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ON DISTRIBUTIONAL SOLUTIONS OF CERTAIN EQUATIONS INVOLVING A RETARDED ARGUMENT

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The present paper deals with a certain type of vector-integral equations with retarded argument. Several theorems on the existence, uniqueness, majorization and boundedness of the solution are proved by using properties of certain operators.

Let $\mathscr{D}$ be the set of all n-vector-valued, infinitely differentiable functions $\varphi(t)$ with compact support, and let $\mathscr{D}'$ be the set of all distributions on $\mathscr{D}$ which vanish on the set $(-\infty, 0)$.

A linear operator A mapping $\mathscr{D}'$ into itself will be called continuous, if for any sequence $x_m \to x$, $x_m \in \mathscr{D}'$ we have $Ax_m \to Ax$. Furthermore, let A_m, $m = 1, 2, \ldots$ be linear continuous operators mapping $\mathscr{D}'$ into itself; if A is an operator from $\mathscr{D}'$ into itself such that $A_m x \to Ax$ for any $x \in \mathscr{D}'$, we shall call the sequence A_m convergent to A and write $A_m \to A$. Analogously, if for a sequence B_i we have $\sum_{i=1}^{m} B_i = S_m \to S$, we shall write $S = \sum_{i=1}^{\infty} B_i$.

Let $T > 0$ and let $\mathfrak{B}_T$ be the class of all operators B mapping $\mathscr{D}'$ into itself and having the following properties:

1. B is linear and continuous.
2. For any $x \in \mathscr{D}'$ such that $x = 0$ on $(-\infty, a)$ we have $Bx = 0$ on $(-\infty, a + T)$.

The class $\mathfrak{B}_T$ is clearly nonempty, because the operator P_T, defined on $\mathscr{D}'$ by $(P_T x, \varphi) = (x, \varphi(t + T))$, $\varphi \in \mathscr{D}$, belongs to $\mathfrak{B}_T$. Moreover, $\mathfrak{B}_T$ with customary operations of sum and product is an algebraic (noncommutative) ring.

Theorem 1. *Let $B \in \mathfrak{B}_T$; then the operator $I + B$ is invertible (I signifies the identity operator), and $(I + B)^{-1} - I = \tilde{B} \in \mathfrak{B}_T$. Moreover,*

$$\tilde{B} = \sum_{i=1}^{\infty} (-1)^i B^i . \tag{1}$$

Proof. First show that the series in (1) converges. Actually, by assumption 2., $B^i x = 0$ on $(-\infty, iT)$ for any i and $x \in \mathscr{D}'$; consequently, if $S_m = \sum_{i=1}^{m} (-1)^i B^i$, then the sequence of numbers $(S_m x, \varphi)$ converges for any chosen $\varphi \in \mathscr{D}$. Since $S_m x \in \mathscr{D}'$ for any m, the sequence $S_m x$ converges to a distribution by a well-known theorem (cf. [1], p. 37). Moreover, this distribution obviously belongs to $\mathscr{D}'$; hence, (1) is meaningful and $\tilde{B}$ maps $\mathscr{D}'$ into itself.

Furthermore, it is clear that $\tilde{B}$ is linear and satisfies condition 2. Next, let $x_m \in \mathscr{D}'$, $m = 1, 2, \ldots$ and $x_m \to x$. Choosing a $\varphi \in \mathscr{D}$ denote r_φ the least integer such that $\sup(\operatorname{supp} \varphi) < r_\varphi T$. Then we have

$$(\tilde{B}x, \varphi) = (\sum_{i=1}^{r_\varphi} (-1)^i B^i x, \varphi) = \sum_{i=1}^{r_\varphi} (-1)^i (B^i x, \varphi),$$

and, for any $m > 0$,

$$(\tilde{B}x_m, \varphi) = (\sum_{i=1}^{r_\varphi} (-1)^i B^i x_m, \varphi) = \sum_{i=1}^{r_\varphi} (-1)^i (B^i x_m, \varphi).$$

Because $(B^i x_m, \varphi) \to (B^i x, \varphi)$ for any $i \geq 1$ by condition 1., it follows that $(\tilde{B}x_m, \varphi) \to \to (\tilde{B}x, \varphi)$. Hence, $\tilde{B}$ is continuous so that $\tilde{B} \in \mathfrak{B}_T$.

Choosing now $x \in \mathscr{D}'$, we have by (1),

$$q_m = (I + S_m)(I + B)x \to (I + \tilde{B})(I + B)x;$$

however, $q_m = x + (-1)^m B^{m+1} x$, and $B^m x \to 0$ as $m \to \infty$ by condition 2. Hence, $q_m \to x$; consequently, $(I + \tilde{B})(I + B) = I$. Conversely, (1) yields for any $x \in \mathscr{D}'$,

$$(I + S_m)x \to (I + \tilde{B})x \quad \text{as} \quad m \to \infty,$$

so that, by continuity of $I + B$,

$$h_m = (I + B)(I + S_m)x \to (I + B)(I + \tilde{B})x.$$

As above, $h_m \to x$, i.e., $I = (I + B)(I + \tilde{B})$. Hence, $I + B$ is invertible, $(I + B)^{-1} = I + \tilde{B}$ and the theorem is proven.

Corollary 1. *If $f \in \mathscr{D}'$ and $B \in \mathfrak{B}_T$, then the equation*

$$x + Bx = f \tag{2}$$

has a unique solution in $\mathscr{D}'$. Moreover, x depends continuously on the right-hand side f, i.e., if $f_m \to f$, $f_m \in \mathscr{D}'$, then the solutions x_m of $x_m + Bx_m = f_m$ converge to x.

Let us now turn our attention to a specific subclass of $\mathfrak{B}_T$. Let the system $\mathfrak{U}$ of operators mapping $\mathscr{D}'$ into itself have the same meaning as in [3], p. 161 except for the fact that members of $\mathscr{D}'$ are vectors here; then we have the following proposition.

Theorem 2. *Let* $A_i \in \mathfrak{U}$, $i = 1, 2, \ldots,$ *and let* $0 < T_1 < T_2 < \ldots,$ $T_i \to \infty$ *as* $i \to \infty$; *then the operator* $B = \sum_{i=1}^{\infty} A_i P_{T_i}$ *belongs to* $\mathfrak{B}_{T_1}$.

Proof. If $x \in \mathscr{D}'$ and $x = 0$ on $(-\infty, a)$, then $P_{T_i}x$ vanishes on $(-\infty, a + T_i)$ and so does $A_i(P_{T_i}x)$ by Theorem 5.4–15 in [3], p. 162. Hence, using the condition $T_i \to \to \infty$ and the same argument as in the proof of Theorem 1, we conclude that the definition of B is meaningful and B satisfies condition 2. The proof of continuity follows the same pattern as above.

If $T > 0$, let $\mathfrak{C}_T$ be the class of all operators $\sum_{i=1}^{m} A_i P_{T_i}$, where $A_i \in \mathfrak{U}$, $T_i \geqq T$ and m is arbitrary, finite.

Clearly, $\mathfrak{C}_T \subset \mathfrak{B}_T$; moreover, we have the proposition:

Theorem 3. $\mathfrak{C}_T$ *with ordinary operations of sum and product is an algebraic ring.*

For the proof the following auxiliary statement will be necessary.

Lemma 1. *Let* $W(t, \tau)$ *be an* $n \times n$ *matrix function defined and infinitely differentiable for* $0 \leqq \tau \leqq t < \infty$, *and let* $T > 0$; *then an* $n \times n$ *matrix function* $\tilde{W}(t, \tau)$, *defined and infinitely differentiable for* $0 \leqq \tau \leqq t < \infty$, *exists such that for any* $x \in \mathscr{D}'$,

$$P_T[Wx] = [\tilde{W}(P_T x)]\,. \tag{3}$$

(For the meaning of $[Wx]$, see [3], p. 154.)

Proof. By definition, for any $x \in \mathscr{D}'$ and $\varphi \in \mathscr{D}$,

$$\mu = (P_T[Wx], \varphi) = ([Wx], \varphi(t + T)) = (x, \{\varphi(t + T)\}_W) =$$
$$= \left(x, -\int_{-\infty}^{t} \overline{W_C(\tau, t)}\, \varphi(\tau + T)\, \mathrm{d}\tau + \left(\int_{-\infty}^{\infty} \overline{W_C(\tau, t)}\, \varphi(\tau + T)\, \mathrm{d}\tau\right) . \int_{-\infty}^{t} \varphi_0(\tau)\, \mathrm{d}\tau\right),$$

where the bar signifies the complex conjugate, $W_C(t, \tau)$ is a smooth extension of $W(t, \tau)$ onto the entire t, τ-plane and $\varphi_0(t)$ is any scalar testing function satisfying the conditions $\int_{-\infty}^{\infty} \varphi_0(t)\, \mathrm{d}t = 1$ and $\operatorname{supp} \varphi_0 \subset (-\infty, 0)$. Since $\varphi_0(t)$ is otherwise arbitrary, let us choose it so as to have $\operatorname{supp} \varphi_0 \subset (-\infty, -T)$.

Introducing the substitution $\tau + T = \sigma$ into the above equality, we obtain

$$\mu = \left(x, -\int_{-\infty}^{t+T} \overline{W_C(\sigma - T, t)}\, \varphi(\sigma)\, \mathrm{d}\sigma + \right.$$
$$\left. + \left(\int_{-\infty}^{\infty} \overline{W_C(\sigma - T, t)}\, \varphi(\sigma)\, \mathrm{d}\sigma\right) \int_{-\infty}^{t+T} \varphi_0(\sigma - T)\, \mathrm{d}\sigma\right).$$

Keeping W_C fixed for the rest of the proof, define for $0 \leqq \tau \leqq t < \infty$ the smooth matrix function $\tilde{W}(t, \tau)$ by $\tilde{W}(t, \tau) = W_C(t - T, \tau - T)$; then for a smooth extension of $\tilde{W}$ onto the entire plane we may clearly take the matrix $W_C(t - T, \tau - T)$ itself, i.e. we can write $\tilde{W}_C(t, \tau) = W_C(t - T, \tau - T)$. Moreover, putting $\tilde{\varphi}_0(t) = \varphi_0(t - T)$ we will have $\operatorname{supp} \tilde{\varphi}_0 \subset (-\infty, 0)$. Introducing these new quantities into the last equation for μ, we obtain by the definition of the product $[Wx]$,

$$\mu = \left(x, -\int_{-\infty}^{t+T} \overline{W_C(\sigma, t + T)}\, \varphi(\sigma)\, d\sigma + \left(\int_{-\infty}^{\infty} \overline{W_C(\sigma, t + T)}\, \varphi(\sigma)\, d\sigma\right) \cdot \right.$$

$$\left. \cdot \int_{-\infty}^{t+T} \tilde{\varphi}_0(\sigma)\, d\sigma\right) = \left(P_T x, -\int_{-\infty}^{t} \overline{\tilde{W}_C(\sigma, t)}\, \varphi(\sigma)\, d\sigma + \left(\int_{-\infty}^{\infty} \overline{\tilde{W}_C(\sigma, t)}\, \varphi(\sigma)\, d\sigma\right) \cdot \right.$$

$$\left. \cdot \int_{-\infty}^{t} \tilde{\varphi}(\sigma)\, d\sigma\right) = (P_T x, \{\varphi(t)\}_{\tilde{W}}) = ([\tilde{W}(P_T x)], \varphi)\,.$$

Hence, the lemma is proved.

Proof of Theorem 3. The conclusion $R, S \in \mathfrak{E}_T \Rightarrow R + S \in \mathfrak{E}_T$ is trivial; thus, prove only that $RS \in \mathfrak{E}_T$ whenever $R, S \in \mathfrak{E}_T$. For this, however, it suffices to show that for any $A_1, A_2 \in \mathfrak{U}$ and $T_1, T_2 > 0$ an operator $A \in \mathfrak{U}$ exists such that $A_1 P_{T_1} A_2 P_{T_2} = A P_{T_1 + T_2}$. Recalling the definition of $\mathfrak{U}$, for A_1, A_2 there exist smooth matrix functions W_1, W_2 and integers k, l such that $A_1 x = [W_1 x]^{(k)}$ and $A_2 x = [W_2 x]^{(l)}$; thus, for any $x \in \mathscr{D}'$ we have

$$(A_1 P_{T_1} A_2 P_{T_2})\, x = [W_1(P_{T_1}[W_2(P_{T_2} x)]^{(l)})]^{(k)} = [W_1(P_{T_1}[W_2(P_{T_2} x)])^{(l)}]^{(k)} =$$
$$= [W_1[\tilde{W}_2(P_{T_1 + T_2} x)]^{(l)}]^{(k)} = A_1 \tilde{A}_2 P_{T_1 + T_2} x$$

due to Lemma 1 with the notation $\tilde{A}_2 x = [\tilde{W}_2 x]$. Finally, $A_1 \tilde{A}_2 \in \mathfrak{U}$ by Theorem 5.4–12 in [3], p. 162; the proof is concluded.

From Theorems 3 and 1 it follows that, for $B \in \mathfrak{E}_T$, $(I + B)^{-1}$ is the limit of a sequence with terms from $\mathfrak{E}_T$.

Remark 1. Observe also the following fact. Since $[Wx]$ is a regular distribution whenever x is (cf. Theorem 5.4–5 and p. 118 in [3]), Theorem 1 yields immediately the proposition: Let $a_i(t)$, $W_i(t, \tau)$ be smooth matrix functions for $t \geqq 0$ and $0 \leqq \tau \leqq t$, respectively, $T_i > 0$, $i = 1, 2, \ldots, m$. If $f \in \mathscr{D}'$ is a regular distribution, then the unique solution $x \in \mathscr{D}'$ of the equation $x + \sum_{i=1}^{m} (a_i P_{T_i} x + [W_i(P_{T_i} x)]) = f$ is also regular.

Example 1. Let $0 < T_1 < T_2 \ldots < T_m$, and let $\tilde{\varphi}(t)$ be a vector function defined and continuous on $[-T_n, 0]$; furthermore, let $A_i(t)$, $i = 1, 2, \ldots, m$ and $A(t)$ be matrix functions which are defined and smooth on $[0, \infty)$, and $\tilde{f}(t)$ be a locally integrable vector function on $[0, \infty)$.

Assume that there exists a vector function $\tilde{x}(t)$, continuous on $[-T_n, \infty)$ and absolutely continuous on $[0, \infty)$ such that

(4) $$\tilde{x}(t) = \tilde{\varphi}(t) \quad \text{on} \quad [-T_n, 0)$$

and

(5) $$\tilde{x}'(t) = A(t)\,\tilde{x}(t) + \sum_{i=1}^{m} A_i(t)\,\tilde{x}(t - T_i) + \tilde{f}(t)$$

almost everywhere in $[0, \infty)$.

It can be readily verified that such $\tilde{x}(t)$ satisfies the equation

(6) $$\tilde{x}(t) = \tilde{\varphi}(0) + \int_0^t A(\tau)\,\tilde{x}(\tau)\,d\tau + \sum_{i=1}^{m} \int_0^t A_i(\tau)\,\tilde{x}(\tau - T_i)\,d\tau + \int_0^t \tilde{f}(\tau)\,d\tau\,, \quad t \geqq 0$$

and vice versa.

Next, introduce new vector functions as follows:

$$x(t) = \tilde{x}(t) \quad \text{on} \quad [0, \infty)\,, \qquad f(t) = \tilde{f}(t) \quad \text{on} \quad [0, \infty)\,,$$
$$= 0 \quad \text{elsewhere}\,, \qquad = 0 \quad \text{elsewhere}\,,$$

$$\varphi_i(t) = \tilde{\varphi}(t - T_i) \quad \text{on} \quad [0, T_i]\,,$$
$$= 0 \quad \text{elsewhere}\,,$$

i $= 1, 2, \dots m$. Then (6) is equivalent to the equation

(7) $$x(t) = \tilde{\varphi}(0)\,H_0 + $$
$$+ \int_0^t A(\tau)\,x(\tau)\,d\tau + \sum_{i=1}^{m} \int_0^t A_i(\tau)\,(\varphi_i(\tau) + x(\tau - T_i))\,d\tau + \int_0^t f(\tau)\,d\tau$$

holding for every t. Since $x, f, \varphi_i \in \mathscr{D}'$ if considered as distributions, (7) can be written as

(8) $$x = [A_\tau x] + \sum_{i=1}^{m} [A_{i\tau} P_{T_i} x] + \tilde{\varphi}(0)\,H_0 + \sum_{i=1}^{m} [A_{i\tau}\varphi_i] + f^{(-1)}\,,$$

(cf. Chapter 5.4, [3]).

However, if, conversely, $x \in \mathscr{D}'$ is regular and satisfies (8) in distributional sense, then equation (7) holds for almost every t; because the right-hand side of (7) is continuous for $t \geqq 0$, $x(t)$ can be chosen such that (7) holds everywhere. Thus, (7) and (8) are equivalent.

Finally, the operator M defined by $Mx = x - [A_\tau x]$ is invertible (cf. Theorem 5.6–5 in [3], p. 181), and $M^{-1} \in \mathfrak{U}$; applying M^{-1} to both sides of (8) we obtain an equivalent equation which has the form considered in the above Remark 1. Hence, (4) and (5) stated as a problem, have a unique solution (in classical sense).

Let us now follow a different trend of considerations, that is problems concerning a majorization of solutions. To this purpose we are going to introduce a partial ordering into $\mathscr{D}'$.

Let a be a matrix; if $a_{ik} \geqq 0$ for all elements of a, a will be called nonnegative and we shall write $a \geqq 0$. If for two matrices a, b we have $a - b \geqq 0$, we shall write $a \geqq b$ or $b \leqq a$. Observe that if A is an $n \times n$ matrix and z an n-vector, then $A \geqq 0$ exactly if $Az \geqq 0$ for every $z \geqq 0$. Other elementary rules are obvious.

Next, let $\varphi \in \mathscr{D}$; we write $\varphi \geqq 0$ if $\varphi(t) \geqq 0$ for every t; furthermore, if $f \in \mathscr{D}'$ is such that $(f, \varphi) \geqq 0$ for every $\varphi \in \mathscr{D}$, $\varphi \geqq 0$, we call f nonnegative and write $f \geqq 0$. As known, nonnegative distributions are in fact nonnegative measures. (Cf. [2], p. 28.) Also, it is clear that, for a regular $f \in \mathscr{D}'$, $f \geqq 0$ exactly if the corresponding vector function $f(t)$ is nonnegative for almost every t.

Finally, let A be a linear continuous operator mapping $\mathscr{D}'$ into itself; A will be called nonnegative, if $Ax \geqq 0$ for every $x \geqq 0$, $x \in \mathscr{D}'$. This fact will be signified by $A \geqq 0$.

The meaning of symbols $A - B \geqq 0$, A, B being operators, or $f - g \geqq 0, f, g \in \mathscr{D}'$ is straightforward.

Let us now state the following simple proposition.

Theorem 4. *If $B_1, B_2 \in \mathfrak{B}_T$ and $0 \leqq B_1 \leqq B_2$, then $0 \leqq (I - B_1)^{-1} \leqq (I - B_2)^{-1}$.*

Actually, from $0 \leqq B_1 \leqq B_2$ we have $0 \leqq B_1^k \leqq B_2^k$ for any integer k, and consequently, $0 \leqq \sum_{k=1}^{m} B_1^k \leqq \sum_{k=1}^{m} B_2^k$ for any m; thus, with $x \geqq 0$ and $\varphi \geqq 0$,

$$0 \leqq \Big(\sum_{k=1}^{m} B_1^k x, \varphi\Big) \leqq \Big(\sum_{k=1}^{m} B_2^k x, \varphi\Big).$$

Letting m tend to infinity and recalling Theorem 1 we conclude the proof.

Corollary 2. *Let $B_1, B_2 \in \mathfrak{B}_T$, $0 \leqq B_1 \leqq B_2$ and let $f_1, f_2 \in \mathscr{D}'$, $0 \leqq f_1 \leqq f_2$. Then for the solutions x_1 and x_2 of equations*

$$x_1 - B_1 x_1 = f_1\,, \quad x_2 - B_2 x_2 = f_2$$

we have $0 \leqq x_1 \leqq x_2$.

(The proof is obvious.)

Lemma 2. Let $W(t, \tau)$ be a smooth matrix function and let $W(t, \tau) \geqq 0$ for every $0 \leqq \tau \leqq t < \infty$; then $[Wx] \geqq 0$ for any $x \geqq 0$, $x \in \mathscr{D}'$.

Proof. Let $x \geqq 0$; then, for any $\varphi \in \mathscr{D}$, we have by definition

$$([Wx], \varphi) = (x, \varphi_W)\,, \tag{9}$$

$$\varphi_W(t) = -\int_{-\infty}^{t} \overline{W_C(\tau, t)}\, \varphi(\tau)\, d\tau + \left(\int_{-\infty}^{\infty} \overline{W_C(\tau, t)}\, \varphi(\tau)\, d\tau\right) \cdot \int_{-\infty}^{t} \varphi_0(\tau)\, d\tau\,, \tag{10}$$

where W_C is a smooth extension of W and φ_0 is any scalar testing function such that $\operatorname{supp} \varphi_0 \subset (-\infty, 0)$, $\int_{-\infty}^{\infty} \varphi_0(\tau)\, d\tau = 1$.

Now, let $\varphi \in \mathscr{D}$, $\varphi \geqq 0$; then for $t \geqq 0$ we have by (10),

$$\varphi_W(t) = \int_t^\infty \overline{W_C(\tau, t)}\, \varphi(\tau)\, d\tau = \int_t^\infty W(\tau, t)\, \varphi(\tau)\, d\tau \tag{11}$$

since W is real and $W_C(\tau, t) = W(\tau, t)$ for $\tau \in [t, \infty)$.

However, from (11) it follows that $\varphi_W(t) \geqq 0$ for $t \geqq 0$. Put $a = (x, \varphi_W)$ and $a_m = (x, \psi_m)$, where $\psi_m(t) = \varphi_W(t + 1/m)$ for every t. Thus, $\psi_m(t) \geqq 0$ on $[-1/m, \infty)$; consequently, for every $m \geqq 1$ there exist vector functions $\psi_m^1, \psi_m^2 \in \mathscr{D}$ such that

1. $\psi_m = \psi_m^1 + \psi_m^2$,
2. $\psi_m^1 \geqq 0$ for every t,
3. $\operatorname{supp} \psi_m^2 \subset (-\infty, 0)$.

Actually, it suffices to construct an infinitely differentiable scalar function $\varkappa_m(t)$ such that $\varkappa_m = 0$ on $(-\infty, -1/m)$, $0 < \varkappa_m < 1$ on $(-1/m, -1/2m)$, $\varkappa_m = 1$ on $[-1/2m, \infty)$ and put $\psi_m^1 = \varkappa_m \psi_m$, $\psi_m^2 = (1 - \varkappa_m)\, \varphi_m$.

Thus, by the assumption $x \geqq 0$, $x \in \mathscr{D}'$ we have $(x, \psi_m^1) \geqq 0$ and $(x, \psi_m^2) = 0$, i.e., $a_m = (x, \psi_m) \geqq 0$.

On the other, hand, $\psi_m - \varphi_W \to 0$ in $\mathscr{D}$ as $m \to \infty$; hence, by continuity of the functional, $a_m - a = (x, \psi_m - \varphi_W) \to 0$, so that $a \geqq 0$. The Lemma is proved.

Lemma 3. *Let $a(t)$ be a smooth matrix function and let $a(t) \geqq 0$ for every $t \geqq 0$; then $ax \geqq 0$ whenever $x \geqq 0$, $x \in \mathscr{D}'$.*

Proof. Let $x \geqq 0$ and $\varphi \geqq 0$, $\varphi \in \mathscr{D}$; then, by definition, (cf. [3], p. 153), $(ax, \varphi) = (x, \bar{a}_C \varphi)$, where a_C is a smooth extension of a onto the entire axis. Clearly, $\bar{a}_C \varphi = a\varphi \geqq 0$ for any $t \geqq 0$. Putting again $\psi_m(t) = \bar{a}_C(t + 1/m)\, \varphi(t + 1/m)$ for every integer $m > 0$ and t, and carrying out a decomposition $\psi_m = \psi_m^1 + \psi_m^2$ as in the proof of Lemma 2, we easily conclude that $(x, \bar{a}_C \varphi) \geqq 0$; hence, the proof.

Theorem 5. *Let $a_i(t)$, $i = 0, 1, 2, \ldots, m$ and $W(t, \tau)$ be smooth matrix functions for $t \geqq 0$ and $0 \leqq \tau \leqq t$, respectively, and let the operator A be defined on $\mathscr{D}'$ by*

$$Ax = a_m x^{(m)} + a_{m-1} x^{(m-1)} + \ldots + a_0 x + [Wx]\,. \tag{12}$$

Then $A \geqq 0$ exactly if

1. *$a_i(t) \equiv 0$ for $i = 1, 2, \ldots, m$, and*
2. *$a_0(t) \geqq 0$ for $t \geqq 0$ and $W(t, \tau) \geqq 0$ for $0 \leqq \tau \leqq t < \infty$.*

Remark 2. As stated in Theorem 5.4–14, [3], p. 162, every operator $A \in \mathfrak{U}$ can be defined by (12) and vice versa; hence, Theorem 5 characterized the nonnegative operators in $\mathfrak{U}$.

Proof of Theorem 5. The sufficiency of conditions 1. and 2. is a trivial consequence of Lemmas 2 and 3. Thus, prove the necessity. To this purpose, introduce the following notation: If J is a constant n-vector and $T \geqq 0$, let $\delta_{T,J}$ be a functional on $\mathscr{D}$ defined by $(\delta_{T,J}, \varphi) = J^{\backprime} \varphi(T)$, ($J^{\backprime}$ denotes the transpose); it is clear that $\delta_{T,J} \in \mathscr{D}'$. Furthermore, if H_T is the scalar function defined by $H_T = 1$ for $t \geqq T$, $H_T = 0$ for $t < T$, then it can be readily verified that $(JH_T)' = \delta_{T,J}$. Observe that $J \geqq 0$ implies that $\delta_{T,J} \geqq 0$.

Thus, assume that $m \geqq 1$ and set $x_0 = (\delta_{T,J})^{-(m-1)}$ with $J \geqq 0$.

Since $u^{-(m-1)} = [U_{m-1}u]$ for any $u \in \mathscr{D}'$, where $U_{m-1}(t, \tau) = I((t - \tau)^{m-2} : : (m-2)!)$ and I is the unit matrix, (cf. [3], p. 158), and $U_{m-1}(t, \tau) \geqq 0$ for $0 \leqq \tau \leqq t$, it follows by Lemma 2 that $x_0 \geqq 0$. Hence, by assumption,

$$(13) \qquad Ax_0 = a_m \delta'_{T,J} + a_{m-1}\delta_{T,J} + a_{m-2}JH_T + \ldots + [Wx_0] \geqq 0\,.$$

Observe that $a_{m-2}JH_T + \ldots + [Wx_0] = Q_T$ is a regular distribution vanishing on $(-\infty, T)$; thus, let $Q_T(t)$ be the corresponding locally integrable vector function vanishing a.e. for $t \leqq T$.

Consequently, for any $\varphi \geqq 0$, $\varphi \in \mathscr{D}$ we have by (13), $(Ax_0, \varphi) = (a_m\delta'_{T,J}, \varphi) + (a_{m-1}\delta_{T,J}, \varphi) + \int_T^\infty Q^{\backprime}_T(t)\, \varphi(t)\, dt \geqq 0$. Expanding the term containing $\delta'_{T,J}$, we obtain

$$(14) \qquad (Ax_0, \varphi) = -J^{\backprime}\bar{a}_m(T)\, \varphi'(T) + J^{\backprime}(\bar{a}_{m-1}(T) - \bar{a}'_m(T))\, \varphi(T) + + \int_T^\infty Q^{\backprime}_T(t)\, \varphi(t)\, dt \geqq 0\,.$$

Next, let $\psi(t)$ be a nonnegative scalar testing function such that $\psi(T) = \psi'(T) = 1$, and let J^* be a nonnegative constant vector. For every $m' \geqq 1$ set $\varphi_{m'}(t) = J^*(1/m') . . \psi(m'(t - T) + T)$; clearly, $\varphi_{m'} \in \mathscr{D}$ and $\varphi_{m'} \geqq 0$. Moreover, $\varphi_{m'}(T) = J^*(1/m')$ and $\varphi'_{m'}(T) = J^*$. Thus, taking $\varphi_{m'}$, for φ in (14), we obtain

$$(15) \qquad (Ax_0, \varphi_{m'}) = -J^{\backprime}\, \bar{a}_m(T)\, J^* + J^{\backprime}(\bar{a}_{m-1}(T) - \bar{a}'_m(T))\, J^* \frac{1}{m'} + + \int_T^\infty Q^{\backprime}_T(t)\, \varphi_{m'}(t)\, dt \geqq 0\,.$$

However, $\int_T^\infty Q^{\backprime}_T(t)\, \varphi_{m'}(t)\, dt \to 0$ as $m' \to \infty$; consequently, letting m' tend to infinity in (15), we get

$$(16) \qquad -J^{\backprime}\, \bar{a}_m(T)\, J^* \geqq 0\,.$$

Repeating now the whole procedure but with ψ such that $\psi(t) \geqq 0$, $\psi(T) = 1$, $\psi'(T) = -1$, we conclude that $J^{\backprime}\, \bar{a}_m(T)\, J^* \geqq 0$. Hence, in view of (16), $J^{\backprime}\, \bar{a}_m(T)\, J^* = 0$ for any $J^{\backprime} \geqq 0$, $J^* \geqq 0$, i.e. $\bar{a}_m(T) = 0$. Thus, 1. is proved and we necessarily have

$$(17) \qquad Ax = a_0 x + [Wx]\,.$$

Put now $x_0 = \delta_{T,J}$ with $J \geqq 0$; then $[W\delta_{T,J}]$ is a regular distribution corresponding to the vector function $W(t, T)\, JH_T$ (cf. [3], p. 119). Hence, for any $\varphi \geqq 0$, $\varphi \in \mathscr{D}$ we have

$$(18) \qquad (Ax_0, \varphi) = J' \bar{a}_0(T)\, \varphi(T) + \int_T^\infty J'\, W'(t, T)\, \varphi(t)\, \mathrm{d}t \geqq 0 .$$

Let $\psi(t)$ be a nonnegative scalar testing function such that $\psi(T) = 1$; putting $\varphi_k(t) = J^*\psi(k(t - T) + T)$ with $J^* \geqq 0$ for every integer $k \geqq 1$, we have $\varphi_k \in \mathscr{D}$ and $\varphi_k \geqq 0$. Clearly, $\int_T^\infty J' W'(t, T)\, \varphi_k(t)\, \mathrm{d}t \to 0$ as $k \to \infty$, so that, by (18), $J' \bar{a}_0(T)\, J^* \geqq \geqq 0$ for any $J' \geqq 0$, $J^* \geqq 0$; hence, $a_0(T) \geqq 0$.

Finally, for the above choice of $x_0 = \delta_{T,J}$, let T^* and T^{**} be any numbers with $T \leqq T^* < T^{**}$; furthermore, let the scalar testing function ψ be defined by

$$\begin{aligned} \psi(t) &= \exp\left(-(t - T^*)^{-1} - (T^{**} - t)^{-1}\right) \quad \text{for} \quad t \in (T^*, T^{**}) , \\ &= 0 \quad \text{elsewhere} . \end{aligned}$$

Choose a constant vector $J^* \geqq 0$ and put $\varphi_k(t) = J^*(\psi(t))^{1/k}$ for any $k \geqq 1$; then clearly $\varphi_k \in \mathscr{D}$ and $\varphi_k \geqq 0$. Introducing φ_k into (18), we obtain

$$(19) \qquad \int_{T^*}^{T^{**}} J'\, W'(t, T)\, J^*(\psi(t))^{1/k}\, \mathrm{d}t \geqq 0 .$$

However, since $(\psi(t))^{1/k} \to 1$ as $k \to \infty$ on (T^*, T^{**}), it follows that

$$\int_{T^*}^{T^{**}} J' W'(t, T)\, J^*\, \mathrm{d}t \geqq 0;$$

thus, necessarily $J' W'(t, T)\, J^* \geqq 0$ for all $0 \leqq T \leqq t$ and $J \geqq 0$, $J^* \geqq 0$. Hence, $W(t, \tau) \geqq 0$ for $0 \leqq \tau \leqq t$ and the theorem is proven.

Corollary . *Let $a_i(t)$ and $W_i(t, \tau)$ be smooth, nonnegative matrix functions for $t \geqq 0$ and $0 \leqq \tau \leqq t < \infty$, respectively, $i = 1, 2, \ldots$, and let $0 < T_1 < T_2 < \ldots$, $T_i \to \infty$ as $i \to \infty$; if the operator A is defined on $\mathscr{D}'$ by*

$$Ax = \sum_{i=1}^{\infty} \{a_i(P_{T_i}x) + [W_i(P_{T_i}x)]\} ,$$

then $A \geqq 0$ and $A \in \mathfrak{B}_{T_1}$.

(The proof follows immediately from Theorems 2, 5 and the fact that $P_{T_i} \geqq 0$.)

Example 2. As an example illustrating the employment of Theorems 4 and 5, let us prove the following proposition on majorization of a solution.

Let $W(t, \tau)$ and $K(t)$ be smooth matrix functions such that $0 \leqq W(t, \tau) \leqq K(t - \tau)$ for $0 \leqq \tau \leqq t < \infty$, and let $\int_0^\infty K(t)\, \mathrm{d}t = M$ with $\|M\| < 1$. If $f \in \mathscr{D}'$ is regular and the corresponding vector function $f(t)$ satisfies the condition $-a_1 \leqq f(t) \leqq a_2$ for all $t \geqq 0$ with some fixed vectors a_1, a_2, then for the solution x (which is regular) of

$$x - [W(P_T x)] = f\,, \quad T > 0\,, \tag{20}$$

we have $-Na_1 \leqq x(t) \leqq Na_2$ for $t \geqq 0$, where $N = I + \sum_{i=1}^{\infty} M^i$.

For proving this, put $f_2(t) = \max[f(t), 0]$ and $f_1(t) = \max[-f(t), 0]$, (the meaning of the symbol max is certainly obvious); then clearly $0 \leqq f_2 \leqq a_2$ and $0 \leqq f_1 \leqq a_1$ for $t \geqq 0$, and $f = f_2 - f_1$. Next, consider the equations

$$x_2 - [W(P_T x_2)] = f_2\,, \quad \xi_2 - [K(P_T \xi_2)] = a_2 H_0\,, \tag{21}$$

$$x_1 - [W(P_T x_1)] = f_1\,, \quad \xi_1 - [K(P_T \xi_1)] = a_1 H_0\,. \tag{22}$$

By Theorem 1, $\xi_2 = a_2 H_0 + \sum_{i=1}^{\infty} A^i(a_2 H_0)$, where $Az = [K(P_T z)]$ for $z \in \mathscr{D}'$. However, since $a_2 H_0$ is regular, we have $0 \leqq A^i(a_2 H_0) \leqq M^i a_2$ for $t \geqq 0$ and any i; actually, assuming the validity of this inequality for some i, we obtain, $A^{i+1}(a_2 H_0) = \int_0^t K(t - \tau)\,(P_T A^i(a_2 H_0))(\tau)\,d\tau \leqq \int_0^t K(t - \tau)\,M^i a_2\,d\tau = \int_0^t K(\tau)\,M^i a_2\,d\tau \leqq M^{i+1} a_2$. Thus, since the series for N converges, we have $0 \leqq \xi_2 \leqq Na_2$ for $t \geqq 0$.

Repeating the same consideration for ξ_1 we conclude that $0 \leqq \xi_1 \leqq Na_1$. Now, applying Theorem 4 or Corollary 2 to the pair of equations (21), we infer that $0 \leqq x_2 \leqq \xi_2 \leqq Na_2$; analogously, (22) yields $0 \leqq x_1 \leqq \xi_1 \leqq Na_1$. Finally, the fact that $x = x_2 - x_1$ is the solution of (20) concludes the proof.

The following proposition appears as a certain counterpart for Theorem 4.

Theorem 6. *Let $A \in \mathfrak{B}_T$, $A \geqq 0$, and let x be the solution of the equation $x + Ax = f$, $f \in \mathscr{D}'$.*

1. *If $(A^k - A^{k+1})f \geqq 0$ for some k even, then*

$$\Big(I + \sum_{i=1}^{k+2n-1} (-1)^i A^i\Big) f \leqq x \leqq \Big(I + \sum_{i=1}^{k+2n} (-1)^i A^i\Big) f \tag{23}$$

for any integer $n \geqq 0$.

2. *If $(A^k - A^{k+1})f \geqq 0$ for some k odd, then*

$$\Big(I + \sum_{i=1}^{k+2n} (-1)^i A^i\Big) f \leqq x \leqq \Big(I + \sum_{i=1}^{k+2n-1} (-1)^i A^i\Big) f \tag{24}$$

for any integer $n \geqq 0$.

Proof. Consider the case 1. By Theorem 1, $x = \big(I + \sum_{i=1}^{k-1+2n} (-1)^i A^i\big) f + s$ with $s = \big(\sum_{i=k+2n}^{\infty} (-1)^i A^i\big) f$. However, it is clear that $s = \sum_{i=0}^{\infty} A^{2n+2i}(A^k - A^{k+1}) f$, and consequently, $s \geqq 0$. From this the first part of inequality (23) follows immediately. The second part and (24) can be proved in the same way.

Example 3. a) Consider the scalar equation $x + Ax = \delta_0$, where $Az = \alpha e^{-t}(P_T x)^{(-1)}$, $T > 0$ and assume that $\alpha \geqq 0$ and $1 - \alpha e^{-T} \geqq 0$. A simple computation yields $A\delta_0 = \alpha e^{-t} H_T$, $A^2\delta_0 = \alpha^2(e^{-t-T} - e^{-2t+T}) H_{2T}$. Since both $A\delta_0$ and $A^2\delta_0$ are regular distributions, we have in usual sense $A\delta_0 - A^2\delta_0 \geqq 0$ on $(0, 2T)$, and for $t \geqq 2T$, $A\delta_0 - A^2\delta_0 = \alpha e^{-t}(1 - \alpha e^{-T}) + \alpha^2 e^{-2t+T} \geqq 0$. Hence, $A\delta_0 - A^2\delta_0 \geqq 0$ in distributional sense, and since $A \geqq 0$, we obtain by (24) with $n = 0$,

$$\delta_0 - \alpha e^{-t} H_T \leqq x \leqq \delta_0 .$$

b) Consider the scalar equation $x + Ax = H_0$, where $Az = \alpha(e^{-t} P_T z)^{(-1)}$, $T > 0$, and assume again that $\alpha \geqq 0$ and $1 - \alpha e^{-T} \geqq 0$. It can be readily verified that $H_0 - AH_0 \geqq 0$. Since x is a regular distribution and $A \geqq 0$, from (23) with $n = 1$ we obtain the following bounds,

$$1 - \alpha e^{-T} \leqq x \leqq 1 - \alpha e^{-T} + \alpha^2 e^{-3T}, \quad t \geqq 2T .$$

Let us now consider more closely the simple equation

$$x + [W P_T x] = f , \tag{25}$$

or, to be more specific, the dependence of x on the matrix function W, provided $f \in \mathscr{D}'$ is regular and the corresponding vector function $f(t)$ is bounded in norm on every finite interval. As shown above, x is then also regular; moreover, it is clear that the corresponding vector function $x(t)$ can be chosen so that $x(t) - f(t)$ is continuous and thus is determined uniquely. In view of this, such $x(t)$ will be meant by the solution of (25) in the sequel.

To this purpose, let us introduce some useful notation. If $k \geqq 0$ is an integer, let $(a)_+^k = a^k$ for $a \geqq 0$ and $(a)_+^k = 0$ for $a < 0$.

Let the scalar functions Φ_T and Ψ_T be defined by

$$\Phi_T(\xi, \eta) = \sum_{i=1}^{\infty} \frac{1}{(i-1)!} (\xi - iT)_+^{i-1} \eta^i , \tag{26}$$

$$\Psi_T(\xi, \eta) = \sum_{i=0}^{\infty} \frac{1}{i!} (\xi - (i+1) T)_+^i \eta^i . \tag{27}$$

Lemma 4. *Let $u(t)$, $v(t)$ and $h(t)$ be nonnegative, locally integrable functions in $[0, \infty)$, let $u(t) = 0$ for $t < 0$ and let $v(t)$ be nondecreasing in $[0, \infty)$; if*

$$u(t) \leqq h(t) + v(t) \int_0^t u(\tau - T) \, d\tau \tag{28}$$

for every $t \geqq 0$, then

$$u(t) \leqq h(t) + \int_0^t \Phi_T(t - \tau, v(t)) \, h(\tau) \, d\tau \tag{29}$$

and

$$(30) \qquad \int_0^t u(\sigma - T)\,\mathrm{d}\sigma \leqq \int_0^t \Psi_T(t - \tau, v(t))\,h(\tau)\,\mathrm{d}\tau$$

for every $t \geqq 0$.

Proof. Choose $T^* > 0$; then, for any $t \in [0, T^*]$, we have by (28)

$$u(t) \leqq h(t) + \varkappa \int_0^t u(\tau - T)\,\mathrm{d}\tau$$

with $\varkappa = v(T^*)$. Thus, there exists a nonnegative function $\nu(t)$ such that

$$(31) \qquad u(t) = h(t) - \nu(t) + \varkappa \int_0^t u(\tau - T)\,\mathrm{d}\tau\,.$$

However, using Theorem 1 and the notation $f = h - \nu$, we obtain

$$(32) \qquad u(t) = f(t) + \sum_{i=1}^{\infty} \varkappa^i P_{iT} \int_0^t \int_0^{t_{i-1}} \dots \int_0^{t_1} f(\sigma)\,\mathrm{d}\sigma =$$

$$= f(t) + \sum_{i=1}^{\infty} \varkappa^i P_{iT} \int_0^t \frac{(t - \sigma)^{i-1}}{(i-1)!} f(\sigma)\,\mathrm{d}\sigma = f(t) + \int_0^t \Phi_T(t - \sigma, \varkappa) f(\sigma)\,\mathrm{d}\sigma\,.$$

Thus, putting $t = T^*$,

$$(33) \qquad u(T^*) = f(T^*) + \int_0^{T^*} \Phi_T(T^* - \sigma, v(T^*)) f(\sigma)\,\mathrm{d}\sigma\,.$$

Finally, because $\Phi_T(\xi, \eta) \geqq 0$ for $\xi, \eta \geqq 0$, (29) follows from (33) immediately.

Furthermore, (32) yields for $t \in [0, T^*]$,

$$\int_0^t u(\tau - T)\,\mathrm{d}\tau = \int_0^t (P_T f)(\sigma)\,\mathrm{d}\sigma + \sum_{i=1}^{\infty} \varkappa^i P_{(i+1)T} \int_0^t \frac{(t - \sigma)^i}{i!} f(\sigma)\,\mathrm{d}\sigma =$$

$$= \int_0^t \Psi_T(t - \sigma, \varkappa) f(\sigma)\,\mathrm{d}\sigma\,.$$

The rest of the proof is obvious.

Let $W(t, \tau)$ be a smooth matrix function for $0 \leqq \tau \leqq t$; for $\sigma \geqq 0$ put $|W|_\sigma = \sup_{0 \leqq \tau \leqq t \leqq \sigma} \|W(t, \tau)\|$. Clearly, $|W|_\sigma$ is a nonnegative nondecreasing function.

Theorem 7. *Let* $W_1(t, \tau)$, $W_2(t, \tau)$ *be smooth matrix functions,* $f \in \mathscr{D}'$ *be regular with* $\|f(t)\|$ *bounded on every finite interval,* $T > 0$. *Then the solutions* x_i *(in the sense indicated above) of the equations* $x_i + [W_i(P_T x_i)] = f$, $i = 1, 2$ *satisfy the* *inequality*

$$(34) \qquad \|x_1(t) - x_2(t)\| \leqq$$
$$\leqq |W_1 - W_2|_t \left(1 + \int_0^t \Phi_T(t - \tau, |W_1|_t \, \mathrm{d}\tau\right) . \int_0^t \Psi_T(t - \tau, |W_2|_t \, \|f(\tau)\| \, \mathrm{d}\tau$$

for $t \geqq 0$.

Proof. Since x_1 and x_2 are regular, we have from the equations defining them,

$$x_1(t) - x_2(t) = - \int_0^t W_1(x_1(\tau - T) - x_2(\tau - T)) \, \mathrm{d}\tau - \int_0^t (W_1 - W_2) \, x_2(\tau - T) \, \mathrm{d}\tau \,, \quad t \geqq 0 \,;$$

consequently,

$$(35) \qquad \|x_1(t) - x_2(t)\| \leqq |W_1|_t \int_0^t \|x_1(\tau - T) - x_2(\tau - T)\| \, \mathrm{d}\tau + $$
$$+ |W_1 - W_2|_t \int_0^t \|x_2(\tau - T)\| \, \mathrm{d}\tau \,.$$

On the other hand, the equation for x_2 yields

$$\|x_2(t)\| \leqq |W_2|_t \int_0^t \|x_2(\tau - T)\| \, \mathrm{d}\tau + \|f(t)\| \,;$$

thus, by Lemma 4,

$$(36) \qquad \int_0^t \|x_2(\tau - T)\| \, \mathrm{d}\tau \leqq \int_0^t \Psi_T(t - \tau, |W_2|_t) \, \|f(\tau)\| \, \mathrm{d}\tau \,.$$

Putting $Q(t) = | W_1 - W_2|_t \int_0^t \Psi_T(t - \tau, |W_2|_t) \, \|f(\tau)\| \, \mathrm{d}\tau$ and using Lemma 4 with (36) for (35), we obtain

$$(37) \qquad \|x_1(t) - x_2(t)\| \leqq Q(t) + \int_0^t \Phi_T(t - \tau, |W_1|_t) \, Q(\tau) \, \mathrm{d}\tau \,.$$

However, since $Q(t)$ is nondecreasing (witness (32) and the following equations), (37) implies (34); hence the proof.

Finally, let us present a simple criterium for the boundedness of a solution of the equation $x + a(P_T x) + [W(P_T x)] = f$.

Theorem 8. *Let* $a(t)$ *and* $W(t, \tau)$ *be smooth matrix functions which are bounded for* $0 \leqq t < \infty$ *and* $0 \leqq \tau \leqq t < \infty$, *respectively. Let the operator B be defined* *on $\mathscr{D}'$ by $Bx = ax + [Wx]$, and $T > 0$. If*

(38) $$\lim_{m\to\infty} \sup_{t\in[mT,\infty]} \left\{\|a(t)\| + \int_{mT}^{t} \|W(t,\tau)\| \,d\tau\right\} = c < 1\,,$$

then a constant $M > 0$ exists such that, for any regular $x \in \mathscr{D}'$ with bounded $\|x(t)\|$ in $[0, \infty)$,

(39) $$\sup_{t\in[0,\infty)} \|(I + BP_T)^{-1}\, x\| \leqq M \sup_{t\in[0,\infty)} \|x\|\,.$$

Proof. For every integer $m \geqq 1$, let

$$\lambda_m = \sup_{t\in[mT,\infty)} \left\{\|a(t)\| + \int_{mT}^{t} \|W(t,\tau)\| \,d\tau\right\}.$$

It can be easily verified that $\lambda_m \geqq \lambda_{m+1}$ for every $m \geqq 1$; thus, by (38), $\lambda_m \to c$ as $m \to \infty$.

Next, prove that $\lambda_1 < \infty$. In view of $\lambda_m \geqq \lambda_{m+1}$, there exists an integer $N > 0$ such that

(40) $$c + 1 \geqq \lambda_N = \sup_{t\in[NT,\infty)} \left\{\|a(t)\| + \int_{NT}^{t} \|W(t,\tau)\| \,d\tau\right\} \geqq c\,.$$

Denote $\alpha = \sup_{[T,\infty)} \|a(t)\| < \infty$ and $\beta = \sup_{T\leqq\tau\leqq t<\infty} \|W(t,\tau)\| < \infty$. If $t \in [T, NT]$, then $\|a(t)\| + \int_T^t \|W(t,\tau)\| \,d\tau \leqq \alpha + (N-1)\,T\beta$. If $t > NT$, then

$$\|a(t)\| + \int_T^t \|W(t,\tau)\| \,d\tau = \int_T^{NT} \|W\| \,d\tau + \|a\| + \int_{NT}^t \|W\| \,d\tau \leqq (N-1)\,T\beta + c + 1$$

due to (40); hence, $\lambda_1 < \infty$.

Next, let $x \in \mathscr{D}'$ and let $\|x(t)\|$ be bounded in $[0, \infty)$; by Theorem 1, $(I + BP_T)^{-1}\, x = x + \sum_{m=1}^{\infty} (-1)^m (BP_T)^m\, x$. We are going to show that, for every $t \geqq 0$,

(41) $$\|(BP_T)^m\, x\| \leqq \prod_{i=1}^{m} \lambda_i \,.\, d$$

with $d = \sup_{t\in[0,\infty)} \|x(t)\|$. Actually, $(BP_T)^m\, x$ vanishes on $(-\infty, mT)$; assuming the validity of (41) for some m, we have with $(BP_T)^m\, x = u$, and $t \geqq 0 : \|(BP_T)^{m+1}\, x\| = \|BP_T u\| \leqq \|a(t)\| \,.\, \|u(t-T)\| + \int_{(m+1)T}^t \|W(t,\tau)\| \,.\, \|u(t-T)\| \,d\tau \leqq (\|a(t)\| + \int_{(m+1)T}^t \|W(t,\tau)\| \,dt) \,.\, \prod_{i=1}^{m} \lambda_i \,.\, d \leqq \prod_{i=1}^{m+1} \lambda_i \,.\, d$. Since (41) is clearly true for $m = 1$, the estimate is proved.

Hence, for any $t \geqq 0$,

$$\|(I + BP_T)^{-1} x\| \leqq (1 + \sum_{m=1}^{\infty} \prod_{i=1}^{m} \lambda_i) d ,$$

and since the series converges due to the assumption $c < 1$, (39) is proved.

Example 4. Referring to Example 1, consider the vector differential equation $x' - Ax = B(t)\, x(t - T)$ with initial condition $x(t) = \varphi(t)$ for $-T \leqq t \leqq 0$. Assume that A is a constant matrix whose eigenvalues have negative real parts, and that $b = \sup_{t \in [0,\infty)} \|B(t)\| < \infty$. We are going to show that, for b sufficiently small, the solution x is bounded in norm on $[0, \infty)$.

Actually, it can be readily verified that, for $t \geqq 0$, the considered equation is equivalent to

$$x(t) = \int_0^t X(t - \tau)\, B(\tau)\, x(\tau - T)\, d\tau + X(t)\, \varphi(0) + \\ + X(t) \int_0^T X(-\tau)\, B(\tau)\, \varphi(\tau - T)\, d\tau , \tag{42}$$

where now $x(t)$ is considered to be zero for $t < 0$, and $X(t)$ is the solution of $X'(t) = AX(t)$, $X(0) = I$. As known, $\|X(t)\| \leqq Ce^{-\lambda t}$ for $t \geqq 0$, $\lambda > 0$. However, (42) is exactly the type of equation admitting the application of Theorem 8. Here we have

$$\|X(t - \tau)\, B(\tau)\| \leqq Cbe^{-\lambda(t-\tau)}$$

so that

$$\lambda_m = \sup_{t \in [mT,\infty)} \int_{mT}^t \|X(t - \tau)\, B(\tau)\|\, d\tau \leqq \sup_{t \in [mT,\infty)} \int_0^{t-mT} Cbe^{-\lambda\sigma}\, d\sigma = \\ = \int_0^{\infty} Cbe^{-\lambda\sigma}\, d\sigma = \frac{Cb}{\lambda} ;$$

hence, for b sufficiently small, we have the boundedness.

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