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ABOUT ONE INVERSE PROBLEM IN THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS WITH MEASURES AS COEFFICIENTS

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Summary. In the paper the following problems PI and PII are studied: for a given sequence $\{z_k\} \subset R^n$ whose elements satisfy some non-restrictive conditions find the matrix $A(\cdot)$, the entries of which are measures, such that $t_k = k$ are the atomic points of these measures and:

$$\text{PI} \quad x(k) = z_k$$

$$\text{PII} \; x(k) - x(k-) = z_k$$

where $x(\cdot)$ denotes the solution of the system $\dot{x} = A(t)\,x$. For $n = 1$ these problems are completely solved while for $n > 1$ a system of linear algebraic equations or a reccurence equation is presented by which we may solve the problems, i.e., we may construct the measure $A(\cdot)$.

Keywords: Linear systems with measures as coefficients, realization of systems.

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1. INTRODUCTION

In my paper [1] concerning linear differential equations of the type (1) below where the elements $a_{ij}(\cdot)$ of the matrix $A(\cdot)$ are measures two reccurence equations were constructed such that: the solution of the first (see [1, (12)]) gives the values of the solution of (1) at the atomic points of the measure $A(\cdot)$, and the solution of the second (see [1, (16)]) denotes the jumps of the solution of (1) at these points. Here we formulate and solve the inverse problem: for a given sequence $\{z_k\}$ the elements of which satisfy some not very restrictive conditions (they will be specified below) we must construct the "simplest possible" equation of the type (1) (i.e., construct a measure $A(\cdot)$ whose atomic points are the natural numbers) such that $z_k = x(k)$ or $z_k = x(k) - x(k-)$ for all $k \in N$. In the scalar case these problems are completely solved while in the multidimensional case we present a system of linear algebraic equations or a reccurence equation the solutions of which solve our problems.

2. PRELIMINARIES, NOTATION AND FORMULATION OF THE PROBLEMS

Let us consider the following system of differential equations:

$$(1) \qquad \dot{x} = A(t)\,x\,, \quad x(t_0+) = x_0\,, \quad t_0 \in (a, b)\,, \quad -\infty \leqq a < b < \infty\,, \quad x \in R^n$$

where the elements of the matrix $A(\cdot)$ are measures, i.e., Stieltjes measures generated by some functions of locally bounded variation:

$$A(\cdot) = \mathscr{A}'(\cdot)\,, \quad \mathscr{A}(\cdot) \in BV_{\mathrm{loc}}(a, b)\,.$$

Here by $BV_{\mathrm{loc}}(a, b)$ we denote the space of all right-continuous functions of locally bounded variation in the interval (a, b).

A function $x(\cdot) \in BV_{\mathrm{loc}}(a, b)$ is a solution of the equation (1) iff $x(\cdot)$ satisfies the following integral equation:

$$x(t) = x_0 + \int_{t_0}^{t} \mathrm{d}\mathscr{A}(s)\, x(s)\,, \quad t \in (a, b) \tag{2}$$

where $\int_c^d f(s)\,\mathrm{d}h(s)$ denotes $\int_{[c,d)} f(s)\,\mathrm{d}h(s)$ and this integral is understood in the Lebesgue-Stieltjes sense.

Under the hypothesis H_1 below the equation (1) (or (2)) has a unique solution which is a picewise – continuous function with jumps at every atomic point of the measure $A(\cdot)$. Denote

$$A(t) = \hat{A}(t) + \sum_{k=1}^{\infty} C_k \delta(t - t_k)$$

where $\hat{A}(\cdot)$ is the continuous part of the measure $A(\cdot)$, $\delta(\cdot)$ is the Dirac's measure, $C\,\delta(\cdot)$ denotes the matrix all elements of which are equal to $c^{ij}\,\delta(s)$. Assume that $a < t_0 < t_1 < \ldots < t_n < \ldots < b$ and that the only accumulation point of the sequence $\{t_n\}$ may be b.

Let $\hat{\Phi}(\cdot)$ be the fundamental matrix of the system

$$\dot{x} = \hat{A}(t)\, x\,, \quad \hat{\Phi}(t_0) = E\,, \tag{3}$$

E is the unit matrix, let $s_k = x(t_k)$ be the value of the solution of (1) at instant t_k and let $\varepsilon_k = s_k - x(t_k-)$ be the jump of this solution at instant t_k. Let us introduce the following hypotheses:

H_1. $\quad \det (E - C_k) \neq 0 \quad$ for all $\quad k \in N$,

H_2. $\quad \det C_k \neq 0 \quad$ for all $\quad k \in N$.

Under the above notation the following relations hold:

$$x(t) = \hat{\Phi}(t)\, \hat{\Phi}^{-1}(t_{k-1})\, s_{k-1} \quad \text{for} \quad t \in [t_{k-1}, t_k)\,,$$

$$x(t) = \hat{\Phi}(t)\, \hat{\Phi}^{-1}(t_0)\, x_0 + \sum_{k: t_k \leqq t} \hat{\Phi}(t)\, \hat{\Phi}^{-1}(t_k)\, \varepsilon_k H(t - t_k)\,,$$

$H(\cdot)$ is the Heaviside function:

$$H(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad t \geqq 0\,, \end{cases}$$

under the hypothesis H_1 – the next three:

$$s_k = (E - C_k)^{-1}\, x(t_k-)\,, \quad \text{for} \quad k = 1, 2, \ldots,$$

(4) $$s_k = (E - C_k)^{-1}\,\hat{\Phi}(t_k)\,\hat{\Phi}^{-1}(t_{k-1})\,s_{k-1}\,,\quad s_0 = x_0\,,\quad k = 1, 2, \ldots,$$

(5) $$\varepsilon_k = (E - C_k)^{-1}\,C_k\,x(t_k -)\,,\quad k = 1, 2, \ldots$$

and – under the hypotheses H_1, H_2 – we have

$$\varepsilon_k = C_k s_k\,,\quad k = 1, 2, \ldots,$$

(7) $$\varepsilon_k = (E - C_k)^{-1}\,C_k\,\hat{\Phi}(t_k)\,\hat{\Phi}^{-1}(t_{k-1})\,C_{k-1}^{-1}\varepsilon_{k-1}\,,\quad k = 2, 3, \ldots$$

where ε_1 is calculated from (5) (for details – see [1]).

Now we formulate two problems which will be solved here. Suppose $b = \infty$, $t_k = k \in N$. Let a sequence $\{z_k\}$, $z_k \in R^n$, $n \geqq 1$ be given the elements of which satisfy some conditions which will be specified later.

Problem PI. Construct the "simples possible" equation of the type (1), the solution $x(\cdot)$ of which satisfies the condition

(7) $$s_k = z_k \quad \text{for all} \quad k \in N\,.$$

Problem PII. Construct the "simplest possible" equation of the type (1) the solution $x(\cdot)$ of which satisfies the condition

(8) $$\varepsilon_k = z_k \quad \text{for all} \quad k \in N\,.$$

By "simplest possible" we understand such an equation for which $\hat{A}(\cdot) \equiv 0$ (because in this case $\hat{\Phi}(\cdot) \equiv E$) or – if this is impossible – one for which the cardinal number of the set

$$V = \{k \in N\colon\ \hat{A}(t) \neq 0 \ \text{ for } \ t \in [t_k, t_{k+1})\}$$

is the smallest possible.

The formulations and solutions of the problems PI, PII may have some practical applications; namely, we can construct the systems with prescribed properties.

For example, if the sequence $\{z_k\}$ tends to the zero-vector then we can construct the stable system with a prescribed regime of motion. Similarly, the linear control system

$$\dot{x} = A(t)\,x + B(t)\,u$$

may be stabilizable in a prescribed way by linear feedback if we define some sequence $z_n \to 0$ and construct the corresponding measure $B(\cdot)$ for a given matrix $A(\cdot)$ whose entries may be functions or measures.

These problems may have also some military applications if we are to bomb some prescribed objects at given instants or to destroy a flying object of the enemy by our rocket.

In one-dimensional case, by selecting the sequences $\{z_k\}$ with some properties we may construct various examples of linear measure-differential equations, the solutions of which have some pathological properties. For example, we may construct a non-negative measure $A(\cdot)$ such that the solutions of the equation (1) are piece-wise – monotonic but not monotonic, or – if $z_{n+1}z_n < 0$ for all n – the solution of the equation (1) will be oscillatory (i.e. will change its sign infinitely many times).

Finally by the próblem PI some sequences may be treated not only as solutions of difference equations but also as solutions of differential equations. Therefore some methods of the qualitative theory of differential equations may be applied to the study of difference equations.

3. SOLUTION OF THE PROBLEM PI

a) *Scalar case*

Let $n = 1$, i.e., $\{z_k\}$ is a sequence of real numbers. Assume that $\{z_k\}$ satisfies the condition

(W.1) $$z_k \neq 0 \quad \text{for} \quad k = 1, 2, \dots .$$

We are looking for a sequence of real numbers $\{C_k\}$ such that $C_k \neq 1$ for all k (see Hyp. H_1) and the solution $x(\cdot)$ of the equation

(9) $$\dot{x} = [\sum_{k=1}^{\infty} C_k \delta(t - k)]\, x\,, \quad x(1) = z_1\,, \quad t \in [1, \infty)\,, \quad x \in R$$

satisfies (7).

For the equation (9) the relation (4) by virtue of (7) has the form

$$z_k = \frac{z_{k-1}}{1 - C_k}$$

from which we may construct the sequence $\{C_k\}$:

(10) $$C_k = \frac{z_k - z_{k-1}}{z_k} = 1 - \frac{z_{k-1}}{z_k}\,.$$

This construction is unique. The condition (W.1) implies that $C_k \neq 1$ for all $k \in N$, hence the hypothesis H_1 is fulfilled. If (W.1) is not fulfilled for some $1 \in N$, then – by (4) – the sequence $\{z_k\}$ must be such that $z_k = 0$ for all $k \geqq 1$.

Remark 1. If we construct the equation (1) without the condition $\hat{A}(\cdot) \equiv 0$ then the corresponding sequence $\{C_k\}$ is given by the formula

(10′) $$C_k = 1 - \frac{\hat{\phi}(k)\, z_{k-1}}{\hat{\phi}(k - 1)\, z_k}$$

and (W.1) implies that the hypothesis H_1 is fulfilled.

b) *Multidimensional case*

Assume that $n > 1$ and let the sequence $\{z_k\}$ of n-dimensional vectors satisfy the condition

(W.1′) $$\|z_k\| > 0 \quad \text{for all} \quad k \in N\,.$$

We are looking for a sequence $\{C_k\}$ of $n \times n$-matrices which satisfy the hypothesis H_1 such that the solution $x(\cdot)$ of the equation (1) under $\hat{A}(\cdot) \equiv 0$ satisfies (7).

The sequence $\{C_k\}$ must be constructed from the relation (4) which leads to the equation

$$C_k z_k = z_k - z_{k-1}\,. \tag{11}$$

This is a system of n linear algebraic equations with n^2 unknowns c_k^{ij}, $i, j = 1, \ldots, n$, therefore the matrix C_k is not uniquely determined. If, for example, all components z_k^i of every vector z_k are different from zero then C_k may be constructed as the diagonal matrix

$$C_k = \operatorname{diag}\left(1 - \frac{z_{k-1}^i}{z_k^i}\right)_{i=1,\ldots,n}\,. \tag{11'}$$

4. SOLUTION OF THE PROBLEM PII

a) *Scalar case*

Let $n = 1$ and let the sequence $\{z_k\}$ of real numbers satisfy the condition (W.1) and

$$z_{k+1} \neq z_k \quad \text{for all} \quad k \in N\,. \tag{W.2}$$

We are looking for a sequence of real numbers $\{C_k\}$ such that $C_k \neq 0$, $C_k \neq 1$ for all $k \in N$ (see Hyp. H_1, H_2) and the corresponding solution $x(\cdot)$ of the equation

$$\dot{x} = \left[\sum_{k=1}^{\infty} C_k \delta(t - k)\right] x\,, \quad x(0) = x_0 = 0\,, \quad t \in [0, \infty)\,, \quad x \in R \tag{9'}$$

satisfies (8).

For the equation (9') relation (6) by virtue of (8) has the form

$$z_k = \frac{C_k}{(1 - C_k)\, C_{k-1}}\, z_{k-1}\,, \quad k = 2, 3, \ldots$$

from which we deduce the reccurence equation for the sequence $\{C_k\}$:

$$C_k = 1 - \frac{z_{k-1}}{z_{k-1} + z_k C_{k-1}}\,, \quad k = 2, 3, \ldots.$$

This equation has the solution

$$C_k = \frac{z_k}{z_1 + \ldots + z_k}\,, \quad k = 2, 3, \ldots. \tag{12}$$

The number C_1 may be calculated from (5):

$$C_1 = 1 - \frac{x_0}{x_0 + z_1}\,. \tag{13}$$

The conditions (W.1), (W.2) imply that the hypotheses H_1, H_2 are fulfilled. The sequence $\{C_k\}$ is uniquely determined.

Assume now that $\{z_k\}$ satisfies (W.1) but not necessarily (W.2). Let us introduce the auxiliary function

$$A(t) = \chi_U(t),$$

the characteristic function of the set

$$U = \bigcup_{k\in\{1:z_1=z_1-1\}} [k-1, k),$$

and let $\hat{\phi}(\cdot)$ be the corresponding fundamental matrix of the system (3). Now from the relation (6) for the equation (1) by virtuc of (8) we obtain the reccurence equation

$$z_k = \frac{\hat{\phi}(k)\, C_k}{(1 - C_k)\, C_{k-1}\hat{\phi}(k-1)}\, z_{k-1}, \quad k = 2, 3, \ldots$$

from which we calculate the elements of the sequence $\{C_k\}$:

$$(12') \qquad C_k = 1 - \frac{\hat{\phi}(k)\, z_{k-1}}{\hat{\phi}(k)\, z_{k-1} + \hat{\phi}(k-1)\, C_{k-1} z_k}, \quad k = 2, 3, \ldots.$$

b) *Multidimensional case*

Now assume that $n > 1$ and let the sequence $\{z_k\}$ of n-dimensional vectors $\{z_k\}$ satisfy (W.1′) and (W.2). Find the sequence $\{C_k\}$ of $n \times n$-matrices which satisfy H_1, H_2 such that the solution $x(\cdot)$ of the equation (1) under $\hat{A}(\cdot) \equiv 0$ satisfies (8).

The sequence $\{C_k\}$ must be constructed from the relation (6) from which we deduce the following reccurence equation:

$$(14) \qquad z_k = C_k z_k + C_k C_{k-1}^{-1} z_{k-1}, \quad k = 2, 3, \ldots$$

or evidently

$$(14') \qquad C_k^{-1} z_k = z_k + C_{k-1}^{-1} z_{k-1}, \quad k = 2, 3, \ldots$$

where the first matrix C_1 is calculated from (5):

$$(15) \qquad z_1 = (E - C_1)^{-1}\, C_1 x_0 .$$

The matrix C_1 is not uniquely determined by (15), hence neither are the other matrices C_k. If the initial condition x_0 and the vector z_1 are such that $x_0^i \neq 0$ for $i = 1, \ldots, n$ and

$$\prod_{i=1}^{n} \left(\frac{x_0^i}{x_0^i + z_1^i}\right) \neq 1$$

then C_1 may be constructed as the diagonal matrix

$$C_1 = \operatorname{diag}\left(\frac{x_0^i}{x_0^i + z_1^i}\right)_{i=1,\ldots,n}.$$

Consequently, if the vectors z_{k-1}, z_k and the diagonal matrix C_{k-1} for $k = 2, 3, \ldots$ are such that $z_k^i \neq 0$ for $i = 1, \ldots, n$ and

$$\prod_{i=1}^{n} \left(\frac{C_{k-1}^i z_k^i}{C_{k-1}^i z_k^i + z_{k-1}^i}\right) \neq 1$$

then we may construct C_k as the diagonal matrix

$$C_k = \operatorname{diag}\left(\frac{C^i_{k-1} z^i_k}{C^i_{k-1} z^i_k + z^i_{k-1}}\right)_{i=1,\ldots,n}, \quad k = 2, 3, \ldots .$$

We summarize our results in the following

Theorem. *If the sequence $\{z_k\}$ satisfies the condition* (W.1) (W.1′) *if $n > 1$) then the equation* (9) *is the solution of the problem* PI *where the numbers C_k are given by* (10) (*the matrices C_k satisfy the equation* (11) *in multidimensional case*).

If the sequence $\{z_k\}$ satisfies the conditions (W.1)–(W.2) (W.1′)–(W.2) *if $n > 1$) then the equation* (9′) *is the solution of the problem* PII *where the numbers C_k are given by* (12)–(13) *in the scalar case* (*the matrices C_k satisfy the reccurence equations* (14)–(15) *in a multidimensional case*).

If $\{z_k\}$ satisfies (W.1) *only but not necessarily* (W.2) *then the solution of the problem* PII*is given by* (12′) *in the one-dimensional case.*

Remark 2. The same problems may be formulated and solved in the same way if two sequences $\{z_k\} \subset R^n$ and $\{t_k\} \subset R$ are given and we are looking for the matrix $A(\cdot)$, the entries of which are measures with the atomic points t_k such that the solution of the equation (1) satisfies $x(t_k) = z_k$ (for PI) or $x(t_k) - x(t_k -) = z_k$ (for PII). Here – for simplicity only – we put $t_k = k$.

References

[1] *Z. Wyderka:* Linear differential equations with measures as coefficients and control theory, (to appear in Časopis pěst. mat.).

Souhrn

O JEDNOM INVERZNÍM PROBLÉMU V TEORII LINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC S MÍRAMI JAKO KOEFICIENTY

ZDZISŁAW WYDERKA

V článku jsou studovány následující problémy PI a PII: pro danou posloupnost $\{z_k\} \subset R^n$ jejíž prvky splňují jisté málo restriktivní podmínky najděte matici $A(\cdot)$, jejíž prvky jsou takové míry, že $t_k = t$ jsou jejich atomické body a

$$\text{PI} \quad x(k) = z_k ,$$

$$\text{PII} \quad x(k) - x(k-) = z_k ,$$

kde $x(\cdot)$ označuje řešení systému $\dot{x} = A(t)\, x$. Pro $n = 1$ jsou oba problémy úplně vyřešeny, zatímco pro $n > 1$ je odvozena soustava algebraických rovnic nebo rekurentní rovnice pomocí kterých lze problém řešit, tj. konstruovat míru $A(\cdot)$.

Резюме

ОБ ОДНОЙ ОБРАТНОЙ ЗАДАЧЕ В ТЕОРИИ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С МЕРАМИ В КАЧЕСТВЕ КОЭФФИЦИЕНТОВ

Zdzisław Wyderka

В статье изучаются следующие проблемы PI и PII: для заданной последовательности $\{z_k\} \subset R^n$, члены которой удовлетворяют некоторым слабым ограничивающим условиям, найти матрицу $A(\cdot)$, состоящую из таких мер, что $t_k = t$ являются их атомическими точками и

$$\text{PI} \quad x(k) = z_k,$$

$$\text{PII} \; x(k) - x(k-) = z_k,$$

где $x(\cdot)$ обозначает решение системы $\dot{x} = A(t)\, x$. Для $n = 1$ обе проблемы полностью решены и для $n > 1$ выведены система алгебраических уравнений или рекуррентные уравнения, при помощи которых можно проблему решить, т. е. построить меру $A(\cdot)$.

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