1}), & {}_{k+1}\alpha^s_{j+1}, & \ldots, & {}_{s+1}\alpha^s_{j+1} \end{vmatrix} \neq 0 \; ^{6})$$

and similarly for a_{-j-1}. Let ${}_pu_{j+1}(t)$, $p = k + 1, k + 2, \ldots, s + 1$ be functions that fulfil:

(i″) they have continuous s-th derivative in $\langle a_j, a_{j+2} \rangle$, $\langle a_{-j-2}, a_{-j} \rangle$ respectively;

[4]) Otherwise a_0 would be an accumulation point of the zero points of the Wronskian which contradicts Lemma.

[5]) This does not contradict (ii′) since in particular $W(x_1, x_2, \ldots, x_{k+1})(a_j + \varepsilon) \neq 0 \neq$ $\neq W(x_1, x_2, \ldots, x_{k+1})(a_{-j} - \varepsilon)$.

[6]) This is possible since the rank of (1) for $t = a_{j+1}$ is k.

(ii″) ${}_p u^{(i)}_{j+1}(a_{j+1}) = {}_p\alpha^i_{j+1}$, ${}_p u^{(i)}_{-j-1}(a_{-j-1}) = {}_p\alpha^i_{-j-1}$ for $i = 0, 1, 2, \ldots, s$, $p = k + 1, k + 2, \ldots, s + 1$.

Again there exists $\varepsilon' > 0$ such that

$$(6)\qquad W(x_1, x_2, \ldots, x_k, {}_{k+1}u_{j+1}, \ldots, {}_{s+1}u_{j+1})(t) \neq 0\,,$$
$$W(x_1, x_2, \ldots, x_k, {}_{k+1}u_{-j-1}, \ldots, {}_{s+1}u_{-j-1})(t) \neq 0$$

holds for all t from the interval $\langle a_{j+1} - \varepsilon', a_{j+1} + \varepsilon' \rangle$, $\langle a_{-j-1} - \varepsilon', a_{-j-1} + \varepsilon' \rangle$, respectively.

Now put

$$v_{j+1}(t) = \sum_{p=1}^{k} \beta_p x_p(t) + \sum_{p=k+1}^{s+1} \beta_p\, {}_p u_{j+1}(t)$$

the constants β_p, $p = 1, 2, \ldots, s + 1$ being the solution of the system

$$\sum_{p=1}^{k} \beta_p x_p^{(i)}(a_{j+1} - \varepsilon') + \sum_{p=k+1}^{s+1} \beta_p\, {}_p u^{(i)}_{j+1}(a_{j+1} - \varepsilon') = Y_+^{(i)}(a_{j+1} - \varepsilon')\,,$$

$i = 0, 1, \ldots, s$.[7]) Since the determinant of the system is nonzero according to (6) there exists a unique solution $\beta_1, \beta_2, \ldots, \beta_{s+1}$. The constants $\beta_{k+1}, \ldots, \beta_{s+1}$ are not simultaneously equal to zero. In fact, if

$$v_{j+1}(t) = \sum_{p=1}^{k} \beta_p x_p(t)$$

then also

$$Y_+^{(i)}(a_{j+1} - \varepsilon') = \sum_{p=1}^{k} \beta_p x_p^{(i)}(a_{j+1} - \varepsilon')$$

$i = 0, 1, 2, \ldots, s$. However, this means that

$$W(x_1, x_2, \ldots, x_k, Y_+)(a_{j+1} - \varepsilon') = 0$$

which is not possible according to the construction of Y_+.

Further, the definition of β_p implies that

$$Y_+^{(i)}(a_{j+1} - \varepsilon') = v^{(i)}_{j+1}(a_{j+1} - \varepsilon')\,,$$

$i = 0, 1, 2, \ldots, s$.

Analogously we define the function $v_{-j-1}(t)$ in the interval $\langle a_{-j-1} - \varepsilon', a_{-j-1} + \varepsilon' \rangle$. (It may be necessary to make ε' smaller.)

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[7]) We have here the derivatives from the left and from the right analogously to the proof of Theorem 1.

Let us now put

$$x_{k+1}(t) = \begin{cases} Y_+(t) & \text{in the interval } \langle a_j + \varepsilon, a_{j+1} - \varepsilon' \rangle \\ v_{j+1}(t) & \text{in the interval } \langle a_{j+1} - \varepsilon', a_{j+1} + \varepsilon' \rangle \\ Y_-(t) & \text{in the interval } \langle a_{-j-1} + \varepsilon', a_{-j} - \varepsilon \rangle \\ v_{-j-1}(t) & \text{in the interval } \langle a_{-j-1} - \varepsilon', a_{-j-1} + \varepsilon' \rangle . \end{cases}$$

Continuing like that we can define $x_{k+1}(t)$ for all $t \in (a, b)$. It is evident that all assertions of Theorem 2 hold, in particular the continuity of the s-th derivative of $x_{k+1}(t)$. Moreover, it is evident that even

$$W(x_1, x_2, \ldots, x_{k+1})(t) \neq 0$$

for all $t \in (a, b)$, $t \neq a_m$, $m = 0, \pm 1, \pm 2, \ldots$

3. From the both Theorems there follows

Corollary. *Let be given functions $x_1(t), x_2(t), \ldots, x_k(t)$ with continuous derivative of n-th order in (a, b), $k < n$. Let the matrix*

$$\begin{pmatrix} x_1(t), & x_2(t), & \ldots, & x_k(t) \\ x_1'(t), & x_2'(t), & \ldots, & x_k'(t) \\ \ldots & \ldots & \ldots & \ldots \\ x_1^{(n-1)}(t), & x_2^{(n-1)}(t), & \ldots, & x_k^{(n-1)}(t) \end{pmatrix}$$

have the rank k for all $t \in (a, b)$.

Then there exists a differential equation

$$x^{(n)} + a_1(t)\, x^{(n-1)} + \ldots + a_{n-1}(t)\, x' + a_n(t)\, x = 0$$

with continuous coefficients $a_i(t)$, $i = 1, 2, \ldots, n$ such that the functions $x_1(t), x_2(t), \ldots \ldots, x_k(t)$ are its solutions on (a, b).

Proof. Completing the system of functions $x_1(t), x_2(t), \ldots, x_k(t)$ according to Theorem 2 $(n - k)$-times we get a system $x_1(t), x_2(t), \ldots, x_n(t)$ with the Wronskian different from zero for all $t \in (a, b)$. The functions $x_{k+1}(t), \ldots, x_n(t)$ have continuous derivatives of the $(n - 1)$-st order.

To be able to write the required differential equation, it is sufficient to use Lemma 6 [2, p. 76]. According to this Lemma, to any function $x(t)$ with continuous k-th derivative in (a, b) and to an arbitrary sequence of numbers $\varepsilon_p > 0$, $p = 0, 1, \ldots$, $\lim_{p \to \infty} \varepsilon_p = 0$, there exists a function $\xi(t)$ analytic in (a, b) and such that for the difference

$$\Delta^{(i)}(t) = |x^{(i)}(t) - \xi^{(i)}(t)|$$

there is $\Delta^{(i)}(t) < \varepsilon_p$, $i = 0, 1, \ldots, k$ for $t \in (a, a_{-p}\rangle \cup \langle a_p, b)$. It is evident that by a proper choice of numbers ε_p it is possible to keep — after replacing $x_{k+q}(t)$ by

$\xi_{k+q}(t)$, $q = 1, 2, \ldots, n - q$ – the inequality for the Wronskian, viz. $W(x_1, x_2, \ldots \ldots, x_k, \xi_{k+1}, \ldots, \xi_n)(t) \neq 0$. Now the required equation can be written in the form

$$W(x_1, x_2, \ldots, x_k, \xi_{k+1}, \ldots, \xi_n, x)(t) = 0$$

whose all coefficients are continuous in (a, b) and the coefficient at $x^{(n)}$ is different from zero since it is equal to $W(x_1, x_2, \ldots, x_k, \xi_{k+1}, \ldots, \xi_n)(t)$.

Author's Note. The paper being already in print, the author's attention was drawn to the paper by Ascoli, G.: *Sulla decomposizione degli operatori differenziali lineari.* Revista (Univ. Nac. Tucuman), Ser. A, 1 (1940), pp. 189–215, where (p. 210) a theorem identical to Corollary of the present paper is proved. However, the method of Ascoli yields just one equation (uniquely determined by the given functions) which has the required properties.

References

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