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ON IDEMPOTENT FILTERS

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In [3], the problem of the existence of idempotent filters was posed, i.e. filters $\mathscr{F}$ isomorphic to the product $\mathscr{F} \cdot \mathscr{F}$. In what follows, a very simple existence proof is given, and a rather complicated construction is described.

1.1. We use the standard terminology and notation, with slight modifications. An ordered pair consisting of x and y will be denoted by $\langle x, y \rangle$. If M is a set, we put $\exp M = \{X : X \subset M\}$, $\mathrm{e}M = \{X : X \subset M, X \text{ is finite}\}$. Letters k, m, n, p, q stand for natural numbers, letters ϑ, δ (possibly with subscripts, etc.) for a natural number or for the ordinal ω. Sequence (on a set M) means a finite or an infinite sequence (of elements of M). A finite sequence will be called *a string* or *a word*. The void string will be denoted by $\emptyset$. The concatenation $\xi \cdot \eta$ of two sequences ξ, η is defined if ξ is finite; in addition, for formal reasons, we put $\xi \cdot \emptyset = \xi$ for any sequence ξ. Given a set M, the set of all strings on M will be denoted by $\mathrm{w}M$.

1.2. Let M, S be classes. If a binary operation $\sigma : D \to S$, where $D \subset M \times M$, is given, we introduce the following binary operations σ' on $\exp M$ and σ'' on $\exp \exp M$. If $X, Y \in \exp M$, then $\sigma'\langle X, Y \rangle = \{\sigma\langle x, y \rangle : x \in X,\ y \in Y\}$; if $\mathscr{X}, \mathscr{Y} \in \exp \exp M$, then $\sigma''\langle \mathscr{X}, \mathscr{Y} \rangle = \{\bigcup(\sigma'\langle\{x\}, fx\rangle : x \in X) : X \in \mathscr{X}, f \in \mathscr{Y}^X\}$.

1.3. The operations just introduced will be used below in two cases: (1) M is a class of sequences and $\sigma\langle \xi, \eta \rangle = \xi \cdot \eta$ is the concatenation; in this case, we shall often write $X \cdot Y$ instead of $\sigma'\langle X, Y \rangle$ and $\mathscr{X} \odot \mathscr{Y}$ instead of $\sigma''\langle \mathscr{X}, \mathscr{Y} \rangle$; (2) M is the universal class and $\sigma\langle x, y \rangle = \langle x, y \rangle$; in this case the standard notation, $X \times Y$, will be used for $\sigma'\langle X, Y \rangle$, and $\sigma''\langle \mathscr{X}, \mathscr{Y} \rangle$ will be denoted by $\mathscr{X} \otimes \mathscr{Y}$.

1.4. In the case (2) just mentioned, $\mathscr{F} \otimes \mathscr{G}$ is a base of a filter (on $A \times B$) whenever $\mathscr{F}$ and $\mathscr{G}$ are filters (on A and B, respectively). The filter generated by $\mathscr{F} \otimes \mathscr{G}$ is the product of filters $\mathscr{F}$ and $\mathscr{G}$, which will be denoted by $\mathscr{F} \cdot \mathscr{G}$ as usual (see e.g. [1], § 7; a different notation was used in [2], where the multiplication of filters was introduced apparently for the first time).

1.5. If $\mathscr{X}, \mathscr{Y}$ are collections of sets, $A = \bigcup \mathscr{X}$, $B = \bigcup \mathscr{Y}$ and there exists a bijective $f : A \to B$ such that $f[\mathscr{X}] = \mathscr{Y}$, then $\mathscr{X}$ is said to be *isomorphic to* $\mathscr{Y}$ (this includes, as a special case, the isomorphism of filters).

2.1. Theorem. *Let $\mathscr{G}$ be a filter on a set A. Assume that $\mu : A \times A \to A$ is bijective and $\mathscr{G} \subset \mu[\mathscr{G} \cdot \mathscr{G}]$. Then there exists exactly one filter $\mathscr{F}$ on A such that* (1) $\mathscr{G} \subset \mathscr{F}$, (2) $\mu[\mathscr{F} \cdot \mathscr{F}] = \mathscr{F}$, (3) *if $\mathscr{H}$ is a filter on A, $\mathscr{G} \subset \mathscr{H}$, $\mu[\mathscr{H} \cdot \mathscr{H}] = \mathscr{H}$, then $\mathscr{F} \subset \mathscr{H}$.*

Proof. Put $\mathscr{G}_0 = \mathscr{G}$. If α is an ordinal, $\alpha > 0$, put $\mathscr{G}_\alpha = \mu[\mathscr{G}_\beta \cdot \mathscr{G}_\beta]$ if $\alpha = \beta + 1$, $\mathscr{G}_\alpha = \bigcup(\mathscr{G}_\beta : \beta < \alpha)$ if α is a limit ordinal. It is easy to see that, for every ordinal α, $\mathscr{G}_\alpha$ is a filter and $\mathscr{G}_\alpha \subset \mathscr{G}_\beta$ whenever $\alpha < \beta$. Hence, $\mathscr{G}_\alpha = \mathscr{G}_{\alpha+1}$ for some α. Put $\mathscr{F} = \mathscr{G}_\alpha$. Then $\mu[\mathscr{F} \cdot \mathscr{F}] = \mathscr{F}$. If $\mathscr{H}$ is as in (3), then we get $\mathscr{G}_\alpha \subset \mathscr{H}$ for all α, hence $\mathscr{F} \subset \mathscr{H}$.

2.2. Theorem. *On every infinite set, there exists an idempotent filter.*

Proof. If A is infinite, put $\mathscr{G} = \{A - X : X \text{ finite}\}$. Let $\mu : A \times A \to A$ be bijective. Clearly $\mathscr{G} \subset \mu[\mathscr{G} \cdot \mathscr{G}]$. Now apply the theorem above.

2.3. An explicit description of an idempotent filter is far more complicated than the existence proof. It is necessarily so, for an idempotent filter cannot be analytic (Souslin), cf. [3]. On the other hand, an explicit construction may provide more insight into properties of such filters.

3.1. The class of all dense linearly ordered sets with a first and no last element will be denoted by $\mathfrak{A}$. As a rule, letters A, B, C, possibly with subscripts, will stand for ordered sets in $\mathfrak{A}$. A set of the form $\{t : t \in A,\ a \leqq t\}$ or $\{t : t \in A,\ a \leqq t < b\}$ will be called an *interval of* A. The set of all nonvoid intervals of A will be denoted by $\mathrm{i}(A)$. We put $\mathrm{i}_0(A) = \mathrm{i}(A) \cup \{\emptyset\}$.

3.2. A pair $\langle B, C\rangle \in \mathrm{i}(A) \times \mathrm{i}(A)$ will be called a *decomposition of* A if $B \cup C = A$ and $x < y$ whenever $x \in B$, $y \in C$. If $\langle B, C\rangle$ is a decomposition of A, we write $B + C = A$.

3.3. A pair $x = \langle T, v\rangle$, where $T \in \mathrm{i}(A)$, $v \subset T$ is finite nonvoid, will be called a *labeled interval* of A. The set of all labeled intervals of A will be denoted by $\mathrm{li}(A)$. If $x = \langle T, v\rangle \in \mathrm{li}(A)$, we put $|x| = T$, $Lx = v$.

3.4. If $\xi = (x_n : n < \vartheta)$ is a sequence on $\mathrm{li}(A)$, we put $|\xi| = \bigcup(|x_n| : n < \vartheta)$, $L\xi = \bigcup(Lx_n : n < \vartheta)$, $\Sigma\xi = \langle|\xi|, L\xi\rangle$. If e.g. $\xi = (x, y)$, we also write $x + y$ instead of $\Sigma\xi$, etc. Clearly, if $\xi \in \mathrm{wli}(A)$ and $|\xi| \in \mathrm{i}(A)$, then $\Sigma\xi \in \mathrm{li}(A)$.

3.5. If $X \subset \text{wli}(A)$, we put $LX = \{L\xi : \xi \in X\}$. If $\mathscr{X} \subset \exp \text{wli}(A)$, we put $L\mathscr{X} = \{LX : X \in \mathscr{X}\}$.

3.6. An idempotent filter may be constructed, roughly speaking, in the following way. Suppose there is defined, for every $A \in \mathfrak{A}$, a collection $\mathscr{K}(A) \subset \exp \text{wli}(A)$ such that (1) if A is isomorphic to B, then $\mathscr{K}(A)$ is isomorphic to $\mathscr{K}(B)$, (2) $L\mathscr{K}(A)$ is a base of a filter, (3) if $B + C = A$, then $\mathscr{K}(B) \odot \mathscr{K}(C) \subset \mathscr{K}(A)$. It may be expected that if A, B, C are mutually isomorphic, then the filter generated by $L\mathscr{K}(A)$ is idempotent, since it is isomorphic to the product of filters generated by $L\mathscr{K}(B)$, $L\mathscr{K}(C)$.

4.1. We are going to construct certain collections with the properties mentioned in 3.6. We shall need a few auxiliary definitions and a number of simple facts concerning subsets of wli(A), etc.

4.2. A sequence $\xi = (x_n : n < \vartheta)$ on li(A) will be called *regular* if (1) either $\xi = \emptyset$ or $\min A \in |x_0|$, (2) the sets $|x_n|$ are disjoint, (3) $|(x_n : n \leqq m)| \in \text{i}(A)$ for every $m < \vartheta$. The set of all regular $\xi \in \text{wli}(A)$ will be denoted by rwli(A).

4.3. Let φ, ψ be mappings of rwli(A) into $\text{i}_0(A)$. Then we put $\psi \leqq \varphi$ iff $\psi(\xi) \subset \varphi(\xi)$ for all $\xi \in \text{rwli}(A)$, and we define $\varphi \wedge \psi$ by putting $(\varphi \wedge \psi)(\xi) = \varphi(\xi) \cap \psi(\xi)$.

4.4. For any $\varphi : \text{rwli}(A) \to \text{i}_0(A)$, $R(\varphi)$ will denote the set of all sequences $\xi = (x_n : n < \vartheta)$ on li(A) such that $|x_n| = \varphi(x_k : k < n)$ for every $n < \vartheta$. We put $S(\varphi) = \{\xi : \xi \in R(\varphi),\ \xi \text{ is finite},\ |\xi| = A\}$.

4.5. A mapping $\varphi : \text{rwli}(A) \to \text{i}_0(A)$ will be called a *transition rule* (*on* A) if (1) $\min A \in \varphi(\emptyset)$, (2) for any $\xi \in \text{rwli}(A)$, $|\xi| = A$ implies $\varphi(\xi) = \emptyset$, $|\xi| \neq A$ implies $\varphi(\xi) \neq \emptyset$, $|\xi| \cap \varphi(\xi) = \emptyset$, $|\xi| \cup \varphi(\xi) \in \text{i}(A)$. The set of all transition rules on A will be denoted by tr(A).

4.6. If $\xi = (x_n : n < \vartheta)$ is a regular sequence on li(A) and $\varphi \in \text{tr}(A)$, then, clearly, there exists exactly one $\eta = (y_k : k < \delta) \in R(\varphi)$ such that (1) every y_k is of the form $\Sigma(x_n : m < n < p)$, (2) if $\eta' = (y'_k : k < \delta')$ satisfies (1), then $\delta' \leqq \delta$, $y'_k = y_k$ for $k < \delta'$. We shall say that η is the *φ-reduction of* ξ. The sequence ζ such that $\xi = \alpha \cdot \zeta$, $|\alpha| = |\eta|$, will be called the *φ-remainder of* ξ. If $|\eta| = |\xi|$, then the φ-reduction of ξ will be called *exact*.

4.7. A transition rule φ on A will be called *regular* if, for any $\xi \in \text{rwli}(A)$ such that $|\xi|$ is a proper subset of $|\eta| \cup \varphi(\eta)$ where η is the φ-reduction of ξ, we have $\varphi(\xi) = \varphi(\eta) - |\xi|$. The set of all regular $\varphi \in \text{tr}(A)$ will be denoted by rtr(A).

4.8. Put $\varphi_0(\xi) = A - |\xi|$ for every $\xi \in \mathrm{rwli}(A)$. Then $\varphi_0 \in \mathrm{rtr}(A)$ (and even $\varphi_0 \in \mathrm{ntr}(A)$, see 5.1 below).

4.9. Let $\varphi \in \mathrm{rtr}(A)$, $\xi \in R(\varphi)$. Then ξ is regular; if $\xi \cdot \zeta \in \mathrm{rwli}(A)$, $|\zeta| \subset \varphi(\xi)$, $|\zeta| \neq \varphi(\xi)$, then ξ is the φ-reduction of $\xi \cdot \zeta$, hence $\varphi(\xi \cdot \zeta) = \varphi(\xi) - |\zeta|$.

4.10. *Let $\varphi \in \mathrm{rtr}(A)$, $\psi \in \mathrm{tr}(A)$, $\psi \leqq \varphi$, $\xi \in R(\psi)$. Let η and ζ be, respectively, the φ-reduction and the φ-remainder of ξ. Then there occurs exactly one of the following cases: (1) η is exact, $\zeta = \emptyset$; (2) η is not exact, ζ is finite, $|\zeta|$ is a proper subset of $\varphi(\eta)$, $\varphi(\xi) = \varphi(\eta) - |\xi|$; (3) η is not exact, ζ is infinite, $|\zeta| \subset \varphi(\eta)$. If ξ is finite, then $|\xi| \cup \varphi(\xi) = |\eta| \cup \varphi(\eta)$. If $\xi \in S(\psi)$, then $\eta \in S(\varphi)$, $L\xi = L\eta \in LS(\varphi)$.*

Proof. Assume that η is not exact. Let $\zeta = (x_m, \ldots)$. By definition (4.6), $|(x_m, \ldots \ldots, x_n)| = \varphi(\eta)$ for no n. Suppose $|(x_m, \ldots, x_n)| \supset \varphi(\eta)$ for some n. Choose the last p such that $\varphi(\eta) - |(x_n : m \leqq n < p)| \neq \emptyset$. Since φ is regular, we have $\varphi(x_n : n < p) = \varphi(\eta) - |(x_n : n < p)|$, hence, due to $\psi \leqq \varphi$, we get $|x_p| = \psi(x_n : n < p) \subset \varphi(\eta)$, $|(x_n : m \leqq n < p + 1)| \subset \varphi(\eta)$, which is a contradiction.

We have shown that every $|(x_m, \ldots, x_n)|$ is a proper subset of $\varphi(\eta)$. The rest of the proof may be omitted.

4.11. Proposition. *If $\varphi_1, \varphi_2 \in \mathrm{rtr}(A)$, then $\varphi_1 \wedge \varphi_2 \in \mathrm{rtr}(A)$.*

Proof. Put $\psi = \varphi_1 \cdot\wedge \varphi_2$. Clearly, $\psi \in \mathrm{tr}(A)$. Let $\xi \in \mathrm{rwli}(A)$, let η denote the ψ-reduction of ξ and let $|\xi| \subset |\eta| \cup \varphi(\eta)$, $|\xi| \neq |\eta| \cup \varphi(\eta)$. We are going to prove that $\psi(\xi) = \psi(\eta) - |\xi|$. Let η_i, $i = 1, 2$, be the φ_i-reduction of η. Clearly, η_i is also the φ_i-reduction of ξ. By 4.10, we have

$$(1) \qquad |\eta| \cup \varphi_i(\eta) = |\eta_i| \cup \varphi_i(\eta_i),$$

hence

$$(2) \qquad |\eta| \cup \psi(\eta) \subset |\eta_i| \cup \varphi_i(\eta_i).$$

This implies

$$(3) \qquad \xi \text{ is a proper subset of } |\eta_i| \cup \varphi_i(\eta_i).$$

Since φ_i are regular, we have

$$(4) \qquad |\xi| \cup \varphi_i(\xi) = |\eta_i| \cup \varphi_i(\eta_i),$$

hence, by (1),

$$(5) \qquad |\xi| \cup \varphi_i(\xi) = |\eta| \cup \varphi_i(\eta).$$

This proves that $|\xi| \cup \psi(\xi) = |\eta| \cup \psi(\eta)$, hence $\psi(\xi) = \psi(\eta) - |\xi|$.

4.12. *If $\varphi \in \text{rtr}(A)$, $\xi \in S(\varphi)$, $\eta \in S(\varphi)$, $L\xi = L\eta$, then $\xi = \eta$.*

Proof. Put $\xi = (x_n : n < p)$, $\eta = (y_k : k < q)$. Clearly, $|x_0| = |y_0|$. Since $L\xi = L\eta$, we get $Lx_0 = Ly_0$, $x_0 = y_0$. The proof proceeds by induction.

5.1. A regular transition rule φ on A will be called *normal* if every $\xi \in R(\varphi)$ is finite. The set of all normal $\varphi \in \text{rtr}(A)$ will be denoted by $\text{ntr}(A)$. The collection of all $S(\varphi)$, $\varphi \in \text{ntr}(A)$, will be denoted by $\mathscr{S}(A)$.

5.2. Proposition. *If $\varphi_1, \varphi_2 \in \text{ntr}(A)$, then $\varphi_1 \wedge \varphi_2 \in \text{ntr}(A)$.*

Proof. Put $\psi = \varphi_1 \wedge \varphi_2$. By 4.11, $\psi \in \text{rtr}(A)$. Suppose that $\xi = (x_n : n < \omega) \in R(\psi)$. For $i = 1, 2$, let η_i and ζ_i be the φ_i-reduction and the φ_i-remainder of ξ, respectively. Since no $\eta \in R(\varphi_i)$ is infinite, 4.10 implies that, for $i = 1, 2$, η_i is not exact, ζ_i is infinite, $|\zeta_i| \subset \varphi_i(\eta_i)$. We may assume $|\eta_1| \subset |\eta_2|$. Let $\beta = (x_0, \ldots, x_p)$, $|\beta| = |\eta_2|$. Then, for $i = 1, 2$, $|\beta| \cup \varphi_i(\beta) = |\eta_i| \cup \varphi(\eta_i) \supset |\xi|$, hence $|\beta| \cup \psi(\beta) \supset |\xi|$, $|(x_0, \ldots, x_p, x_{p+1})| \supset |\xi|$, which is a contradiction.

5.3. Proposition. *If $\varphi \in \text{ntr}(A)$, then $S(\varphi) \neq \emptyset$.*

Proof. Choose a mapping f of the set $\{\xi : \xi \in \text{rwli}(A), |\xi| \neq A\}$ into $\text{li}(A)$ such that $|f(\xi)| = \varphi(\xi)$. Define a sequence $\zeta = (z_n)$ as follows: $z_n = f(z_k : k < n)$ provided $|(z_k : k < n)| \neq A$; if $|(z_0, \ldots, z_p)| = A$, then $\zeta = (z_0, \ldots, z_p)$. Clearly, $\zeta \in R(\varphi)$, hence ζ is finite, $|\zeta| = A$.

5.4. Proposition. *For any $A \in \mathfrak{A}$, $L\mathscr{S}(A)$ (see 5.1, 3.5) is a base of a filter.*
This follows at once from 4.8, 5.3, 5.2, 4.10 (last assertion).

5.5. If $A \in \mathfrak{A}$, then the filter on eA (see 1.1) generated by $L\mathscr{S}(A)$ will be denoted by $\mathscr{F}(A)$.

6.1. Let $B + C = A \in \mathfrak{A}$ (see 3.2). Let $\varphi : \text{rwli}(B) \to \text{i}_0(B)$ and, for every $\xi \in S(\varphi)$, let $\psi_\xi : \text{rwli}(C) \to \text{i}_0(C)$. For every $\xi = (x_n : n < p) \in \text{rwli}(A)$ define $\tau(\xi)$ as follows: (1) if $B - |\xi| \neq \emptyset$, put $\tau(\xi) = \varphi(\xi)$; (2) if $|\xi| \supset B$ and, for some η, ζ, we have $|\zeta| = B$, $\xi = \eta \cdot \zeta$, put $\tau(\xi) = \psi_\eta(\zeta)$; (3) if $|\xi| \supset B$ and $B = |(x_n : n < m)|$ for no m, put $\tau(\xi) = A - |\xi|$. Then τ is a mapping of $\text{rwli}(A)$ into $\text{i}_0(A)$, which will be denoted by $\varphi * (\psi_\xi)$.

6.2. *Let $B + C = A \in \mathfrak{A}$. Let $\varphi \in \text{tr}(B)$ and, for every $\xi \in S(\varphi)$, let $\psi_\xi \in \text{tr}(C)$. Put $\tau = \varphi * (\psi_\xi)$. Then (1) $\tau \in \text{tr}(A)$; (2) if φ, ψ_ξ are regular (normal), then so is τ; (3) if $\xi \in S(\varphi)$, $\eta \in R(\psi_\xi)$, then $\xi \cdot \eta \in R(\tau)$; (4) if $\zeta \in R(\tau)$, $|\zeta| - B \neq \emptyset$, then $\zeta = \xi \cdot \eta$, where $\xi \in S(\varphi)$, $\eta \in R(\psi_\xi)$.*

Proof. We omit the straightforward proof of (1)–(3) and prove (4) only. Put $\zeta = (z_n : n < \vartheta)$ and consider the last p such that $|z_p| \subset B$. Then $|(z_0, \ldots, z_p)| = B$, for otherwise we should have $z_{p+1} = \varphi(z_n : n \leqq p)$, hence $|z_{p+1}| \subset B$. Put $\xi = (z_0, \ldots, z_p)$, $\eta = (z_{p+1}, \ldots)$.

6.3. *Let $\langle B, C \rangle$ be a decomposition of $A \in \mathfrak{A}$. Then* (1) $\mathscr{S}(B) \odot \mathscr{S}(C)$ (see 1.3) *is equal to the collection of all $S(\tau)$ where $\tau = \varphi * (\psi_\xi)$, $\varphi \in \operatorname{ntr}(B)$, $\psi_\xi \in \operatorname{ntr}(C)$ for every $\xi \in S(\varphi)$;* (2) $\mathscr{S}(B) \odot \mathscr{S}(C) \subset \mathscr{S}(A)$.

Proof. Let $X \in \mathscr{S}(B) \odot \mathscr{S}(C)$. Then there exists a transition rule $\varphi \in \operatorname{ntr}(B)$ and a mapping $g : S(\varphi) \to \operatorname{ntr}(C)$ such that X consists of all $\xi \cdot \eta$ where $\xi \in S(\varphi)$, $\eta \in S(g\xi)$. Put $\psi_\xi = g\xi$, $\tau = \varphi * (\psi_\xi)$. Then, by 6.2, $X = S(\tau)$. Since, by 6.2, $\tau \in \operatorname{ntr}(A)$, we have $\mathscr{S}(B) \odot \mathscr{S}(C) \subset \mathscr{S}(A)$.

6.4. For any collections V, W, Z of sets such that $v \cup w \in Z$ whenever $v \in V$, $w \in W$, we denote by u the mapping $u : V \times W \to Z$ defined by $u\langle v, w \rangle = v \cup w$.

6.5. Proposition. *Let $\langle B, C \rangle$ be a decomposition of $A \in \mathfrak{A}$. Then $u[L\mathscr{S}(B) \otimes L\mathscr{S}(C)] = L[\mathscr{S}(B) \odot \mathscr{S}(C)]$.*

Proof. I. Let $X \in \mathscr{S}(B) \odot \mathscr{S}(C)$. Let φ, g, ψ_ξ be as in the proof of 6.3. Then, clearly, $LX = \{L\xi \cup L\eta : \xi \in S(\varphi), \eta \in S(\psi_\xi)\}$. For every $x \in LS(\varphi)$ there is, by 4.12, exactly one $\xi \in S(\varphi)$ such that $L\xi = x$; put $\xi = fx$. Then $LX = u\{\langle x, y \rangle : x \in LS(\varphi), y \in LS(\psi_{fx})\}$, hence $LX \in u[L\mathscr{S}(B) \otimes L\mathscr{S}(C)]$. – II. If $Z \in u[L\mathscr{S}(B) \otimes L\mathscr{S}(C)]$, then, clearly, there is a $\varphi \in \operatorname{ntr}(B)$ and a mapping $g : LS(\varphi) \to L\mathscr{S}(C)$ such that $Z = \{x \cup y : x \in LS(\varphi), y \in g(x)\} = \{L\xi \cup L\eta : \xi \in S(\varphi), \eta \in S(\psi_\xi)\}$ where $LS(\psi_\xi) = g(L\xi)$. Hence $Z = \{L(\xi \cdot \eta) : \xi \in S(\varphi), \eta \in S(\psi_\xi)\}$ and therefore $Z \in L[\mathscr{S}(B) \odot \mathscr{S}(C)]$. This proves the proposition.

6.6. *Let $B + C = A \in \mathfrak{A}$, $\tau \in \operatorname{rtr}(A)$. Define τ' as follows: for $\xi = (x_n : n < p) \in \operatorname{rwli}(A)$ put $\tau'(\xi) = \tau(\xi)$ if $B \cap \tau(\xi) = \emptyset$, $\tau'(\xi) = B \cap \tau(\xi)$ if $B \cap \tau(\xi) \neq \emptyset$. Then* (1) $\tau' \in \operatorname{rtr}(A)$; (2) *if $\xi \in \operatorname{rwli}(B)$, $|\xi| \neq B$, then $\tau'(\xi) \subset B$;* (3) *every finite $\xi \in R(\tau')$ is of the form $\xi = \eta \cdot \zeta$ where $|\eta| \subset B$, $|\zeta| \subset C$;* (4) *if $\xi = (x_n : n < \vartheta) \in R(\tau')$, then either $\xi \in R(\tau)$ or, for some m, $|x_{m-1}| \subset B$, $|x_m| \subset C$, $(x_0, \ldots, x_{m-1} + x_m, \ldots) \in R(\tau)$;* (5) $LS(\tau') \subset LS(\tau)$; (6) *if τ is normal, then so is τ',* (7) $\tau' = \varphi * (\psi_\xi)$, *for some $\varphi \in \operatorname{ntr}(B)$, $\psi_\xi \in \operatorname{ntr}(C)$.*

The proof is straightforward and may be omitted.

6.7. Proposition. *Let $\langle B, C \rangle$ be a decomposition of $A \in \mathfrak{A}$. Then for every $\tau \in \operatorname{ntr}(A)$ there exists a set $X \in \mathscr{S}(B) \odot \mathscr{S}(C)$ such that $LX \subset LS(\tau)$.*

Proof. Let τ' be as in 6.6. Put $X = S(\tau')$. Then, by 6.6, $LX \subset LS(\tau)$. By 6.6, (7), we have $S(\tau') \in \mathscr{S}(B) \odot \mathscr{S}(C)$.

7.1. Proposition. *If $\langle B, C\rangle$ is decomposition of $A \in \mathfrak{A}$, then $u[\mathscr{F}(B) \cdot \mathscr{F}(C)] = \mathscr{F}(A)$, hence $\mathscr{F}(B) \cdot \mathscr{F}(C)$ is isomorphic to $\mathscr{F}(A)$.*

Proof. I. By definition (5.5), $L\mathscr{S}(B)$ and $L\mathscr{S}(C)$ generate the filters $\mathscr{F}(B)$ and $\mathscr{F}(C)$, respectively. Hence $L\mathscr{S}(B) \otimes L\mathscr{S}(C)$ generates $\mathscr{F}(B) \cdot \mathscr{F}(C)$. By 6.5, $u[L\mathscr{S}(B) \otimes L\mathscr{S}(C)] = L[\mathscr{S}(B) \odot \mathscr{S}(C)]$. By 6.3, $L[\mathscr{S}(B) \odot \mathscr{S}(C)] \subset L\mathscr{S}(A)$. Hence $L[\mathscr{S}(B) \odot \mathscr{S}(C)]$ generates $u[\mathscr{F}(B) \cdot \mathscr{F}(C)]$, and $u[\mathscr{F}(B) \cdot \mathscr{F}(C)] \subset \mathscr{F}(A)$. II. By 6.7, for every $P \in L\mathscr{S}(A)$ there exists a set $Q \in L[\mathscr{S}(B) \odot \mathscr{S}(C)]$ such that $Q \subset P$. Hence, for every $P \in \mathscr{F}(A)$ there exists a set $Q \in u[L\mathscr{S}(B) \otimes L\mathscr{S}(C)] \subset u[\mathscr{F}(B) \cdot \mathscr{F}(C)]$ such that $Q \subset P$. This implies $\mathscr{F}(A) \subset u[\mathscr{F}(B) \cdot \mathscr{F}(C)]$, which proves the proposition.

7.2. Theorem. *There exists a mapping $\mathscr{F}$ of the class $\mathfrak{A}$ of all dense linearly ordered sets with a first and with no last element into the class of all filters such that (1) $\mathscr{F}(A)$ is a filter on eA, (2) if $\langle B, C\rangle$ is a decomposition of A, then $\mathscr{F}(B) \cdot \mathscr{F}(C)$ is isomorphic to $\mathscr{F}(A)$, (3) if A_1 is isomorphic (as an ordered set) to A_2, then the filters $\mathscr{F}(A_1)$, $\mathscr{F}(A_2)$ are isomorphic. If $A \in \mathfrak{A}$ has a decomposition $\langle B, C\rangle$ such that A, B, C are mutually isomorphic, then $\mathscr{F}(A)$ is an idempotent filter.*

This follows at once from 7.1.

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