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ON ASSOCIATED PARTITIONS

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1. Introduction. In [1], BORŮVKA studied those partitions in the cartesian square of any set S which are naturally induced by partitions in S, and more generally, he defined a partition in the cartesian product of two sets S_1, S_2 which is naturally induced by given partitions in S_1 and S_2. As the properties of such induced partitions are not investigated in [1], these will be the subject of our considerations. But first we recapitulate briefly some notions and results, in particular those of [1, chapter III].*)

2. Partitions in sets. Let S be a fixed non-void set. A *partition* $\mathscr{P}$ *in* S is defined as a non-void set of pairwise disjoint non-void subsets (called the *blocks* of $\mathscr{P}$ or the $\mathscr{P}$-*blocks*) in S. If the set-union of all $\mathscr{P}$-blocks is S, then $\mathscr{P}$ is said to be a *partition on* S. If $\boldsymbol{\mathscr{P}} = (\mathscr{P}_\iota)_{\iota \in I}$ is a family of partitions in S, then we shall define the set of the so-called $\boldsymbol{\mathscr{P}}$-blocks as the set $\{X \mid X \in \mathscr{P}_\iota, \iota \in I\}$. In the set $\mathfrak{S}(S)$ of all partitions in S, one may introduce an ordering $\leqq$ as follows: if $\mathscr{P}_1, \mathscr{P}_2$ are partitions in S then $\mathscr{P}_1 \leqq \mathscr{P}_2$ means that each $\mathscr{P}_1$-block is contained in some $\mathscr{P}_2$-block. Then $\mathfrak{S}(S)$ is a complete semi-lattice (in the sense of [2], pp. 20 and 33): to each family $\boldsymbol{\mathscr{P}} = (\mathscr{P}_\iota)_{\iota \in I}$ of partitions in S there exists its supremum, which may be characterised also by the notion of a *chaining* (cf. [1], p. 16) in $\boldsymbol{\mathscr{P}}$ between two $\boldsymbol{\mathscr{P}}$-blocks A, B; such a chaining is any finite sequence of $\boldsymbol{\mathscr{P}}$-blocks $A = A_0 \between A_1 \between \dots \between A_n = B$ (the symbol $\between$ means here and in the following text the non-void intersecting of two sets.) Every sup $\boldsymbol{\mathscr{P}}$-block is then characterized as the set-join of the maximal set of $\boldsymbol{\mathscr{P}}$-blocks which are mutually chained in $\boldsymbol{\mathscr{P}}$. The infimum of a family $\boldsymbol{\mathscr{P}}$ need not exist; if it exists, then the corresponding inf $\boldsymbol{\mathscr{P}}$-blocks have the form $\bigcap_{\iota \in I} A_\iota \neq \emptyset$, where $A_\iota \in \mathscr{P}_\iota$ for all $\iota \in I$. All partitions on S form in $\mathfrak{S}(S)$ a complete sublattice (in the sense of [2], p. 34) which will be denoted by $\mathfrak{L}(S)$. In $\mathfrak{L}(S)$ there also exists the inf $\boldsymbol{\mathscr{P}}$ for each family $\boldsymbol{\mathscr{P}}$ of partitions on S. For all these results cf. [1, pp. 14–19].

*) The author wishes to express his gratitude to Prof. M. KOLIBIAR for his improvements of the first version of the text.

3. The associated mapping. If $\mathfrak{A}(\leqq_{\mathfrak{A}})$, $\mathfrak{B}(\leqq_{\mathfrak{B}})$ are ordered sets, then the corresponding cardinal product $\mathfrak{A} \times \mathfrak{B}(\leqq_{\mathfrak{A}\times\mathfrak{B}})$ is ordered in the usual manner: $(\alpha, \beta) \leqq \leqq_{\mathfrak{A}\times\mathfrak{B}}(\gamma, \delta)$ means that both $\alpha \leqq_{\mathfrak{A}} \gamma$ and $\beta \leqq_{\mathfrak{B}} \delta$ for $\alpha, \gamma \in \mathfrak{A}$ and $\beta, \delta \in \mathfrak{B}$, [2, pp. 14–15]. In particular, if $\mathfrak{A}(\leqq_{\mathfrak{A}})$, $\mathfrak{B}(\leqq_{\mathfrak{B}})$ are (complete) semi-lattices or (complete) lattices, then also $\mathfrak{A} \times \mathfrak{B}(\leqq_{\mathfrak{A}\times\mathfrak{B}})$ is a (complete) semi-lattice or a (complete) lattice respetively. If $\mathfrak{A}(\leqq_{\mathfrak{A}})$, $\mathfrak{B}(\leqq_{\mathfrak{B}})$ are complete semi-lattices or complete lattices and if $\mathscr{F} = ((a_\iota, b_\iota))_{\iota\in I}$ is a family of elements in $\mathfrak{A} \times \mathfrak{B}$ and $\mathscr{A} = (a_\iota)_{\iota\in I}$, $\mathscr{B} = (b_\iota)_{\iota\in I}$, then $\sup \mathscr{F} = (\sup \mathscr{A}, \sup \mathscr{B})$, $\inf \mathscr{F} = (\inf \mathscr{A}, \inf \mathscr{B})$; cf. [2], pp. 51–52.

Let S, T be fixed non-empty sets. The mapping soc : $\mathfrak{S}(S) \times \mathfrak{S}(T) \to \mathfrak{S}(S \times T)$ assigns to each $(\mathscr{A}, \mathscr{B}) \in \mathfrak{S}(S) \times \mathfrak{S}(T)$ the partition soc $(\mathscr{A}, \mathscr{B}) \in \mathfrak{S}(S \times T)$ which consists of the blocks $A \times B$ with $A \in \mathscr{A}$, $B \in \mathscr{B}$. We shall speak about the associated mapping, and for the image in this mapping the term *socius* will be used; cf. [1], p. 25. If $\mathscr{F}$ is a family of elements $(\mathscr{A}_\iota, \mathscr{B}_\iota) \in \mathfrak{S}(S) \times \mathfrak{S}(T)$, $\iota \in I$ then we denote by soc $\mathscr{F}$ the family of the partitions soc $(\mathscr{A}_\iota, \mathscr{B}_\iota) \in \mathfrak{S}(S \times T)$, $\iota \in I$; cf. [1], p. 26.

4. Fundamental properties of the associated mapping. In the sequel, S and T will be fixed non-void sets, and $\mathscr{A} = (\mathscr{A}_\iota)_{\iota\in I}$ and $\mathscr{B} = (\mathscr{B}_\iota)_{\iota\in I}$ arbitrary families of partitions in S and T respectively, with a common index set I. The corresponding family $\mathscr{F} = ((\mathscr{A}_\iota, \mathscr{B}_\iota))_{\iota\in I}$ will be termed *admissible*.

A. *The associated mapping is not surjective, but it is injective and both-sided isotone.*

One can easily find a partition of $\mathfrak{S}(S \times T)$ which is not a socius of any $(\mathscr{A}, \mathscr{B}) \in \in \mathfrak{S}(S) \times \mathfrak{S}(T)$. Let $(\mathscr{A}, \mathscr{B}) \leqq (\mathscr{C}, \mathscr{D})$ for $(\mathscr{A}, \mathscr{B},) (\mathscr{C}, \mathscr{D}) \in \mathfrak{S}(S) \times \mathfrak{S}(T)$. Thus to arbitrary $A \in \mathscr{A}$, $B \in \mathscr{B}$ there exist $C \in \mathscr{C}$, $D \in \mathscr{D}$ such that $A \subseteq C$, $B \subseteq D$, and consequently $A \times B \subseteq C \times D$, soc $(\mathscr{A}, \mathscr{B}) \leqq$ soc $(\mathscr{C}, \mathscr{D})$. Conversely, if soc $(\mathscr{A}, \mathscr{B}) \leqq \leqq$ soc $(\mathscr{C}, \mathscr{D})$, then to each $(\mathscr{A}, \mathscr{B})$-block $A \times B$ there exists a soc $(\mathscr{C}, \mathscr{D})$-block $C \times D \supseteq A \times B$, which implies $A \subseteq C$, $B \subseteq D$, $(\mathscr{A}, \mathscr{B}) \leqq (\mathscr{C}, \mathscr{D})$.

B$_1$. *For every admissible family $\mathscr{F}$ there is*

$$\sup \operatorname{soc} \mathscr{F} \leqq \operatorname{soc} \sup \mathscr{F} . \tag{1}$$

Let $A \times B \in \sup \operatorname{soc} \mathscr{F}$; then $A \times B$ is a set-union of a maximal set of mutually chained (in soc $\mathscr{F}$) soc $\mathscr{F}$-blocks. If $A_0 \times B_0$ is a soc $\mathscr{F}$-block contained in $A \times B$, then any other soc $\mathscr{F}$-block $A_* \times B_*$ contained in $A \times B$ is characterized by the existence of a chaining $A_0 \times B_0 \between A_1 \times B_1 \between \ldots \between A_n \times B_n = A_* \times B_*$ in soc $\mathscr{F}$, where $A_i \times B_i$, $i = 1, \ldots, n - 1$ are, of course, suitable soc $\mathscr{F}$-blocks. Consequently there are chainings $A_0 \between A_1 \between \ldots \between A_n = A_*$ in $\mathscr{A}$ and $B_0 \between B_1 \between \ldots \between B_n = B_*$ in $\mathscr{B}$. But $A_0 \in \mathscr{A}_0$, $B_0 \in \mathscr{B}_0$, $(\mathscr{A}_0, \mathscr{B}_0) \leqq \sup \mathscr{F} = (\sup \mathscr{A}, \sup \mathscr{B})$, so that there exist $F \in \sup \mathscr{A}$, $G = \sup \mathscr{B}$ with $A_0 \subseteq F$, $B_0 \subseteq G$. From the existence of preceding two chainings one obtains that $A_* \subseteq F$, $B_* \subseteq G$; hence $A \times B \subseteq F \times G$, proving (1). An admissible family $\mathscr{F}$ will be termed *regular* if to any two elements (a_0, b_0), (a, b)

of an arbitrary soc sup $\mathscr{F}$-block, there exists a chaining $A_0 \times B_0 \between A_1 \times B_1 \between \ldots \between A_n \times B_n$ in soc $\mathscr{F}$ such that $(a_0, b_0) \in A_0 \times B_0$, $(a, b) \in A_n \times B_n$. A simple example of a non-regular admissible family $\mathscr{F}$ is given in [1], p. 27.

B$_2$. *If $\mathscr{F}$ is an admissble family, then*

$$\sup \operatorname{soc} \mathscr{F} \geqq \operatorname{soc} \sup \mathscr{F} \tag{2}$$

iff $\mathscr{F}$ is regular.

Let (2) hold, so that according to B$_1$ there is sup soc $\mathscr{F}$ = soc sup $\mathscr{F}$. Let $F \times G$ be a soc sup $\mathscr{F}$-block, $(a_0, b_0), (a, b) \in F \times G$. Since $F \times G \in$ sup soc $\mathscr{F}$, it follows from the definition of suprema that there exists the chaining in soc $\mathscr{F}$ required by the definition of regularity. Conversely, let $\mathscr{F}$ be regular and $F \times G \in$ soc sup $\mathscr{F}$. Choose $(a_0, b_0) \in F \times G$. Then, from the regularity of the family $\mathscr{F}$, it follows that for each $(a, b) \in F \times G$, the couples $(a_0, b_0), (a, b)$ lie in the same sup soc $\mathscr{F}$-block. Thus (2) holds.

C. *If $S = T$ and $\mathscr{A}_\iota = \mathscr{B}_\iota$ (for all $\iota \in I$) in an admissible family $\mathscr{F}$, then $\mathscr{F}$ is regular if and only if, to each pair of* sup *$\mathscr{A}$-blocks and to each choice of elements a_0, b_0 in the first block and of elements a, b in the second block, there exist chainings $A_0 \between A_1 \between \ldots \between A_n$, $B_0 \between B_1 \between \ldots \between B_n$ in $\mathscr{A}$ of the same length and such that $a_0 \in A_0$, $b_0 \in B_0$, $a \in A_n$, $b \in B_n$ and $A_i, B_i \in \mathscr{A}_i$ in $\mathscr{A}$ for $i = 0, 1, \ldots, n$.*

The proof follows on using the characteristic property of regular families given preceding theorem B$_2$ for the special case considered. Hence one obtains as corollaries some results of [1], pp. 26–28.

D. *If $\mathscr{F}$ is an admissible family with the family $\mathscr{A}$ of partitions on S and with the family $\mathscr{B}$ of partitions on T, then $\mathscr{F}$ is regular.*

First note that the assumptions imply that, for each $\iota \in I$, there exists a soc $\mathscr{A}_\iota$-block containing a given element of S, and also a soc $\mathscr{B}_\iota$-block containing a given element of T. Choose arbitrary elements $(a_0, b_0), (a, b)$ in any $F \times G$ = soc sup $\mathscr{F}$. We know that sup $\mathscr{F}$ = (sup $\mathscr{A}$, sup $\mathscr{B}$), so that $F \in$ sup $\mathscr{A}$, $G \in$ sup $\mathscr{B}$. Thus there exist chainings $A_0 \between A_1 \between \ldots \between A_n$ in $\mathscr{A}$ and $B_0 \between B_1 \between \ldots \between B_m$ in $\mathscr{B}$ such that $a_0 \in A_0$, $a \in A_m$, $b_0 \in B_0$, $b \in B_m$; it may be supposed without the loss of generality that the length of these two chainings is the same. Using the remark at the beginning of this proof one may construct easily enlarged chainings

$$A_0 \between A_1 \between A_1^* \between A_2 \between A_2^* \between \ldots \between A_m\,,$$

$$B_0 \between B_1^* \between B_1 \between B_2^* \between B_2 \between \ldots \between B_m$$

such that $A_0 \times B_0$, $A_1 \times B_1^*$, $A_1^* \times B_1$, $A_2 \times B_2^*$, $A_2^* \times B_2, \ldots, A_m \times B_m$ are all the soc $\mathscr{F}$-blocks. Thus $\mathscr{F}$ is regular. Cf. [1], p. 29.

E. *Let $\mathscr{F}$ be an admissible family. Then* inf $\mathscr{F}$ *exists iff* soc inf $\mathscr{F}$ *exists. If* inf $\mathscr{F}$

exists then

$$(3) \qquad \inf \operatorname{soc} \mathscr{F} = \operatorname{soc} \inf \mathscr{F} .$$

The first assertion is obvious. As for the second, note only that if $\inf \mathscr{F}$ exists then both $\operatorname{soc} \inf \mathscr{F}$-blocks and $\inf \operatorname{soc} \mathscr{F}$-blocks have the common form $A \times B$, $A \in \inf \mathscr{A}$, $B \in \inf \mathscr{B}$.

Corollary to $\mathbf{B}_{1-2}$, D, E. $\operatorname{soc}(\mathfrak{L}(S) \times \mathfrak{L}(T)) \subseteq \mathfrak{L}(S \times T)$, *and the portion of the associated mapping with the domain* $\mathfrak{L}(S) \times \mathfrak{L}(T)$ *is a lattice-monomorphism.*

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Výtah

O ASOCIOVANÝCH ROZKLADECH

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Jsou-li S, T pevné neprázdné množiny, pak označíme $\mathfrak{S}(S)$, $\mathfrak{S}(T)$, $\mathfrak{S}(S \times T)$ polosvazy všech rozkladů v S, v T a v $S \times T$. V článku jsou studovány základní vlastnosti zobrazení, přiřazujícího každému páru $(\mathscr{A}, \mathscr{B}) \in \mathfrak{S}(S) \times \mathfrak{S}(T)$ rozklad $\operatorname{soc}(\mathscr{A}, \mathscr{B})$, jehož bloky jsou tvaru $A \times B$, $A \in \mathscr{A}$, $B \in \mathscr{B}$. Je ukázáno, že jde o injektivní oboustranně isotonní zobrazení úplného polosvazu $\mathfrak{S}(S) \times \mathfrak{S}(T)$ do úplného polosvazu $\mathfrak{S}(S \times T)$, které zachovává infimum (pokud toto infimum existuje), avšak supremum zachovává pouze pro tzv. regulární rodiny dvojic $(\mathscr{A}_\iota, \mathscr{B}_\iota) \in \mathfrak{S}(S) \times \mathfrak{S}(T)$, $\iota \in I$. Jsou uvedeny některé postačující podmínky proto, aby rodina dvojic $(\mathscr{A}_\iota, \mathscr{B}_\iota) \in \mathfrak{S}(S) \times \mathfrak{S}(T)$, $\iota \in I$, byla regulární. Jde o tématiku, jejíž základy položil prof. O. Borůvka ([1]).

Резюме

ОБ АССОЦИИРОВАННЫХ РАЗЛОЖЕНИЯХ

ВАЦЛАВ ГАВЕЛ (Václav Havel), Брно

Если S, T – фиксированные непустые множества, то символами $\mathfrak{S}(S)$, $\mathfrak{S}(T)$, $\mathfrak{S}(S \times T)$ обозначим полуструктуры всех разложений в S, в T и в $S \times T$. В статье изучаются основные свойства отображения, которое каждой паре $(\mathscr{A}, \mathscr{B}) \in \mathfrak{S}(S) \times \mathfrak{S}(T)$ статвит в соответствие разложение soc $(\mathscr{A}, \mathscr{B})$, блоки которого имеют вид $A \times B$, $A \in \mathscr{A}$, $B \in \mathscr{B}$. Показано, что имеем дело с инъекционным двусторонне изотонным отображением полной полуструктуры $\mathfrak{S}(S) \times \mathfrak{S}(T)$ в полную полуструктуру $\mathfrak{S}(S \times T)$, которое сохраняет инфимум (если оно существует), но супремум (верхнюю грань) оно сохраняет только для т. наз. регулярных семейств пар $(\mathscr{A}_\iota, \mathscr{B}_\iota) \in \mathfrak{S}(S) \times \mathfrak{S}(T)$, $\iota \in J$. Приведены некоторые достаточные условия для того, чтобы семейство пар $(\mathscr{A}_\iota, \mathscr{B}_\iota) \in \mathfrak{S}(S) \times \mathfrak{S}(T)$, $\iota \in J$ было регулярным. Эти проблемы касаются тематики, основоположником которой является проф. О. Борувка (1).