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LATTICE ORDERED GROUPS WITH CYCLIC LINEARLY ORDERED SUBGROUPS

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In this note a solution is given to a problem proposed by CONRAD and MONTGOMERY [3] on lattice ordered groups G with the property that each linearly ordered subgroup of G is cyclic.

Let G be an archimedean lattice ordered group. Consider the following conditions for G:

(a) G is singular;

(b) each linearly ordered subgroup of G is cyclic.

In [3] it was proved that (a) implies (b) while the problem whether (a) is implied by (b) remained open. We shall show that *the answer is negative in general; nonetheless*, (b) $\Rightarrow$ (a) is valid if G is complete.

For the basic notions and notations cf. BIRKHOFF [1] and FUCHS [4]. Let G be a lattice ordered group. An element $0 \leqq g \in G$ is called singular, if $x \wedge (g - x) = 0$ for each $x \in G$ with $0 \leqq x \leqq g$. It is easy to verify that a strictly positive element $g \in G$ is singular if and only if the interval $[0, g]$ is a Boolean algebra. The l-group G is singular, if for each $0 < g \in G$ there is a singular element $h \in G$ such that $0 < h \leqq g$. Singular lattice ordered groups were investigated in the papers [2], [5], [6], [7], [8].

The following theorem is known (cf. [2]):

(A) *Let G be a complete l-group. Then there are l-subgroups A, B of C such that A is singular, B is a vector lattice and $G = A \times B$.*

(The symbol $A \times B$ denotes the direct sum of l-groups A and B.)

Now let G be a complete l-group that is not singular. According to (A) we have $B \neq \{0\}$ and hence there is b, $0 < b \in B$. Let R be the set of all reals; since B is a vector lattice, for each $r \in R$ there exists $rb \in B$. Denote $B_1 = \{rb : r \in R\}$. Then B_1 is a linearly ordered subgroup of G that fails to be cyclic. Therefore (a) is implied by (b) whenever G is a complete lattice ordered group.

The following example shows that an archimedean lattice ordered group fulfilling (b) need not be singular.

Let Q be the set of all rational numbers and let G_0 be the set of all real functions defined on Q. For $f, g \in G_0$ we put $f \leqq g$ if $f(x) \leqq g(x)$ for all $x \in Q$. Then $(G_0; +, \leqq)$ is an archimedean lattice ordered group. Let φ be a one-to-one mapping of the set N of all positive integers onto the set Q. Further, let G be the set of all $f \in G_0$ with the following properties:

(i) $2^{n-1} f(\varphi(n))$ is an integer for all $n \in N$;

(ii) there are irrational numbers $\alpha_1 < \beta_1 \leqq \alpha_2 < \beta_2 \leqq \ldots \leqq \alpha_m < \beta_m$ such that f is a constant on each set $Q \cap [\alpha_i, \beta_i]$ $(i = 1, \ldots, m)$ and $f(x) = 0$ for each $x \in Q \setminus \bigcup[\alpha_i, \beta_i]$ $(i = 1, \ldots, m)$. Then G is an l-subgroup of G_0.

Let $H \neq \{0\}$ be a linearly ordered subgroup of G. For each $h \in H$ put

$$s(h) = \{x \in Q : h(x) \neq 0\}.$$

Lemma 1. *Let* $0 \neq h_i \in H$ $(i = 1, 2)$. *Then* $s(h_1) = s(h_2)$.

Proof. Suppose that $s(h_1) \neq s(h_2)$. Then we can assume that there is $x \in s(h_1) \setminus s(h_2)$. We have $|h_i| \in H$, $s(|h_i|) = s(h_i)$ $(i = 1, 2)$. The elements $|h_1|, |h_2|$ are comparable and $|h_1|(x) > 0 = |h_2|(x)$. Since $h_2 \neq 0$, there is $y \in s(h_2)$ and hence $|h_2|(y) > 0$. There is a positive integer n with $n|h_2|(y) > |h_1|(y)$. Since $n|h_2| \in H$, the elements $n|h_2|$ and $|h_1|$ are comparable, thus $n|h_2| > |h_1|$. But

$$0 = n|h_2|(x) < |h_1|(x)$$

and this is a contradiction.

For $x \in Q$ let

$$F_x = \{h(x) : h \in G\}.$$

Obviously F_x is an additive group.

Lemma 2. *Let* $0 \neq h_0 \in H$, $x \in s(h_0)$. *The mapping*

$$\varphi_1 : h \to h(x)$$

is an isomorphism of H *into* F_x.

Proof. If $h_1, h_2 \in H$ and $\circ \in \{+, \wedge, \vee\}$, then

$$\varphi_1(h_1 \circ h_2) = h_1(x) \circ h_2(x),$$

thus φ_1 is a homomorphism of H into F_x. Let $\varphi_1(h_1) = \varphi_1(h_2)$ and suppose that $h_1 \neq h_2$. Then $h = h_1 - h_2 \in H$, $h \neq 0$ and $h(x) = 0 \neq h_0(x)$. Thus $s(h) \neq s(h_0)$, which contradicts Lemma 1. Therefore $h_1 = h_2$ and hence φ_1 is an isomorphism.

Lemma 3. *The l-group* H *is cyclic.*

Proof. Let $x \in Q$, $\varphi^{-1}(x) = n$. There exist irrational numbers α, β such that $x \in [\alpha, \beta]$ and $\varphi^{-1}(y) \geqq n$ for each $y \in [\alpha, \beta] \cap Q$. Let $f \in G_0$ such that $f(z) = 2^{1-n}$

for each $z \in [\alpha, \beta] \cap Q$ and $f(z) = 0$ otherwise. Then $f \in G_0$ and hence $2^{1-n} \in F_x$. Thus by (i), 2^{1-n} is a generator of the group F_x and therefore F_x is cyclic. Hence each subgroup of F_x is cyclic; by Lemma 2, H is cyclic.

Lemma 4. *Let* $0 < f \in G_0$. *Then* f *is not singular.*

Proof. Suppose that f is singular. Then each $f_1 \in G_0$, $0 < f_1 < f$ is singular. There exist irrational numbers α_1, β_1 and a real $c \neq 0$ such that $f(x) = c$ for each $x \in [\alpha_1, \beta_1] \cap Q$. Let $f_1 \in G_0$ such that $f_1(x) = f(x) = c$ for each $x \in Q \cap [\alpha_1, \beta_1]$ and $f_1(x) = 0$ otherwise. Clearly $f_1 \in G$ and $0 < f_1 \leqq f$. Let

$$N_1 = \{\varphi^{-1}(x) : x \in Q \cap [\alpha_1, \beta_1]\} .$$

Let k be the least element of N_1. According to (i) and (ii), $2^{k-1}c$ is an integer. We can choose irrational numbers $\alpha < \beta$ such that $[\alpha, \beta] \subset [\alpha_1, \beta_1]$ and $\varphi(k) \notin [\alpha, \beta]$. Let $y \in [\alpha, \beta] \cap Q$. Put $\varphi^{-1}(y) = t$. Since $t > k$, we infer that $2^{k-1}(\frac{1}{2}c)$ is an integer. Thus the function $g \in G_0$ defined by

$$g(x) = \tfrac{1}{2}c \quad \text{if} \quad x \in [\alpha, \beta] \cap Q \quad \text{and} \quad g(x) = 0 \quad \text{otherwise}$$

belongs to G_0. We have $0 < 2g < f_1$, hence $g < f_1 - g$ and therefore

$$g \wedge (f_1 - g) = g > 0 ;$$

thus f_1 cannot be singular. This shows that f is not singular.

From Lemma 2 and Lemma 4 it follows that there exists an archimedean lattice ordered group fulfilling (b) with no singular elements.

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