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A FURTHER NOTE ON THE P.N.T. ERROR TERM

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Summary. It is shown that the author's method from the previous paper makes it possible to obtain much better estimate of the error term in the Prime Number Theorem.

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1. INTRODUCTION

In [1], Čížek proceeds from simple estimates of Riemann's zeta function, ζ, and by way of Fourier transform theory he obtains the class of error terms

$$\pi(x) - \operatorname{li} x = O(x \ln^{-n} x) \quad \text{as} \quad x \to \infty\,, \quad n \in \mathbb{Z}_+\,. \tag{1}$$

In [2] we note that the O-relation in (1) is not uniform with respect to n, but by examining the nature of the non-uniformity we deduce the improved result

$$\pi(x) - \operatorname{li} x = O\left[x \exp\left\{-\frac{1}{25}\frac{(\ln \ln x)^2}{\ln \ln \ln x}\right\}\right] \quad \text{as} \quad x \to \infty\,. \tag{2}$$

The purpose of this note is to show that our method easily yields a much better error term than that in (2).

2. THE ERROR TERM

In [2] (14) we quote the underlying Fourier transform theorem in the form

$$|a| \geqq d_r \Rightarrow |\hat{f}(a)| \leqq c_r |a|^{-r} \quad (r = 1, 2, \ldots) \tag{3}$$

where f is a function involving ζ. It emerges that we can choose $d_r = 2$, $c_r = K(Cr)^{12r}$ for some $K > 0$ and $C > 9$. The procedure in [2] was to make the positive integer r depend on a, say $r = \varrho(|a|)$. Then (3) yields

$$|a| \geqq 2 \Rightarrow |\hat{f}(a)| \leqq c_{\varrho(|a|)} |a|^{-\varrho(|a|)}\,, \tag{4}$$

where upon we choose the function ϱ to give an estimate (4) which leads to (2). If ϱ were able to take any real value in $[1, \infty)$, the expression

$$K(C\varrho(b))^{12\varrho(b)}\, b^{-\varrho(b)}$$

would be minimised for every b if we put $\varrho(b) = (Ce)^{-1}\, b^{1/12} = \varrho_1(b)$, say. The integer part of ϱ_1 will legitimately serve as ϱ in (4) and we have

$$(5)\qquad |\hat{f}(a)| \leqq c_{[\varrho_1(|a|)]}|a|^{-[\varrho_1(|a|)]} = A\,, \quad \text{say}\,,$$

$$\leqq c_{[\varrho_1(a)]}\,|a|^{-[\varrho_1(|a|)]} \leqq c_{\varrho_1(|a|)-1}|a|^{1-\varrho_1(|a|)} =$$

$$= K\{C(\varrho_1(|a|) - 1)\}^{12(\varrho_1(|a|)-1}\,|a|^{1-\varrho_1(|a|)} =$$

$$= K\exp\left[12(\varrho_1(|a|) - 1)\ln\{C(\varrho_1(|a|) - 1)\} + (1 - \varrho_1(|a|)\ln|a|\right] =$$

$$= K\exp\left[12\left(\frac{|a|^{1/12}}{Ce} - 1\right)\ln\left\{C\left(\frac{|a|^{1/12}}{Ce} - 1\right)\right\} + \right.$$

$$\left. + \left(1 - \frac{|a|^{1/12}}{Ce}\right)\ln|a|\right] =$$

$$(6)\qquad = B\,, \quad \text{say}\,.$$

To assess the last expression we first note that A in (5) is $K\exp\{-12|a|^{1/12}/(Ce)\}$. We could reasonably expect B in (6) to be about the same as A for large $|a|$ and indeed,

$$\frac{B}{A} = \exp\left[12\frac{|a|^{1/12}}{Ce}\left\{\ln\left(C\left(\frac{|a|^{1/12}}{Ce} - 1\right)\right) - \ln\left(C\frac{|a|^{1/12}}{Ce}\right)\right\} - \right.$$

$$\left. - 12\ln\left\{C\left(\frac{|a|^{1/12}}{Ce} - 1\right)\right\} + \ln|a|\right].$$

Using the inequalities

$$-x^{-1} \geqq \ln(C(x - 1)) - \ln Cx \geqq -(x - 1)^{-1}$$

we obtain

$$\frac{B}{A} \leqq \exp\left[12\frac{|a|^{1/12}}{Ce}\left\{-\frac{Ce}{|a|^{1/12}}\right\} - 12\ln\left(C\frac{|a|^{1/12}}{Ce}\right) + \right.$$

$$\left. + 12\left(\frac{|a|^{1/12}}{Ce} - 1\right)^{-1} + \ln|a|\right] = \exp\left[12\left(\frac{|a|^{1/12}}{Ce} - 1\right)^{-1}\right].$$

If we just write $B/A = 1 + o(1)$ as $|a| \to \infty$, (6) becomes

$$|\hat{f}(a)| \leqq K\exp\left(-\frac{12}{Ce}|a|^{1/12}\right)\{1 + o(1)\} = O\{\exp(-L|a|^{1/12})\} \quad \text{as} \quad |a| \to \infty$$

for some $L > 0$, and it follows as in [1] and [2] that

$$(7)\qquad \pi(x) - \operatorname{li} x = O\{x\exp(-L\ln^{1/12} x)\} \quad \text{as} \quad x \to \infty\,.$$

(7), unlike (2), is an error term of a familiar type, normally obtained by consideration of zero-free regions in the critical strip — or by "elementary" methods.

Finally we remark that if we have

$$\int_3^\infty \left|\frac{\mathrm{d}^r}{\mathrm{d}t^r}\left\{\frac{\zeta'}{\zeta}(1 + it)(1 + it)^{-2}\right\}\right| \mathrm{d}t \leqq K(Cr)^{Gr}$$

for some $G \geqq 1$, our method implies that

$$\pi(x) - \operatorname{li} x = O\{x \exp(-L \ln^{1/G} x)\} .$$

References

[1] *J. Čížek:* On the Proof of the Prime Number Theorem. Časopis pěst. mat. *106* (1981) 395—401.

[2] *A. Grant:* Fourier Transforms and the P.N.T. Error Term. Časopis pěst. mat. 337—347.

Souhrn

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ЕЩЁ ОДНО ЗАМЕЧАНИЕ ОБ ОСТАТКЕ В ТЕОРЕМЕ О ПРОСТЫХ ЧИСЛАХ

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