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MAPPINGS THAT PRESERVE CAUCHY SEQUENCES

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Summary. Mappings preserving Cauchy sequences and their relation to continuous and uniformly continuous mappings are investigated.

Keywords: Cauchy sequences, continuous mappings, uniformly continuous mappings.

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Let (X, d_X), (Y, d_Y) be metric spaces. A sequence S in X is a mapping of the set N of all positive integers into X; R denotes the set of reals and $\mathrm{cl}\,A$ denotes the closure of A. Let F_X denote the set of all Cauchy sequences in X. Further, let $C(X, Y)$ and $U(X, Y)$ denote the set of all continuous and uniformly continuous mappings $f: X \to Y$, respectively. Let $F(X, Y)$ be the set of all mappings $f: X \to Y$ preserving Cauchy sequences, i.e.

$$F(X, Y) = \{f: X \to Y: S \in F_X \Rightarrow f \circ S \in F_Y\} .$$

R. F. Snipes in [2] has investigated the properties of $F(X, Y)$. He has shown that

$$U(X, Y) \subset F(X, Y) \subset C(X, Y) . \tag{1}$$

We recall some properties of $F(X, Y)$ by [2].

Let (X, d_X), (Y, d_Y), (Z, d_Z) be metric spaces. Then

(2) if $f \in F(X, Y)$ and $g \in F(Y, Z)$, then $g \circ f \in F(X, Z)$;

(3) if $f \in F(X, Y)$ and $A \subset X$, then $f|_A \in F(A, Y)$;

(4) if (X, d_X) is a complete metric space, then $F(X, Y) = C(X, Y)$;

(5) if (Y, d_Y) is a complete metric space, A a subset of X and $f \in F(A, Y)$, then there is $g \in F(\mathrm{cl}\,A, Y)$ such that $g|_A = f$;

(6) if Y is a normed linear space, then $F(X, Y)$ is a linear space.

In this paper we investigate on what assumptions the equalities in (1) hold.

We recall that a subset A of a metric space (X, d) is discrete if for every $x \in A$ there is a positive ε such that $d(x, y) > \varepsilon$ whenever $y \in A$ and $y \neq x$. A set A is uniformly discrete if ε does not depend on x.

Definition 1. Let (X, d) be a metric space. A set $A \subset X$ is called *F-discrete* if every totally bounded subset B of A is finite.

Remark 1. Every uniformly discrete set is F-discrete and every F-discrete set is discrete.

In fact, let A be a uniformly discrete set. Then there is a positive ε such that $d(x, y) > \varepsilon$ for any $x, y \in A$, $x \neq y$. Let $B \subset A$ be a totally bounded set. Then there is a finite set $C \subset A$ such that $B \subset \bigcup_{x \in C} S(x, \varepsilon)$. By virtue of the uniform discreteness of A we have $B = C$.

If A is not discrete, then there is a point $a \in A$ and a sequence S in A converging to a such that $\{S(n): n \in N\}$ is infinite. But $\{S(n): n \in N\}$ is a totally bounded set and hence A is not F-discrete.

Definition 2. We say that a metric space (X, d) has the property (V) if there is a sequence S in X such that

(7) the sequence S has no Cauchy subsequence,

(8) for every positive ε there are positive integers m, n such that $0 < d(S(m), S(n) < < \varepsilon$.

Remark 2. Evidently any totally bounded space does not possess the property (V).

Lemma 1. *Let (X, d) be a metric space. Then the following three conditions are equivalent:*

(9) *(X, d) is a complete metric space;*

(10) *if A and B are disjoint closed subsets of X, then there is a function $f \in F(X, R)$ such that $f|_A = 0$ and $f|_B = 1$;*

(11) *every closed discrete set in X is F-discrete.*

Proof. (9) $\Rightarrow$ (10): In view of the normality of X there is a function $f \in C(X, R)$ such that $f|_A = 0$ and $f|_B = 1$. According to (4), $f \in F(X, R)$.

(10) $\Rightarrow$ (11): Let us assume that there is a closed discrete set M in X which is not F-discrete. Therefore there is an infinite totally bounded set $P \subset M$. Hence there is a one-to-one Cauchy sequence S in P.

From the closedness and discreteness of M we obtain that the sequence S does not converge. Then any subsequence of S does not converge, either. Let us denote $A = \{S(2n - 1): n \in N\}$ and $B = \{S(2n): n \in N\}$. Then A and B are closed disjoint sets in X. Let $f: X \to R$ be a function such that $f|_A = 0$ and $f|_B = 1$. Then $S \in F_X$, but $f \circ S \notin F_R$; i.e. $f \notin F(X, R)$.

(11) $\Rightarrow$ (9): Let us assume that (X, d) is not a complete metric space. Then there is a Cauchy sequence S in X which does not converge. Then any subsequence of S

does not converge, either. It is easy to verify that $A = \{S(n): n \in N\}$ is a closed, discrete, totally bounded and infinite set. Therefore A is not F-discrete.

Lemma 2. *Let (X, d) be a metric space. Then (X, d) does not possess the property* (V) *if and only if every F-discrete subset of X is uniformly discrete.*

Proof. Necessity. Let (X, d) fail to have the property (V). Let us assume that there is an F-discrete set A in X which is not uniformly discrete. Then for $n \in N$ there are $x_n, y_n \in A$ such that

$$0 < d(x_n, y_n) < 1/n . \tag{12}$$

Let S be a sequence defined by $S(2n - 1) = x_n$ and $S(2n) = y_n$. Then S satisfies (8) and hence S does not satisfy (7). Therefore there is a Cauchy subsequence T of S. Then $B = \{T(n): n \in N\}$ is a totally bounded subset of A. In view of the F-discreteness of A the set B is finite. Thus there is $\varepsilon > 0$ such that $d(T(m), T(n)) \notin (0, \varepsilon)$ for all $m, n \in N$, which contradicts (12).

Sufficiency. Let (X, d) have the property (V). Then there is a sequence S in X satisfying (7) and (8). Let $A = \{S(n): n \in N\}$. Then in view of (8), A is not a uniformly discrete set. If B is a totally bounded subset of A, then by (7) B must be a finite set. Therefore A is a F-discrete set.

Lemma 3. *Let (X, d) be a metric space. Let A be a subset of X and $f: A \to \langle 0, 1\rangle$ a function such that $f \in F(A, \langle 0, 1\rangle)$. Then there is a function $g: X \to \langle 0, 1\rangle$ such that $g \in F(X, \langle 0, 1\rangle)$ and $g|_A = f$.*

Proof. Let (X^*, d_{X^*}) be the completion of (X, d_X). Therefore (X^*, d_{X^*}) is a complete metric space and there is a one-to-one mapping $j: X \to X^*$ such that $j \in U(X, j(X))$, $j^{-1} \in U(j(X), X)$ and $j(X)$ is dense in X^*. By (1) and (2) we have $f \circ j^{-1} \in F(j(A), \langle 0, 1\rangle)$. According to (5) there is $h \in F(\mathrm{cl}\, j(A), \langle 0, 1\rangle)$ such that $h|_{j(A)} = f \circ j^{-1}$. By Tietze's theorem there is a continuous function $h^*: X^* \to \langle 0, 1\rangle$ such that $h^*|_{\mathrm{cl}\, j(A)} = h$. In view of (4), $h^* \in F(X, \langle 0, 1\rangle)$.

Let $g = h^*|_{j(X)} \circ j$. Then $g \in F(X, \langle 0, 1\rangle)$ by (3) and (2). It is easy to see that $g|_A = f$.

Now let Y be a normed linear space. If $Y = \{0\}$ then obviously the equality in (1) holds. Hence we shall assume that $Y \neq \{0\}$.

Theorem 1. *Let (X, d) be a metric space and let $(Y, \|\cdot\|)$, $Y \neq \{0\}$, be a normed linear space. Then $F(X, Y) = C(X, Y)$ if and only if (X, d) is a complete metric space.*

Proof. Necessity. Let (X, d) fail to be a complete metric space. Then by Lemma 1 there is a closed discrete set M in X which is not F-discrete. Hence there is a one-to-one Cauchy sequence S in M. Let $g: M \to \langle 0, 1\rangle$ be defined by

$$g(x) = 1 \quad \text{if} \quad x = S(2n) \quad \text{for some} \quad n \in N \quad \text{and} \quad g(x) = 0 \quad \text{otherwise} .$$

In view of the discreteness of M, g is a continuous function. Hence by Tietze's theorem there is a continuous function $g^*: X \to \langle 0, 1\rangle$ such that $g^*|_M = g$. Let $y \in Y$, $y \neq 0$. It is easy to see that $h: \langle 0, 1\rangle \to Y$, defined by $h(t) = ty$, is a uniformly continuous mapping. Hence the mapping $f = h \circ g^*$ is continuous. However, $f \notin F(X, Y)$, because $S \in F_X$ and $f \circ S \notin F_Y$.

Sufficiency follows by (4).

Theorem 2. *Let (X, d) be a metric space and let $(Y, \|\cdot\|)$, $Y \neq \{0\}$, be a normed linear space. Then $F(X, Y) = U(X, Y)$ if and only if (X, d) fails to have the property (V).*

Proof. Necessity. Let (X, d) have the property (V). Then by Lemma 2 there is an F-discrete set M in X which is not uniformly discrete. It is easy to see that there are sequences S, T in M such that

$$0 < d(S(n), T(n)) < 1/n \quad \text{for all} \quad n \in N \tag{13}$$

and

$$S(i) \neq T(j) \quad \text{for each} \quad i, j \in N\,. \tag{14}$$

Let us define a function $g: M \to \langle 0, 1\rangle$ as follows:

$$g(x) = 1 \quad \text{if} \quad x = S(n) \quad \text{for some} \quad n \in N \quad \text{and}$$
$$g(x) = 0 \quad \text{otherwise}\,.$$

Then obviously $g(T(n)) = 0$ for $n \in N$.

Let P be a Cauchy sequence in M. Then $\{P(n): n \in N\}$ is a totally bounded subset of M and hence it is finite. The sequence P is thus eventually constant and hence $f \circ P$ is an eventually constant sequence, too. Hence $g \circ P \in F_Y$ and $g \in F(M, \langle 0, 1\rangle)$. By Lemma 3 there exists $g^* \in F(X, \langle 0, 1\rangle)$ such that $g^*|_M = g$. Let $y \in Y$, $y \neq 0$, and let $h: \langle 0, 1\rangle \to Y$ be defined by $h(t) = ty$. Then (2) implies $f = h \circ g^* \in F(X, Y)$. However, f is not uniformly continuous because for any $n \in N$ we have

$$d(S(n), T(n)) < 1/n \quad \text{and} \quad \|f(S(n)) - f(T(n))\| = \|y\|\,.$$

Sufficiency. Let (X, d) fail to have the property (V). We will assume that there is $f \in F(X, Y) - U(X, Y)$. Then there are sequences S, T in X and $\varepsilon > 0$ such that

$$0 < d(Sn), T(n)) < 1/n \tag{15}$$

and

$$\|f(S(n)) - f(T(n))\| \geqq \varepsilon\,. \tag{16}$$

The set $A = \{S(n): n \in N\} \cup \{T(n): n \in N\}$ is not uniformly discrete. Hence according to Lemma 2, the set A is not F-discrete. Therefore there is an infinite totally bounded set $B \subset A$. Let e.g. the set $B \cap \{S(n): n \in N\}$ be infinite. Then the sequence S

has a Cauchy subsequence P, i.e. there is an increasing function $u: N \to N$ such that $P = S \circ u$. Let Q be a sequence in M defined by

$$Q(2n) = S(u(n)) \quad \text{and} \quad Q(2n - 1) = T(u(n)) \quad \text{for all} \quad n \in N .$$

Then it is easy to verify (by (15)) that Q is a Cauchy sequence. Hence $f \circ Q$ is a Cauchy sequence, too. However, this contradicts (16).

Corollary. *Let (X, d) be a metric space and let $(Y, \|\cdot\|)$, $Y \neq \{0\}$, be a normed linear space. Then $U(X, Y) = C(X, Y)$ if and only if every closed discrete subset of X is uniformly discrete.*

Let X be a set and $(Y, \|\cdot\|)$ a normed linear space. We recall that a mapping $f: X \to Y$ is bounded if there is a positive k such that $\|f(x)\| \leqq k$ for each $x \in X$. If $P(X, Y) \subset Y^X$, then $bP(X, Y)$ denotes all bounded mappings belonging to $P(X, Y)$. If $P(X, Y)$ is a linear space, then $bP(X, Y)$ is a linear normed space with the sup-norm.

Lemma 4. *Let X be a set and $(Y, \|\cdot\|)$ a normed linear space. Let $P(X, Y)$, $Q(X, Y) \subset Y^X$ be linear spaces. Let $P(X, Y)$ be a closed subset of $Q(X, Y)$ with respect to the topology of uniform convergence. If $bP(X, Y) \neq bQ(X, Y)$, then $P(X, Y)$ is a nowhere dense set in $Q(X, Y)$.*

Proof. The uniform convergence in $Q(X, Y)$ is metrizable with the metric

$$\varrho(f, g) = \min \{1, \sup \{\|f(x) - g(x)\|: x \in X\}\} \quad \text{for} \quad f, g \in Q(X, Y) .$$

In view of the closedness of $P(X, Y)$, it is sufficient to show that $P(X, Y)$ is a boundary set.

Let $\varepsilon > 0$ and $f \in Q(X, Y)$. We shall show that there is $g \in Q(X, Y) - P(X, Y)$ such that $\varrho(f, g) < \varepsilon$. Since $bP(X, Y)$ is a closed linear subspace of $bQ(X, Y)$ and $bP(X, Y) \neq bQ(X, Y)$, according to [1] the set $bP(X, Y)$ is nowhere dense in $bQ(X, Y)$. Hence there is $h \in bQ(X, Y) - bP(X, Y)$ such that $\|h(x)\| < \varepsilon/2$ for all $x \in X$. If $f \notin P(X, Y)$, then we put $g = f$. If $f \in P(X, Y)$, then we put $g = f + h$. Then $g \in Q(X, Y) - P(X, Y)$ and $\varrho(f, g) < \varepsilon$.

Lemma 5. *Let (X, d_X), (Y, d_Y) be metric space. Let F, $F(n) = f_n$, be a sequence of functions belonging to $F(X, Y)$ which uniformly converges to f. Then f belongs to $F(X, Y)$.*

Proof. Let $f_n \in F(X, Y)$ for all $n \in N$ and let F converge uniformly to f. We shall show that $f \in F(X, Y)$. Let $S \in F_X$ and $\varepsilon > 0$. Then there is $m \in N$ such that $d_Y(f_m(x), f(x)) < \varepsilon/3$ for each $x \in X$.

Since $f_m \circ S \in F_Y$ by the hypotheses, there is $p \in N$ such that $d_Y(f_m(S(i)), f_m(S(j))) < \varepsilon/3$ for $i, j \geqq p$. Then for any $i, j \geqq p$ we have

$$d_Y(f(S(i)), f(S(j))) \leqq d_Y(f(S(i)), f_m(S(i))) + \\ + d_Y(f_m(S(i)), f_m(S(j))) + d_Y(f_m(S(j)), f(S(j))) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon .$$

Therefore $f \circ S \in F_Y$, i.e. $f \in F(X, Y)$.

Theorem 3. *Let (X, d) be a metric space and $(Y, \|\cdot\|)$, $Y \neq \{0\}$ a normed linear space. If (X, d) is not a complete metric space, then $F(X, Y)$ is a nowhere dense set in $C(X, Y)$ with respect to the topology of uniform convergence. If (X, d) has the property (V), then $U(X, Y)$ is a nowhere dense set in $F(X, Y)$ with respect to the topology of uniform convergence.*

Proof. Obviously $C(X, Y)$, $U(X, Y)$ and (by (6)) $F(X, Y)$ are linear spaces. The set $U(X, Y)$ is closed in $F(X, Y)$ and by Lemma 5 also $F(X, Y)$ is closed in $C(X, Y)$. If (X, d) is not a complete metric space, then by Theorem 1 and its proof and Lemma 4 we see that $F(X, Y)$ is a nowhere dense set in $C(X, Y)$. Similarly, if (X, d) has the property (V), then by Theorem 2 and its proof and Lemma 4 we see that $U(X, Y)$ is a nowhere dense set in $F(X, Y)$.

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Súhrn

ZOBRAZENIA, KTORÉ ZACHOVÁVAJÚ FUNDAMENTÁLNE POSTUPNOSTI

JÁN BORSÍK

V práci sa vyšetrujú zobrazenia, ktoré zachovávajú fundamentálne postupnosti a ich vzťah ku spojitým a rovnomerne spojitým zobrazeniam.

Резюме

ОТОБРАЖЕНИЯ, СОХРАНЯЮЩИЕ ПОСЛЕДОВАТЕЛЬНОСТИ КОШИ

JÁN BORSÍK

В статье исследуется отношение отображений, сохраняющих последовательности Коши, к непрерывным и равномерно непрерывным отображениям.

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