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ON THE LINEAR CONTROL PROBLEM $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$

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NOTATIONS AND DEFINITIONS

Let K be a compact subset of E_k (E_k is the k-dimensional Euclidian space). $(,)$ is the scalar product in E_k. The hyperplane $(\boldsymbol{\varphi}, \mathbf{x}) = \gamma$ will be called the support hyperplane of K if $(\boldsymbol{\varphi}, \mathbf{y}) \leqq \gamma$ for all $\mathbf{y} \in K$ and if there is a $\mathbf{z} \in K$ such that $(\varphi, \mathbf{z}) = \gamma$ (then we write $\gamma = \max_{\mathbf{x} \in K} (\boldsymbol{\varphi}, \mathbf{x})$). For any $\boldsymbol{\varphi} \in E_k$, $\boldsymbol{\varphi} \neq 0$ a support hyperplane of K is determined, namely the hyperplane $(\boldsymbol{\varphi}, \mathbf{x}) = \max_{\mathbf{y} \in K} (\boldsymbol{\varphi}, \mathbf{y})$. The point $\mathbf{p} \in K$ will be called an exposed point of K if there is a $\boldsymbol{\varphi} \in E_k$ such that $(\boldsymbol{\varphi}, \mathbf{p}) = \gamma$ and $(\boldsymbol{\varphi}, \mathbf{y}) < \gamma$ for all $\mathbf{y} \in K$, $\mathbf{y} \neq \mathbf{p}$. The set of all exposed points of K will be denoted by $A(K)$; further $\mathbf{conv}\, K$ let be the convex hull of K and ∂K the boundary of K. For a set $M \subset E_k$, $\overline{M}$ is the closure of M in E_k. If K is a convex set then to each point of ∂K there is a support hyperplane which passes through this point. This fact is known in the case that K contains an interior point in E_k; if the dimension of K is less than k then the whole set K is contained in any hyperplane of the form $(\boldsymbol{\varphi}, \mathbf{x}) = \gamma$, $\boldsymbol{\varphi} \neq 0$. Evidently is $A(K) \subset \partial K$. STRASZEWICZ in [1] proved the following properties of the convex hull and the exposed points:
1. $\mathbf{conv}\, K = \mathbf{conv}\, A(K)$; 2. $A(K) = A(\mathbf{conv}\, K)$; 3. the minimal set (in the sense of inclusion) in the system of all compact sets with the property that their convex hull is $\mathbf{conv}\, K$ is the set $\overline{A(K)}$.

In this note we consider the linear control system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{1}$$

where $\mathbf{x} \in E_n$, $\mathbf{u} \in U \subset E_r$, $\mathbf{A}$ is an $n \times n$ matrix, $\mathbf{B}$ is an $n \times r$ matrix and the set $U \subset E_r$ is compact. We suppose that $T > 0$ is fixed.

The control $\mathbf{u}(t) : 0 \leqq t \leqq T$ will be called admissible if the function $\mathbf{u}(t)$ is measurable and $\mathbf{u}(t) \in U$ for almost all $t \in \langle 0, T \rangle$. The set of all admissible controls (with values in U for almost all $t \in \langle 0, T \rangle$) is denoted by $\Omega(U)$.

Definition. The control $u \in \Omega(U)$ will be called an extremal control corresponding to $\psi \in E_n$, $\psi \neq 0$ if

$$(2) \qquad (e^{-A'\tau}\psi, Bu(\tau)) = \max_{u \in U} (e^{-A'\tau}\psi, Bu)$$

holds for almost all $\tau \in \langle 0, T \rangle$ (A' is the transposed matrix to A).

Remark. Each $\psi \in E_n$, $\psi \neq 0$ determines at least one extremal control which corresponds to ψ; this control certainly need not be unique.

In the following we consider the set

$$\Lambda_T(U) = \left\{ y \in E_n,\ y = \int_0^T e^{-A\tau} Bu(\tau)\, d\tau,\ u \in \Omega(U) \right\}.$$

By means of the set $\Lambda_T(U)$ we can express the set $S_T(U)$ of all points in E_n that can be reached from the point x_0 in the time T with some control from $\Omega(U)$ in the following way:

$$S_T(U) = e^{AT}(x_0 + \Lambda_T(U)).$$

PROPERTIES OF THE SET $\Lambda_T(U)$

L. W. Neustadt in [2] proved the following

Proposition 1. *$\Lambda_T(U)$ is convex and compact.*

Let us now introduce

Proposition 2. *Let $y^* \in \Lambda_T(U)$ and let $(\psi, x) = \gamma$, $\psi \neq 0$ be a support hyperplane of $\Lambda_T(U)$ where $(\psi, y^*) = \gamma$. Then*

$$(3) \qquad y^* = \int_0^T e^{-A\tau} Bu^*(\tau)\, d\tau$$

where u^* is an extremal control corresponding to ψ.

If conversely y^* is given by (3) where u^* is an extremal control corresponding to any $\psi \in E_n$, $\psi \neq 0$ then y^* is a common point of the set $\Lambda_T(U)$ and the support hyperplane of $\Lambda_T(U)$ which is determined by ψ.

Proof. Since $y^* \in \Lambda_T(U)$ there is $y^* = \int_0^T e^{-A\tau} Bu^*(\tau)\, d\tau$ with $u^* \in \Omega(U)$. It holds

$$(\psi, y^*) = \int_0^T (e^{-A'\tau}\psi, Bu^*(\tau))\, d\tau = \gamma = \max_{x \in \Lambda_T(U)} (\psi, x).$$

If (2) is not fulfilled by $u^*(\tau)$ on any part of $\langle 0, T \rangle$ with positive measure then (ψ, y^*) cannot reach its maximal value γ in $\Lambda_T(U)$. Hence u^* must be an extremal control corresponding to ψ.

Conversely if $\boldsymbol{u}^*$ is an extremal control corresponding to $\boldsymbol{\psi}$ then $\boldsymbol{y}^*$ given by (3) is contained in $\Lambda_T(U)$. For an arbitrary $\boldsymbol{y} \in \Lambda_T(U)$ there is $\boldsymbol{y} = \int_0^T e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau$ where $\boldsymbol{u} \in \Omega(U)$ and $(e^{-\boldsymbol{A}'\tau}\boldsymbol{\psi}, \boldsymbol{B}\boldsymbol{u}(\tau)) \leqq (e^{-\boldsymbol{A}'\tau}\boldsymbol{\psi}, \boldsymbol{B}\boldsymbol{u}^*(\tau))$ holds for almost all $\tau \in \in \langle 0, T \rangle$. Hence

$$(\boldsymbol{\psi}, \boldsymbol{y}) = \int_0^T (e^{-\boldsymbol{A}'\tau}\boldsymbol{\psi}, \boldsymbol{B}\boldsymbol{u}(\tau))\,\mathrm{d}\tau \leqq \int_0^T (e^{-\boldsymbol{A}'\tau}\boldsymbol{\psi}, \boldsymbol{B}\boldsymbol{u}^*(\tau))\,\mathrm{d}\tau = (\boldsymbol{\psi}, \boldsymbol{y}^*) = \gamma$$

and therefore $\boldsymbol{y}^*$ is contained in the hyperplane $(\boldsymbol{\psi}, \boldsymbol{x}) = \gamma$ which supports the set $\Lambda_T(U)$.

Remark. J. Kurzweil in [3] (cf. Theorem 3 in [3]) proved similarly an analogous statement for the set of all points which can be transfered in to the origin in time less or equal T in the case of a convex set U which contains 0 as its interior point. As for each point of $\partial\Lambda_T(U)$ there is at least one support hyperplane of $\Lambda_T(U)$ passing through it, it is possible – by Proposition 2 – to express each point of $\partial\Lambda_T(U)$ in the form (3) where $\boldsymbol{u}^*$ is some extremal control.

We prove

Lemma 1. $\Lambda_T(U) = \Lambda_T(\mathbf{conv}\ U)$.

Proof. Evidently $\Lambda_T(U) \subset \Lambda_T(\mathbf{conv}\ U)$. The converse inclusion will be proved by contradiction. Let exist $\boldsymbol{y} \in \Lambda_T(\mathbf{conv}\ U)$ such that $\boldsymbol{y} \in \Lambda_T(U)$. By the strict separation theorem for a compact convex set and a closed set (see [4]) there is a $\boldsymbol{\psi} \in E_n$ such that $\gamma = \max_{\boldsymbol{x} \in \Lambda_T(U)} (\boldsymbol{\psi}, \boldsymbol{x}) < (\boldsymbol{\psi}, \boldsymbol{y})$; $(\boldsymbol{\psi}, \boldsymbol{x}) = \gamma$ is a support hyperplane of $\Lambda_T(U)$. We can write $\boldsymbol{y} = \int_0^T e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)\,\mathrm{d}\tau$ where $\boldsymbol{u} \in \Omega(\mathbf{conv}\ U)$. Further evidently $\max_{\boldsymbol{u} \in U} (e^{-\boldsymbol{A}'\tau}\boldsymbol{\psi}, \boldsymbol{B}\boldsymbol{u}) = \max_{\boldsymbol{u} \in \mathbf{conv} U} (e^{-\boldsymbol{A}'\tau}\boldsymbol{\psi}, \boldsymbol{B}\boldsymbol{u})$. We determine $\boldsymbol{u}^* \in \Omega(U)$ such that (2) is fulfilled and write $\boldsymbol{y}^* = \int_0^T e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}^*(\tau)\,\mathrm{d}\tau$. Hence $\boldsymbol{y}^* \in \Lambda_T(U)$ and $(\boldsymbol{\psi}, \boldsymbol{y}^*) = (\boldsymbol{\psi}, \boldsymbol{y}) > \gamma$. This contradiction gives $\Lambda_T(U) \supset \Lambda_T(\mathbf{conv}\ U)$.

From Lemma 1 $\Lambda_T(U) = \Lambda_T(U_1)$ follows for such U_1 that $\mathrm{conv}\, U_1 = \mathrm{conv}\, U$ holds. According to results of Straszewicz (see 3. page 141) the minimal compact set with this property is the set $\overline{A(U)}$ therefore $\Lambda_T(U) = \Lambda_T(\overline{A(U)})$ holds. Hence $S_T(U) = S_T(\overline{A(U)})$, too.

We have the following

Theorem. *A point which can be reached from the point $\boldsymbol{x}_0 \in E_n$ by any control $\boldsymbol{u} \in \Omega(U)$ in the time T can be reached by a control $\boldsymbol{u}^* \in \Omega(\overline{A(U)})$, too.*

Remark. This theorem is an analogon of the well known bang-bang principle of LaSalle (see J. P. LaSalle: The time optimal control problem, Contr. to the Theory of Nonlinear Oscillations, Vol. 5), Actually: if U is the unit cube $|u_i| \leqq 1$, $i = 1, \ldots, r$ then $\overline{A(U)} = V$ where V are the vertices of the cube U.

We suppose in the following that $\mathbf{B}\mathbf{u} = 0$ iff $\mathbf{u} = 0$. Under this condition the following propositions are known (see [5]):

Proposition 3. *For almost all $\psi \in E_n$ (in the sense of the n-dimensional Lebesgue measure) the extremal control corresponding to ψ is given uniquely almost everywhere in $\langle 0, T \rangle$.*

Evidently if $\mathbf{u}^*$ is an extremal control which is given uniquely almost everywhere in $\langle 0, T \rangle$ then $\mathbf{u}^* \in \Omega(\overline{A(U)})$ with respect to the property of $\mathbf{B}$. Further similarly as in [5] holds

Proposition 4. *Let the extremal control $\mathbf{u}^*$ corresponding to $\psi \in E_n$ be given uniquely almost everywhere in $\langle 0, T \rangle$ and let $\mathbf{y}^*$ be given by (3). Then $\mathbf{y}^* \in$ $\in A(\Lambda_T(U))$.*

and also the converse

Proposition 5. *If $\mathbf{y}^* \in A(\Lambda_T(U))$ then it is possible to write $\mathbf{y}^*$ in the form (3) where $\mathbf{u}^*$ is an extremal control which corresponds to some $\psi \in E_n$ and is uniquely determined almost everywhere in $\langle 0, T \rangle$.*

Remark. Proposition 4 holds even if the above condition for the matrix $\mathbf{B}$ is not fulfilled.

Since by the quoted results of [1] is $\Lambda_T(U) = \mathbf{conv}\, \Lambda_T(U) = \mathbf{conv}\, \overline{A(\Lambda_T(U))}$ we receive from Propositions 4 and 5 the following

Theorem. *The set $\Lambda_T(U)$ is the convex hull of the closure of all points $\mathbf{y}^*$ which can be written in the form (3), with an extremal control uniquely determined almost everywhere in $\langle 0, T \rangle$.*

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