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ON THE PROOF OF A THEOREM OF V. G. BOLTIANSKIJ

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The proof of sufficient conditions for optimality of the time-optimal control problem is simplified and completed.

INTRODUCTION

In his book [1] BOLTIANSKIJ has formulated and proved two theorems giving sufficient conditions for optimality of the time-optimal control problem (Theorems IV. 4., p. 226 and IV. 7., p. 238). We are going to deal with the first of them; it will be referred to as *ThB*. (The second theorem is a non-trivial consequence of the first one.)

The proof of *ThB* is based on the following proposition: Let A_k and B_n be sufficiently smooth manifolds with boundary of dimension k and n respectively. Suppose that $k \leqq n$ and that $\Phi : A_k \to B_n$ is a C^1-mapping. Denote S the set of singular points of Φ. Then $\Phi(S)$ is a set of the first category in B_n.

For the proof of this proposition the reader of [1] is referred to [2] if $k < n$, and to [3] if $k = n$. However, in the proof of *ThB* this proposition can be replaced by the following well-known fact: If $k = n$, A_k is an open subset of R^n and $B_n = R^n$, then $\Phi(S)$ has a zero Lebesgue measure. Particularly, we need not speak about manifolds with boundary. To show it is the objective of this paper.

(In his interesting paper [3] Dubovickij deals mainly with the non-trivial case $k > n$. He shows that in this case Φ has to be a C^{k-n+1}-mapping.)

ASSUMPTIONS AND NOTATIONS
FORMULATION OF LEMMA 1

Denote $\mathscr{L}(R^k, R^n)$ the space of all linear mappings of R^k into R^n with its usual topology. We shall write simply $\mathscr{L}(R^n)$ instead of $\mathscr{L}(R^n, R^n)$. Furthermore, we shall identify the space $\mathscr{L}(R, R^n)$ with R^n by means of the canonical isomorphism. Denote ϱA the rank of the operator $A \in \mathscr{L}(R^k, R^n)$.

Let H be a neighborhood of $x \in R^k$. The derivative $DF(x)$ of a function $F : H \to R^n$ is an element of $\mathscr{L}(R^k, R^n)$ (if it exists). Also, if $k = 1$, $\dot{F}(t)$ will be written instead of $DF(t)$. If, for example, $H \subset R^k \times R^l \times R^m$ and $x = (x_1, x_2, x_3)$, then $D_2F(x_1, x_2, x_3)$ denotes the partial derivative of F with respect to the second variable at the point x. It is an element of $\mathscr{L}(R^l, R^n)$ ([4]).

Consider the following time-optimal control problem:

$$\dot{x} = f(x, u)\,, \tag{1}$$

where $f : G \times U \to R^n$ is continuous, G is open in R^n, and U is a Hausdorff space.

Suppose that the derivative $D_1f(x, u)$ exists for all $(x, u) \in G \times U$ and that $D_1f : G \times U \to \mathscr{L}(R^n)$ is also continuous.

We consider only the piece-wise continuous strategies u defined on a compact interval in R. If the strategy $u : \langle t_1, t_2 \rangle \to U$ is fixed, the trajectory x is a solution of (1), i.e. it is a continuous function defined on some interval $I \subset \langle t_1, t_2 \rangle$ which satisfies (1) at all points of continuity of u. If $I = \langle t_1, t_2 \rangle$, denote the pair of x and u by $(x, u)_{t_1}^{t_2}$ and call it a process.

ThB is an immediate consequence of the next lemma (see [1], p. 225):

Lemma 1. *Let M be a piece-wise smooth subset of G of dimension less than n. Let $\omega : G \to R$ be continuous in G and have a derivative* $D\omega(x)$ *at all points $x \in G - M$. Suppose that the inequality* $D\omega(x) \circ f(x, u) \leqq 1$ *holds for all $x \in G - M$ and for all $u \in U$. Then, for every process $(x, u)_{t_1}^{t_2}$ we have $t_2 - t_1 \geqq \omega(x_2) - \omega(x_1)$, $x_i = x(t_i)$, $i = 1, 2$.*

We shall recall now the definition of a piece-wise smooth set, introducing simultaneously some useful notation.

A set K is called an s-dimensional polyhedron iff K is a convex compact polyhedron in R^s with a non-empty interior. A mapping $\varphi : K \to R^n$ is said to be of the class C^1 iff φ is the restriction of a C^1-mapping ψ defined on an open neighborhood of K. It is easy to see that for $\xi \in K$ the value of $D\psi(\xi) \in \mathscr{L}(R^s, R^n)$ depends only on φ; we can therefore denote it by $D\varphi(\xi)$. A C^1-mapping φ is called regular iff $\varrho D\varphi(\xi) = s$ for all $\xi \in K$.

A curvilinear polyhedron $L = \varphi(K)$ of dimension s in R^n is, by definition, a triple (L, φ, K) where K is an s-dimensional polyhedron, $\varphi : K \to R^n$ is a regular injection and L is a subset of R^n, $L = \varphi(K)$. We shall call it simply the polyhedron L, or polyhedron $L = \varphi(K)$ provided there is no danger of misunderstanding. The points of the set $L \subset R^n$ will be referred to as the points of polyhedron L. Similarly, the union of a family of polyhedra is the union of the corresponding family of subsets of R^n etc.

Let us keep for a while the meaning of letters L, φ, K and s. Let $x \in L$, $\xi = \varphi^{-1}(x)$ and $A = D\varphi(\xi)$. Then AR^s is a vector subspace of dimension s in R^n; it will be denoted by L_x and called the tangent space of L at the point x.

Definition 1. A set $M \subset G$ is called *a piece-wise smooth set of dimension less* *than n in G* iff M is a union of a locally finite family of curvilnear polyhedra of dimension less than n.

Of course, M need not be a piece-wise smooth set in some $G' \supset G$.

PROOF OF LEMMA 1

We shall take advantage of the following form of the theorem on continuous dependence on the initial conditions for equation (1):

Theorem 1. *Let $(x, u)_{t_1}^{t_2}$ be a process and $G' \subset G$ be an open neighborhood of the point-set*

$$[x] = x(\langle t_1, t_2 \rangle) . \tag{2}$$

For a fixed strategy u let us denote H the set of all initial conditions $(\tilde{x}, \tilde{\tau}) \in G \times \times \langle t_1, t_2 \rangle$ such that the trajectory $y(t) = y(t, \tilde{x}, \tilde{\tau})$, $y(\tilde{\tau}) = \tilde{x}$ can be defined on the entire interval $\langle t_1, t_2 \rangle$ and assumes its values in G'. Denote $\psi(\tilde{x}, \tilde{\tau}) = y(t_1, \tilde{x}, \tilde{\tau})$. Finally, let σ be a finite subset of $\langle t_1, t_2 \rangle$ containing t_1, t_2 and all points of discontinuity of the strategy u. Then

1° *H is open in $G' \times \langle t_1, t_2 \rangle$.*

2° *$\psi : H \to G'$ is piece-wise smooth with respect to σ. (See Definition 2.)*

3° *The linear operator $A(\tilde{x}, \tilde{\tau}) = D_1\psi(\tilde{x}, \tilde{\tau}) \in \mathscr{L}(R^n)$ is regular for all $(\tilde{x}, \tilde{\tau}) \in H$.*

4° *The following formula is valid:*

$$D_2\psi(\tilde{x}, \tilde{\tau}) = -A(\tilde{x}, \tilde{\tau}) \circ D_1 y(\tilde{\tau}, \tilde{x}, \tilde{\tau}) \quad \text{if} \quad \tilde{\tau} \notin \sigma . \tag{3}$$

The proof of the theorem follows the standard pattern. For example, it can be easily obtained by a slight modification of the proofs of Theorems 7.1. and 7.2. in [5]. For the sake of completness a proof of relation (3) will be given here, because it is not contained in [5]: We have $y(t, \tilde{x}, \tilde{\tau}) = y(t, y(s, \tilde{x}, \tilde{\tau}), s)$ and particularly $\psi(\tilde{x}, \tilde{\tau}) = \psi(y(s, \tilde{x}, \tilde{\tau}), s)$. Hence, $D_2\psi(\tilde{x}, \tilde{\tau}) = D_1\psi(y(s, \tilde{x}, \tilde{\tau}), s) \circ D_3 y(s, \tilde{x}, \tilde{\tau})$ and it sufficies to put $s = \tilde{\tau}$ because $D_3 y(\tilde{\tau}, \tilde{x}, \tilde{\tau}) = -D_1 y(\tilde{\tau}, \tilde{x}, \tilde{\tau})$ (see the proof of Theorem 7.2. in [5]).

Next, the definition of a piece-wise smooth function will be given:

Definition 2. Let I be a compact interval in R and σ be a finite subset of I containing its end-points. Let Ω be open in $R^k \times I$. Denote $\tilde{\Omega} = \Omega \cap (R^k \times (I - \sigma))$, which is an open subset of $R^k \times R$. A function $\Phi : \Omega \to R^n$ will be called *piece-wise smooth with respect to σ* iff the following conditions hold:

1° Φ is continuous.

2° $\Phi_{\tilde{\Omega}} : \tilde{\Omega} \to R^n$ is a C^1-mapping.

3° The mappings $x \to \Phi(x, t)$ are of the class C^1 on the corresponding sets $\Omega(t)$ for all $t \in \sigma$. (Or, equivalently, for all $t \in I$.)

Here $\Omega(t)$ denotes the section of Ω, i.e.

(4) $$\Omega(t) = \{x \in R^k : (x, t) \in \Omega\} .$$

Besides Theorem 1, we shall need some consequences of the following well-known theorem of advanced calculus:

Theorem 2. *Let Ω be open in R^n and $\Phi : \Omega \to R^n$ be of the class C^1. Let $S = \{\xi \in \Omega : \varrho D\Phi(\xi) < n\}$ be the set of all singular points of Φ. Then $|\Phi(S)| = 0$.* ($|X|$ denotes the Lebesgue measure of the set X.)

For a simple proof see [6].

The next corollaries are trivial but useful consequences of Theorem 2:

Corollary 1. *The statement of Theorem 2 remains true for $\Omega \subset R^k$, $k \leqq n$.* (It sufficies to apply Theorem 2 to the function Ψ, where $\Psi(\xi_1, \xi_2, \ldots, \xi_k, \xi_{k+1}, \ldots \ldots, \xi_n) = \Phi(\xi_1, \ldots, \xi_k)$ defined on the cylinder $\Omega \times R^{n-k}$ in R^n.)

Corollary 2. *Using the notation of Definition 2, $\Phi : \Omega \to R^n$ is a piece-wise smooth function with respect to σ. Suppose that $k + 1 \leqq n$. Let*

(5) $$S = S_1 \cup S_2 ,$$

where

(6) $$S_1 = \{\xi \in \tilde{\Omega} : \varrho D\Phi(\xi) < n\}$$

and

(7) $$S_2 = \bigcup_{t \in \sigma} \Omega(t) \times \{t\} .$$

Then $|\Phi(S)| = 0$.

(Obvious.)

Let us finally turn to the proof of Lemma 1. As shown in [1] it sufficies to prove the following assertion:

Lemma 2. *Let Γ be a family of (curvilinear) polyhedra of dimension less than n, which is locally finite in G. Let $(x, u)_{t_1}^{t_2}$ be a process. Then in every neighborhood of $x_1 = x(t_1)$ a point y_1 exists such that:*

1° There exists a process $(y, u)_{t_1}^{t_2}$ satisfying $y(t_1) = y_1$.

2° The trajectory $y : \langle t_1, t_2\rangle \to G$ of this process intersects only a finite number of polyhedra of Γ.

3° If $y(t) \in L \in \Gamma$ for some $t \in \langle t_1, t_2\rangle$, then the strategy u is continuous in t and $\dot{y}(t) \notin L_{y(t)}$.

Proof. The set $[x] \subset G$ (see (2)) is compact. Thus there exists an open, relatively compact set G'; $[x] \subset G' \subset \bar{G}' \subset G$. Then G' intersects only a finite number of polyhedra of the family Γ. Denote them by $L_1, L_2, \ldots, L_m$. Assume that the sets

$\sigma \subset \langle t_1, t_2 \rangle$, $H \subset G' \times \langle t_1, t_2 \rangle$ and the functions $y(t, \tilde{x}, \tilde{\tau})$ and ψ are defined as in Theorem 1. Let $W = H(t_1)$ be the section of H (see (4)). Then $W \subset G'$ is an open neighborhood of x_1. Conditions 1° and 2° of our lemma are satisfied for every $y_1 \in W$. Denote $\tilde{N}$ the set of these $y_1 \in W$ for which condition 3° is not fulfilled. Similarly, denote N_i the set of those $y_1 \in W$ for which 3° is not fulfilled provided the family Γ is reduced to $\{L_i\}$. Thus we have $\tilde{N} = N_1 \cup N_2 \cup \ldots \cup N_m$.

We have to prove that in every neighborhood of x_1 there exists a point $y_1 \in W - \tilde{N}$. It is sufficient to show that $|N_i| = 0$ for $1 \leqq i \leqq m$. The subscript i once fixed will be omitted in the following.

Let $L = \varphi(K)$. We may assume that φ is defined and belongs to the class C^1 on some open neighborhood K' of K. Let s be the dimension of L. Define a continuous mapping $g : K' \times \langle t_1, t_2 \rangle \to G' \times \langle t_1, t_2 \rangle$ by the formmula $g(\xi, t) = (\varphi(\xi), t)$. Denote $\Omega = g^{-1}(H)$. Ω is open in $K' \times \langle t_1, t_2 \rangle$ and the function $\Phi : \Omega \to G'$; $\Phi(\xi, t) = \psi(\varphi(\xi), t) = (\psi \circ g)(\xi, t)$ is piece-wise smooth in Ω with respect to σ.

Let S be defined as in Corollary 2 (see (5)). It is sufficient to show that $N \subset \Phi(S)$. Consider now an arbitrary $y_1 \in N$. Then $y(t_0, y_1, t_1) = y_0 \in L$ for some $t_0 \in \langle t_1, t_2 \rangle$ and at least one of the following relations holds: $t_0 \in \sigma$ or $\dot{y}(t_0) \in L_{y_0}$. Denote $\xi_0 = \varphi^{-1}(y_0)$. Then $y_1 = \psi(y_0, t_0) = \Phi(\xi_0, t_0)$. It can be immediately seen that $y_1 \in \Phi(S)$ if $t_0 \in \sigma$ (see (7)). Suppose $t_0 \in \langle t_1, t_2 \rangle - \sigma$. Let $A = D_1\psi(y_0, t_0) \in \mathscr{L}(R^n)$ and $B = D\Phi(\xi_0, t_0) = D(\psi \circ g)(\xi_0, t_0) \in \mathscr{L}(R^{s+1}, R^n)$. To find the rank of B we shall determine the dimension of the vector space $BR^{s+1} \subset R^n$. According to the theorem on the derivative of a composed function we have immediately $BR^{s+1} = AL_{y_0} + \{D_2\psi(y_0, t_0)\}_v$, where $\{a\}_v$ denotes the vector space spanned by a vector a. However, $D_2\psi(y_0, t_0) = -A\dot{y}(t_0)$ (see (3)) and consequently, $BR^{s+1} = AL_{y_0} + \{A\dot{y}(t_0)\}_v = A(L_{y_0} + \{\dot{y}(t_0)\}_v)$. Therefore, by the regularity of A (Theorem 1) we have $\varrho B = \dim (L_{y_0} + \{\dot{y}(t_0)\}_v)$. If now $y_1 \in N$, then $L_{y_0} + \{\dot{y}(t_0)\}_v = L_{y_0}$ and $\varrho B = s \leqq n - 1$. Hence, $(\xi_0, t_0) \in S$ (see (5), (6)) and $y_1 = \Phi(\xi_0, t_0) \in \Phi(S)$.

Thus we have $N \subset \Phi(S)$, and, by Corollary 2, $|N| = 0$. This proves our lemma.

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