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FOURIER TRANSFORMS AND THE P.N.T. ERROR TERM

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Summary. This paper shows that under suitable conditions the property $f(x) = O(x^{-n})$ as $x \to \infty$ for every n implies $f(x) = O\,[\exp\{-\varphi(x) \ln x\}]$ for some φ such that $\varphi(x) \to \infty$ with $x \to \infty$. As an application it is shown that from $\pi(x) - \operatorname{li} x = O(x \ln^{-n} x)$ as $x \to \infty$ for every n we can derive $\pi(x) - \operatorname{li} x = O\left[x \exp\left\{-\frac{1}{25} \frac{(\ln \ln x)^2}{\ln \ln \ln x}\right\}\right]$ as $x \to \infty$.

Keywords: Prime Number Theorem, Error term of Prime Number Theorem, Fourier Transform.

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1. INTRODUCTION

J. Čížek [2] has shown that the Prime Number Theorem error term can be written in terms of a Fourier transform and that a well-known theorem yields the form

$$(*) \qquad \pi(x) - \operatorname{li} x = O(x \ln^{-n} x) \quad \text{as} \quad x \to \infty \quad \text{for any} \quad n \in \mathbb{Z}_+ .$$

He conjectures that no better error term can be obtained by the same method. It is shown here that an improvement on the type of error term in $(*)$ is obtainable in quite general circumstances and that using only those properties of ζ employed by Čížek we can obtain

$$\pi(x) - \operatorname{li} x = O\left[x \exp\left\{-\frac{1}{25} \frac{(\ln \ln x)^2}{\ln \ln \ln x}\right\}\right] \quad \text{as} \quad x \to \infty .$$

2. THE ASYMPTOTIC BEHAVIOUR OF FOURIER TRANSFORMS

We aim to improve on Lemma 1 in [2]. It is convenient to reformulate it in terms of the families of functions defined below:

Definition 1. *For each $n \in \mathbb{Z}_+^*$ let $\mathcal{A}_n$ denote the set of all functions $f\colon \mathbf{R} \to \mathbf{C}$ such that*

(i) $f, f', \ldots, f^{(n)}$ *are all in* $\mathscr{L}_1(R)$ *and*

(ii) $f', f'', \ldots, f^{(n-1)}$ *are all absolutely continuous on* R (*i.e. they are all absolutely continuous on every closed interval*).

Definition 2. *Let* $\mathscr{A} = \bigcap_{n=1}^{\infty} \mathscr{A}_n$.

Remark. $\mathscr{A}, \mathscr{A}_1, \mathscr{A}_2, \ldots$ are distinct and non-empty. For example if $a \in \mathscr{L}_1(R)$ is a discontinuous step function, the convolution $a*a* \ldots *a$ with n factors is in $\mathscr{A}_{n-1} - \mathscr{A}_n$; and $\exp(-x^2)$ is in $\mathscr{A}$.

Definition 3. *Let* $\varphi: R_+ \to R_+$ *be monotonic non-decreasing with* $\varphi(t) = 0$ *for* $t \in [0, 1)$. *Then we denote by* $\mathscr{B}_\varphi$ *the set of all functions* $g: R \to C$ *such that*

$$g(t) = O[\exp\{-\varphi(|t|) \ln |t|\}] \quad \textit{as} \quad t \to \pm\infty .$$

We can now write Lemma 1 in [2] as

$$f \in \mathscr{A}_n \Rightarrow \hat{f} \in \mathscr{B}_n .$$

Here $\hat{f}$ means the Fourier transform $\hat{f}: R \to C$ given by $\hat{f}(y) = \int_{-\infty}^{\infty} e(-ty) f(t) \, dt$ ($e(a) = \exp 2\pi i a$) and on the right hand side n denotes the constant function $n(u) = n$. In proving Theorem 1 in [2] Čížek uses this in the form

$$f \in \mathscr{A} \Rightarrow \hat{f} \in \bigcap_{n=1}^{\infty} \mathscr{B}_n . \tag{1}$$

We might hope to improve on this by finding a φ as in Definition 3 with the further property

$$\lim_{u \to +\infty} \varphi(u) = +\infty , \tag{2}$$

and such that

$$f \in \mathscr{A} \Rightarrow \hat{f} \in \mathscr{B}_\varphi . \tag{3}$$

Unfortunately, such a function does not exist, since we have

Theorem 1. *For every* φ *as in definition* 3 *and satisfying* (2) *there exists a function* $f \in \mathscr{A}$ *such that* $\hat{f} \notin \mathscr{B}_\varphi$.

Proof. Let φ be as required by the theorem. We define the functions G, T, g and f, all with domain R, as follows:

$$G(y) = \begin{cases} \exp\{-\frac{1}{2}\varphi(|y| - \frac{1}{2}) \ln(|y| - \frac{1}{2})\} & (|y| \geqq 1), \\ 0 & (|y| < 1). \end{cases} \tag{4}$$

$$(5) \qquad T(y) = \begin{cases} 0 & (|y| \geqq 1), \\ \exp\left(\dfrac{1}{y^2 - 1}\right) & (|y| < 1). \end{cases}$$

$$(6) \qquad g = G * T.$$

$$(7) \qquad f(y) = \hat{g}(-y).$$

The following properties follow from definitions (4) to (7):

$$(8) \qquad G(-y) = G(y), \quad G \in \mathscr{L}_1(\mathbb{R}), \quad G \searrow \text{ on } [\tfrac{3}{2}, \infty), \quad \hat{G}(y) = o(1) \text{ as } y \to \pm\infty,$$

$$(9) \qquad T \in \mathscr{L}_1(\mathbb{R}), \quad T \in C^\infty(\mathbb{R}), \quad T \searrow \text{ on } [0, 1],$$

$$(10) \qquad f(y) = \hat{G}(-y)\,\hat{T}(-y).$$

First we show that $f \in \mathscr{A}$. Differentiating n times under the integral sign ($n \geqq 0$) we have

$$(11) \qquad [\hat{G}(-y)]^{(n)} = \left[\int_{-\infty}^{\infty} e(ty)\,G(t)\,\mathrm{d}t\right]^{(n)} = \int_{-\infty}^{\infty} e(ty)\,(2\pi i t)^n\,G(t)\,\mathrm{d}t$$

and

$$(12) \qquad [\hat{T}(-y)]^{(n)} = \int_{-\infty}^{\infty} e(ty)\,(2\pi i t)^n\,T(t)\,\mathrm{d}t.$$

(4) and (5) imply that $t^n\,G(t)$ and $t^n\,T(t)$ both belong to $\mathscr{L}_1(\mathbb{R})$ and that $t^n\,T(t)$ is continuous, so the integrals in (11) and (12) all exist. Hence the basic properties of Fourier transforms give us

$$[\hat{G}(-y)]^{(n)} = o(1) \quad \text{as} \quad y \to \pm\infty$$

and

$$[\hat{T}(-y)]^{(n)} \in \mathscr{L}_1(\mathbb{R}).$$

When we differentiate (10) N times we deduce that $f^{(N)}$ exists for $N = 0, 1, 2, \ldots$ with

$$(13) \qquad f^{(N)}(y) = \sum_{n=0}^{N} \binom{N}{n} [\hat{G}(-y)]^{(n)}\,[\hat{T}(-y)]^{(N-n)} \in \mathscr{L}_1(\mathbb{R}).$$

It follows from (13) that $f \in \mathscr{A}$. Theorem 1 will be proved if we show that $\hat{f} \notin \mathscr{B}_\varphi$. From (8) and (9) both G and T belong to $\mathscr{L}_1(\mathbb{R})$ and hence so does their convolution g. Therefore we can apply the Fourier inversion theorem to (7) and obtain

$$\begin{aligned} \hat{f}(y) &= g(y) \\ &= (G * T)(y) \\ &= \int_{-\infty}^{\infty} G(y - t)\,T(t)\,\mathrm{d}t \end{aligned}$$

$$= \int_{-1}^{1} G(y - t)\, T(t)\, \mathrm{d}t$$

$$\geqq \int_{-1/2}^{1/2} G(y - t)\, T(t)\, \mathrm{d}t \quad \text{because } G \text{ and } T \text{ are non-negative}$$

$$\geqq \int_{-1/2}^{1/2} G(|y| + \tfrac{1}{2})\, T(\tfrac{1}{2})\, \mathrm{d}t \quad \text{if} \quad |y| > 2$$

by the monotonicity properties in (8) and (9)

$$= \exp(-\tfrac{4}{3})\, G(|y| + \tfrac{1}{2})$$
$$= \exp(-\tfrac{4}{3}) \exp\{-\tfrac{1}{2}\varphi(|y|) \ln |y|\}\,.$$

It follows that $\hat{f} \notin \mathscr{B}_\varphi$ and Theorem 1 is proved. □

Although we have ruled out a theorem which asserts (3), we can at least prove the following which is our main result.

Theorem 2. *For every $f \in \mathscr{A}$ there exists a φ as in Definition* 3 *and satisfying* (2) *such that $\hat{f} \in \mathscr{B}_\varphi$.*

Proof. We have (1), from Čížek's Lemma 1, whence given $f \in \mathscr{A}$ there exist sequences (c_r) and (d_r) $(r = 1, 2, \ldots)$ of positive numbers such that

$$|a| \geqq d_r \Rightarrow |\hat{f}(a)| \leqq c_r |a|^{-r} \quad (r = 1, 2, \ldots)\,. \tag{14}$$

We shall suppose that (c_r) and (d_r) are both non-decreasing and that $c_r = d_r$. We are certainly free so to choose them. We distinguish two cases:

(i) (c_r) bounded as $r \to \infty$ and

(ii) (c_r) unbounded.

Case (i). Let C be an upper bound of the sequence (c_r) with $C > 1$. Then (14) gives

$$|a| \geqq C \Rightarrow |\hat{f}(a)| \leqq C |a|^{-r} \quad (r = 1, 2, \ldots)\,,$$

so $\hat{f}$ is zero outside the interval $(-C, C)$, hence

$$\hat{f} \in \mathscr{B}_\varphi$$

for every φ.

Case (ii). Define $\varphi\colon R_+ \to R_+$ by

$$\varphi(t) = \max\{r \in Z\colon c_r \leqq t\}\,, \quad \text{taking} \quad c_0 = 0\,. \tag{15}$$

This φ satisfies (2). For any a such that $|a| \geqq c_1$ we let $r = \varphi(|a|)$ and get from (14) and (15) that

$$|\hat{f}(a)| \leqq c_r |a|^{-r} \leqq |a|\, |a|^{-r} = |a|^{1-\varphi(|a|)}$$
$$= \exp[\{1 - \varphi(|a|)\} \ln |a|]$$

$$\doteq O[\exp\{-(1-\varepsilon)\,\varphi(|a|)\ln|a|\}] \quad \text{as} \quad a \to \pm\infty$$

for any $\varepsilon > 0$, i.e.

$$\hat{f} \in \mathscr{B}_{(1-\varepsilon)\varphi} \quad \text{for any} \quad \varepsilon \in (0, 1]\,. \qquad \square$$

Our proof of Theorem 2 is a pure existence proof but will nevertheless help us in constructing an error term for the Prime Number Theorem. Obviously the theorems also hold for other definitions of the Fourier transform such as

$$\hat{f}(y) = \alpha \int_{-\infty}^{\infty} \exp(-ity) f(t)\, dt \quad (\alpha = 1 \text{ or } (2\pi)^{-1/2})\,.$$

Čížek deals with the function $f\colon \mathbb{R} \to \mathbb{C}$ given by

$$f(t) = (1 + it)^{-2} \left[\frac{\zeta'}{\zeta}(1 + it) + (it)^{-1}\right] (f(1) = f(1+))\,.$$

Applying our Theorem 2 to this function we shall obtain

Corollary. *There is a function* φ *as in Definition* 3 *and satisfying* (2) *such that*

$$g(x) = \sum_{n \leq x} \Lambda(n) \ln \frac{x}{n} = O[x \exp\{-\varphi(\ln x) \ln\ln x\}] \quad \text{as} \quad x \to \infty\,.$$

(cf. [2], p. 396, Theorem 1.)

Proof of Corollary. In this proof we follow Čížek and define the Fourier transform by

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(t) \exp(-ity)\, dt\,.$$

In proving his Theorem 1, Čížek shows that $f \in \mathscr{A}$ and deduces that

$$\hat{f}(y) = (-iy)^{-m} \int_{-\infty}^{\infty} f^{(m)}(t) \exp(-ity)\, dt$$

for every $m \in \mathbb{Z}_+$. So for the sequence (c_r) in the proof of our Theorem 2 we can take

$$c_r \geqq \max\left\{1, \int_{-\infty}^{\infty} |f^{(r)}(t)|\, dt\right\}, \quad r = 1, 2, 3, \ldots,$$

with (c_r) non-decreasing. By Theorem 1 there is a $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, non-decreasing and unbounded, such that

(17) $$\hat{f}(y) = O[\exp\{-\varphi(y)\ln y\}] \quad \text{as} \quad y \to \infty\,.$$

Hence, as in the proof of Čížek's Theorem 1,

(18) $$g(x) = O[x \exp\{-\varphi(\ln x)\ln\ln x\}] \quad \text{as} \quad x \to \infty\,. \qquad \square$$

In the next section we find a φ for which (18) holds.

We shall need two lemmas:

Lemma 1. *Let $A \in \mathbb{R}_+^*$ be fixed and let f be any function analytic on $[-A, A]$. Then there is a number $\beta > 0$ such that*

$$\int_{-A}^{A} |f^{(r)}(t)|\,\mathrm{d}t = O(r!\,\beta^r) \quad \text{as} \quad r \to \infty\,, \tag{19}$$

where r takes the values $1, 2, 3, \ldots$.

Remark. This lemma does not hold for every C^∞ function on $[-A, A]$. See for example [5] p. 418 ex. 13, which shows how to construct a C^∞ function for which (19) does not hold.

Proof of Lemma 1. Our proof is similar to that of Mandelbrojt ([4] p. 49) who is proving a related theorem.

For every $a \in [-A, A]$ there is an open disc $B(a; \varrho) = \{s \in C;\ |s - a| < \varrho\}$ in which f is analytic. If we choose one such disc for every a, they form an open covering of the interval $[-A, A]$, which is compact in C. Hence there is a finite subcover comprising, say, the open discs

$$B(a_1; \varrho_1), \ldots, B(a_\lambda; \varrho_\lambda)\,,$$

with $a_0 = -A \leqq a_1 \leqq \ldots \leqq a_\lambda \leqq A = a_{\lambda+1}$. The boundary of the union of these discs comprises arcs of a finite number of circles and does not intersect $[-A, A]$. Hence it is the image of a closed loop $\gamma\colon [0, 1] \to C$. γ is continuous, hence so are $\operatorname{Re}\gamma$, $\operatorname{Im}\gamma$, $|\gamma - A|$ and $|\gamma + A|$. The distance from a point of im γ to the nearest point of $[-A, A]$ is given by the function $\delta\colon [0, 1] \to \mathbb{R}_+^*$ with

$$\delta(u) = \begin{cases} |\gamma(u) + A| & \text{if} \quad \operatorname{Re}\gamma(u) < -A\,, \\ |\operatorname{Im}\gamma(u)| & \text{if} \quad -A \leqq \operatorname{Re}\gamma(u) \leqq A\,, \\ |\gamma(u) - A| & \text{if} \quad \operatorname{Re}\gamma(u) > A\,. \end{cases}$$

δ is continuous and since $[0, 1]$ is compact, δ has a positive minimum, say $\delta(u) \geqq \varepsilon$. There is therefore a loop Γ with $[-A, A]$ in its interior and with f analytic on im Γ. For example we take im Γ to be the closed curve which is everywhere distant $\frac{1}{2}\varepsilon$ from the nearest point of $[-A, A]$. We integrate around Γ to get

$$a \in [-A, A] \Rightarrow f^{(r)}(a) = \frac{r!}{2\pi\mathrm{i}} \int_\Gamma \frac{f(w)}{(w - a)^{r+1}}\,\mathrm{d}w$$

$$\Rightarrow |f^{(r)}(a)| \leqq \frac{(4A + \varepsilon\pi)\,r!}{2\pi} M(\tfrac{1}{2}\varepsilon)^{-r-1}\,,$$

where $4A + \varepsilon\pi$ is the length of im Γ, $\frac{1}{2}\varepsilon$ is $\inf |w - a|$ for $w \in \operatorname{im}\Gamma$, and we have defined

$$M = \sup_{w \in \operatorname{im} \Gamma} |f(w)| .$$

Hence

$$\int_{-A}^{A} |f^{(r)}(t)| \, dt \leqq \frac{2A(4A + \varepsilon\pi)}{2\pi} r! \; M(\tfrac{1}{2}\varepsilon)^{-r-1} = Br! \; \beta^r , \quad \text{say} ,$$

i.e.

$$\int_{-A}^{A} |f^{(r)}(t)| \, dt = O(r! \; \beta^r) \quad \text{as} \quad r \to \infty . \qquad \square$$

Lemma 2. *Let* T_n; $n = 0, 1, 2, \ldots$ *be the Bell numbers defined by* $T_0 = 1$, $T_n = \sum_{k=1}^{n} S_k^n (n \geqq 1)$, *where* S_k^n *are Stirling numbers of the second kind* ($S_1^n = S_n^n = 1$, $S_k^n = kS_k^{n-1} + S_{k-1}^{n-1}$; $k = 2, 3, \ldots, n-1$; $n \geqq 3$ – see [6] p. 230 ex 6.94). *Then we have* $T_n < n!$ *for all* n *greater than some number* N.

Proof. A well-known generating function is

$$\exp(e^z - 1) = \sum_{n=0}^{\infty} \frac{T_n}{n!} z^n .$$

See for exampe [1] p. 216. As the left-hand side is an entire function, Hadamard's formula for the radius of convergence of a Taylor series gives

$$\sqrt[n]{\frac{T_n}{n!}} \to 0 \quad \text{as} \quad n \to \infty .$$

It follows that $T_n/n! < 1$ for all large enough n. $\square$

We cannot extend Lemma 1 directly to the interval in (16) because the lemma does not in general hold uniformly for $A \in [A_0, +\infty)$, but we shall use the lemma in conjunction with a separate estimate of $\int_A^{\infty} |f^{(r)}(t)| \, dt$. The fact that for our particular function f

$$f(-t) = \overline{f(t)}$$

implies that $|f^{(r)}(-t)| = |f^{(r)}(t)|$ for $r = 0, 1, 2, \ldots$, so we do not need to consider $\int_{-\infty}^{-A} |f^{(r)}(t)| \, dt$ separately. We shall take $A = 3$ and first deal with the second term on the right-hand side of the following inequality:

$$(20) \qquad \int_3^{\infty} |f^{(r)}(t)| \, dt \leqq \int_3^{\infty} \left| \frac{d^r}{dt^r} \left\{ \frac{\zeta'}{\zeta} (1 + it)(1 + it)^{-2} \right\} \right| dt + \int_3^{\infty} \left| \frac{d^r}{dt^r} \{ (it)^{-1} (1 + it)^{-2} \} \right| dt .$$

We have

$$(21)\qquad \int_3^\infty \left| \frac{\mathrm{d}^r}{\mathrm{d}t^r} \{(\mathrm{i}t)^{-1} (1 + \mathrm{i}t)^{-2}\} \right| \mathrm{d}t =$$

$$= \int_3^\infty \left| \sum_{v=0}^{r} \binom{r}{v} \frac{\mathrm{d}^v}{\mathrm{d}t^v} (\mathrm{i}t)^{-1} \frac{\mathrm{d}^{r-v}}{\mathrm{d}t^{r-v}} (1 + \mathrm{i}t)^{-2} \right| \mathrm{d}t =$$

$$= \int_3^\infty \left| \sum_{v=0}^{r} \binom{r}{v} \mathrm{i}^{-1}(-1)^r v!\, t^{-v-1} \mathrm{i}^{r-v} (r - v + 1)!\, (1 + \mathrm{i}t)^{-r+v-2} \right| \mathrm{d}t \leqq$$

$$\leqq \int_3^\infty \sum_{v=0}^{r} \binom{r}{v} v!\, (r - v + 1)!\, t^{-r-3}\, \mathrm{d}t \leqq (r + 2)! \int_3^\infty t^{-r-3}\, \mathrm{d}t = (r + 1)!\, 3^{-r-2},$$

which is similar in magnitude to the integral in Lemma 1. Finally we apply a similar method to the first term on the right-hand side of (20). We write

$$\left(\frac{\zeta'}{\zeta}\right)^{(k-1)} = (\ln \circ \zeta)^{(k)}; \quad k = 1, 2, 3, \ldots.$$

It is easily shown by induction that

$$(22)\qquad (\ln \circ \zeta)^{(k)} = \sum_{j=1}^{k} S_j^k F_{k+1-j} \ln^{(j)} \circ \zeta,$$

where S_j^k are Stirling numbers of the second kind and the function F_n is a weighted average of products of the type

$$\zeta^{(\alpha)} \zeta^{(\beta)} \ldots \zeta^{(\nu)}$$

where $\alpha + \beta + \ldots + \nu = k$ and $\max(\alpha, \beta, \ldots, \nu) \leqq n$. We use the same estimates as Čížek in [2] p. 396, namely

$$(23)\qquad \left.\begin{aligned} &|\zeta^{(k)}(s)| \leqq p_k \ln^{k+1} t \quad \text{and} \\ &\left|\frac{1}{\zeta(s)}\right| \leqq q \ln^7 t \end{aligned}\right\} \quad (s = 1 + \mathrm{i}t,\ t \geqq 3)$$

for some constants $q, p_1, p_2, \ldots$. Ingham's method of deriving the first inequality in (23) (see [3] p. 28), using Cauchy's integral formula, shows that we may take $p_k = k!\, p_1$. We substitute (23) into (22) to obtain

$$|\{\ln \circ \zeta(s)\}^{(k)}| \leqq T_k p_1^k k! \ln^{2k} t |\ln^{(K)} \circ \zeta(s)|,$$

where K is that value of j which gives the largest value of the last factor on the right hand side, and T_k is the kth Bell number.

We also have $\ln^{(k)} \circ \zeta = (-1)^{k-1} (k - 1)!\, \zeta^{-k}$, so

$$|\ln^{(K)} \circ \zeta(s)| \leqq (K - 1)!\, q \ln^{7K} t \leqq (k - 1)!\, q \ln^{7k} t.$$

Thus we get

$$(24)\qquad \begin{aligned} |\{\ln \circ \zeta(s)\}^{(k)}| &\leqq T_k k!\, p_1^k \ln^{2k} t (k - 1)!\, q \ln^{7k} t \leqq \\ &\leqq r p_1^k q (k!)^2 (k - 1)! \ln^{9k} t \quad \text{for some } r > 0 \end{aligned}$$

by Lemma 2, where r does not depend on k. Now we return to (20) and estimate

$$[\{(\ln \circ \zeta)'(1+it)\}(1+it)^{-2}]^{(n)} =$$

$$= \sum_{k=0}^{n} \binom{n}{k} i^k (\ln \circ \zeta)^{(k+1)}(1+it)(-i)^{n-k}(n-k+1)!(1+it)^{-n+k-2}.$$

We have, using (24),

$$|[\{(\ln \circ \zeta)'(1+it)\}(1+it)^{-2}]^{(n)}| \leqq$$

$$\leqq \sum_{k=0}^{n} \binom{n}{k} p_1^{k+1} qr\{(k+1)!\}^2 k! \ln^{9k+9} t (n-k+1)! |(1+it)^{-n+k-2}| \leqq$$

$$\leqq \sum_{k=0}^{n} \binom{n}{k} p_1^{k+1} qr\{(k+1)!\}^2 k! \ln^{9k+9} t (n-k+1)! \, t^{-n+k-2}.$$

$$(t \geqq 3)$$

So

$$\int_3^\infty |[\{(\ln \circ \zeta)'(1+it)\}(1+it)^{-2}]^{(n)}| \, dt \leqq$$

$$\leqq \sum_{k=0}^{n} \binom{n}{k} p_1^{k+1} qr\{(k+1)!\}^2 k! (n-k+1)! \int_3^\infty \ln^{9k+9} t \, t^{-n+k-2} \, dt.$$

Because

$$\int_3^\infty (\ln^{9k+9} t) \, t^{-n+k-2} \, dt \leqq \int_1^\infty (\ln^{9k+9} t) \, t^{-n+k-2} \, dt \leqq$$

$$\leqq \int_1^\infty (\ln^{9k+9} t) \, t^{-2} \, dt = (9k+9)!, \quad 0 \leqq k \leqq n,$$

we have

$$\int_3^\infty |[\{(\ln \circ \zeta)'(1+it)\}(1+it)^{-2}]^{(n)}| \, dt \leqq$$

$$\leqq \sum_{k=0}^{n} \binom{n}{k} p_1^{k+1} qr\{(k+1)!\}^2 k! (n-k+1)! \int_3^\infty (\ln^{9k+9} t) \, t^{-n+k-2} \, dt =$$

$$= qrn! \sum_{k=0}^{n} p_1^{k+1} \{(k+1)!\}^2 (n-k+1)(9k+9)! \leqq$$

$$\leqq (n+2) p_1^{n+1} qr\{(n+1)!\}^3 (9n+9)! \quad (\text{assuming that } p_1 \geqq 1) =$$

$$= O\{(Cn)^{12n}\} \quad \text{as} \quad n \to \infty, \tag{25}$$

which is enough to absorb the other terms (21) and (19). So in (16) we can choose $c_r = K(Cr)^{12r}$ for some constants $C > 9$ and K. We have case (ii) in the proof of theorem 2 and we look for a function φ^* such that $\varphi^*(t) \leqq \varphi(t)$ for all large enough t, with φ defined by (15) and φ^* having an elementary form. It is not difficult to see that a suitable expression is

$$\varphi^*(y) = \frac{2}{25} \frac{\ln y}{\ln \ln y}. \tag{26}$$

(17) and (18) are therefore satisfied for $\varphi = (1 - \varepsilon)\,\varphi^*$.

At this stage we can make use of Walfisz's Lemmas 14–16 in V. 3 [7]. For this purpose we need to verify that our function $\varphi(\ln x) \ln \ln x$ (see (18)) has the same properties as Walfisz's $2\,\omega(x)$, so we define

$$\omega(x) = \tfrac{1}{2}(1 - \varepsilon)\,\varphi^*(\ln x) \ln \ln x = (1 - \varepsilon) \frac{1}{25} \frac{(\ln \ln x)^2}{\ln \ln \ln x}$$

and note that for all $x \geqq \exp \exp e$ we have both

(a) $0 < \omega(x) \leqq \dfrac{1}{48} \ln x$

and (b) $\dfrac{1}{48} \ln x - \omega(x)$ is increasing.

We then obtain from (18) our

Theorem 3.

$$\pi(x) - \operatorname{li} x = O\left[x \exp\left\{-\frac{1}{25} \frac{(\ln \ln x)^2}{\ln \ln \ln x}\right\}\right] \quad \text{as} \quad x \to \infty. \qquad \square$$

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Souhrn

FOURIEROVA TRANSFORMACE A ZBYTEK V PRVOČÍSELNÉ VĚTĚ

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V práci je dokázáno, že za jistých podmínek z vlastnosti $f(x) = O(x^{-n})$ pro $x \to \infty$ pro všechna n plyne $f(x) = O[\exp g - \varphi(x) \ln x\}]$ pro jisté φ, $\varphi(x) \to \infty$ pro $x \to \infty$. Jako aplikace je ukázáno, že ze vztahu $\pi(x) - \operatorname{li} x = O(x \ln^{-n} x)$ pro $x \to \infty$ pro všechna n plyne $\pi(x) - \operatorname{li} x = O\left[x \exp\left\{-\frac{1}{25} \frac{(\ln \ln x)^2}{\ln \ln \ln x}\right\}\right]$ pro $x \to \infty$.

Резюме

ПРЕОБРАЗОВАНИЕ ФУРЬЕ И ОСТАТОК В ТЕОРЕМЕ О ПРОСТЫХ ЧИСЛАХ

ANDREW GRANT

В статье доказано, что при некоторых условиях из свойства $f(x) = O(x^{-n})$ при $x \to \infty$ для всех n следует $f(x) = O[\exp\{g - \varphi(x) \ln x\}]$ для некоторой функции φ со свойством $\varphi(x) \to \infty$ при $x \to \infty$. В качестве приложения этого результата показано, что из соотношения $\pi(x) - \operatorname{li} x = O(x \ln^{-n} x)$ при $x \to \infty$ для всех n следует $\pi(x) - \operatorname{li} x = O\left[x \exp\left\{-\frac{1}{25} \frac{(\ln \ln x)^2}{\ln \ln \ln x}\right\}\right]$ при $x \to \infty$.

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