Časopis pro pěstování matematiky

Pavel Krbec

On nonparasit generalized solutions of differential relations

Časopis pro pěstování matematiky, Vol. 106 (1981), No. 4, 368--372

Persistent URL: `http://dml.cz/dmlcz/108495`

Časopis pro pěstování matematiky, roč. 106 (1981), Praha

ON NONPARASIT GENERALIZED SOLUTIONS OF DIFFERENTIAL RELATIONS

PAVEL KRBEC, Praha

(Received July 5, 1979)

0. Introduction

In [5] Sentis introduced generalized solutions of the differential relation $\dot{x} \in F(t,x)$, F being an upper-semicontinuous but not necessarily convex mapping. It appears, that the set of the generalized solutions depends on the behaviour of F on $M \times R^n$, the Lebesgue measure of M being zero. We modify the definition of the generalized solutions to obtain independence with respect to such M.

1. Definitions and Notation

Let F be a mapping from $Q = [0, 1] \times B_3$, $B_3 \subset R^n$ being the closed ball with center at origin and radius 3, into the set $\mathscr{K}$ of all compact nonempty subsets of the unit ball $B_1 \subset R^n$. For $M \subset R$ the set $\{(t, x, y) \in Q \times B_1 |\ t \notin M,\ y \in F(t, x)\}$ is denoted by $G_M F$. Thus $G_M F$ is the graph of $F|_{([0,1]-M) \times B_3}$, F being considered as a multivalued mapping into R^n. For M empty we shall write GF instead of $G_M F$. A mapping $F : Q \to \mathscr{K}$ is upper-semicontinuous (u.s.c.) if GF is closed in R^{2n+1} (see Kuratowski [3]). We say that a mapping Φ from $[0, 1]$ into the set of all compact subsets of a ball B in R^m is approximately continuous at a point $t \in [0, 1]$ if there exists a measurable set $A \subset [0, 1]$, $t \in A$, such that $\lim_{h \to 0^+} (\mu((t - h, t + h) \cap A)/2h) = 1$ and $\Phi|_A$ is continuous in the relative topology of A and the Hausdorff topology on compact subsets of B.

The set $h = \{0 = h^0 < h^1 < h^2 < \ldots < h^{m+1} = 1\}$ is called a division of $[0, 1]$, $|h| = \max_{i=0,1,\ldots,m} |h^{i+1} - h^i|$, $\nu(h) = m$ and $\mu(M)$ stands for the Lebesgue measure of $M \subset R$.

Definition 1 (Sentis [5]). A function $y(\cdot) : [0, 1] \to R^n$ is a *g-solution of the differential relation*

(1) $$\dot{x} \in F(t, x), \quad x(0) = x_0 \in B_1$$

on $[0, 1]$ if there exists a sequence $\{y_n\}_{n=1}^{\infty}$ of piecewise linear functions and a sequence $\{h_n\}_{n=1}^{\infty}$ of divisions such that (denote $y_n(h_n^k)$ by x_n^k and $v(h_n)$ by v_n)

i) $\lim_{n\to\infty} |h_n| = 0$,

ii) $x_n^0 = x_0$,

iii) for every positive integer n and $k = 0, 1, \ldots, v_n$ there are $a_n^k \in F(h_n^k, x_n^k)$ and $\varepsilon_n^k \in R^n$ such that $x_n^{k+1} = x_n^k + a_n^k(h_n^{k+1} - h_n^k) + \varepsilon_n^k$ and $y_n(\cdot)$ is linear on every $[h_n^k, h_n^{k+1})$, $k = 0, 1, \ldots, v_n$,

iv) $\lim_{n\to\infty} \sum_{k=1}^{v_n} \|\varepsilon_n^k\| = 0$,

v) $\lim_{n\to\infty} y_n = y$ uniformly on $[0, 1]$.

2.

Sentis introduced this definition to cover the case (cl stands for closure)

$$F(t, x) = \bigcap_{\delta>0} \bigcap_{\substack{N\subset R^{n+1}\\ \mu(N)=0}} \operatorname{cl} f(B_\delta(t, x) - N),$$

$f : R^{n+1} \to R^n$ being possibly discontinuous, and his definition works well for such right-hand sides, see [5]. The following example shows that in general (i.e. for F being only u.s.c.) the definitions of g-solutions should be modified.

Example 1. For $R^n = R$ set $F_1(t, x) = \{-1\}$ for $x < 0$ and every t, $F_1(t, x) = \{-1, 1\}$ for $x = 0$ and every t and $F_1(t, x) = \{1\}$ for $x > 0$ and every t, $F_2(t, x) = F_1(t, x)$ for t dyadically irrational and every x. For $t = (k/2^m)$, k odd set $F_2(t, x) = F_1(t, x)$ for $x \notin [-1/2^m, 1/2^m]$ and $F_2(t, x) = \{-1, 1\}$ for $x \in [-1/2^m, 1/2^m]$. Then both F_1 and F_2 are u.s.c. mappings and $\mu\{t \in [0, 1] \mid \exists_x F_1(t, x) \neq F_2(t, x)\} = 0$. The function $y(\cdot)$, identically equal to zero on $[0, 1]$, is not a g-solution of $\dot{x} \in F_1(t, x)$, $x(0) = 0$ (see Sentis [5]) but it is a g-solution of the relation $\dot{x} \in F_2(t, x)$, $x(0) = 0$ on $[0, 1]$. The sequence $\{y_n\}_{n=1}^{\infty}$ can be constructed as follows: $h_n = \{0, 1/2^n, 2/2^n, \ldots, (2^n - 1)/2^n, 1\}$, $x_n^k = 1/2^n$ for k odd, $x_n^k = 0$ for k even, $y_n(\cdot)$ is linear on every $[h_n^k, h_n^{k+1}]$. It is easy to see that $\{y_n\}$ and y fulfil the conditions (i), …, (v).

3.

To avoid this discrepancy we will define generalized solutions of $\dot{x} \in F(t, x)$ (we will call them regular g-solutions or rg-solutions) through certain regular right-hand side F^*. To obtain F^* we set $G^*F = \bigcap_{M\subset[0,1],\mu(M)=0} \operatorname{cl} G_M F$ and define F^* by means of the relation $GF^* = G^*F$. Let $\pi : R^{2n+1} \to R^{n+1}$, $\pi(t, x, y) = (t, x)$ be the projection.

Lemma 1. *Let $F : Q \to \mathcal{K}$ be a u.s.c. mapping. Then there exists a set $M_0 \subset \subset [0, 1]$ such that $\mu(M_0) = 0$, $G^*F = \text{cl}\, G_{M_0}F$ and $\pi(G^*F) = Q$.*

Proof. It will be helpful to introduce the mapping Φ, $\Phi : t \in [0, 1] \to \Phi(t) = \{(t, x, y) \in R^{2n+1} \mid (t, x) \in Q,\ y \in F(t, x)\}$. The upper semicontinuity of F implies that Φ is a u.s.c. mapping into the set of compact subsets of $Q \times B_1$. Therefore, there is a set $M_0 \subset [0, 1]$ such that Φ is approximately continuous at all points of $[0, 1] - M_0$ and $\mu(M_0) = 0$ (see Hermes [1]). For this M_0 the set $\{(t, x, y) \in R^{2n+1} \mid t \notin M_0,\ (t, x) \in Q,\ (t, x, y) \in \Phi(t)\}$ will be denoted by $G_0\Phi$. We have $G_0\Phi = G_{M_0}F$ and we shall prove $G^*F \supset \text{cl}\, G_0\Phi$.

Let $(t, x, y) \in \text{cl}\, G_0\Phi$. Then there exists a sequence $\{(t_n, x_n, y_n)\} \to (t, x, y)$ for $n \to \infty$ such that $t_n \notin M_0$ and $y_n \in F(t_n, x_n)$. Let $\mu(M) = 0$. In virtue of the approximate continuity of Φ we can find a sequence $\{\tau_n, \xi_n, \psi_n\}$ such that $\tau_n \notin M$, $(\tau_n, \xi_n, \psi_n) \to (t, x, y)$ for $n \to \infty$ and $\psi_n \in F(\tau_n, \xi_n)$. Hence $(t, x, y) \in \text{cl}\, G_MF$, i.e. $\text{cl}\, G_0\Phi \subset \text{cl}\, G_MF$ and since M was an arbitrary null set we conclude $\text{cl}\, G_0\Phi \subset G^*F$. Since the converse inequality is obvious we have $\text{cl}\, G_{M_0}F = G^*F$ and $\pi(G^*F) = Q$.

Remark. The upper-semicontinuity of F is not necessary. The proof is still valid if we suppose F to be only Scorza-Dragonian, i.e., u.s.c. except for sets whose projection to the t-axis has "arbitrarily small" measure (for the precise definition of the Scorza-Dragonian property see Jarník, Kurzweil [2]), due to the fact that the Scorza-Dragonian property implies Borel measurability of Φ (see Rzeżuchowski [4]).

For $F : Q \to \mathcal{K}$ let us define the mapping F^* by means of the relation $F^*(t, x) = \{y \in R^n \mid (t, x, y) \in G^*F\}$. Then as a consequence of Lemma 1 we obtain $F^* : Q \to \mathcal{K}$ and since $GF^* = G^*F$ and G^*F is closed we have that F^* is u.s.c. Moreover, $F^* \subset F$ and since the mapping Φ from Lemma 1 is approximately continuous at all points of $[0, 1] - M_0$, it follows immediately that $\{t \in [0, 1] \mid \exists_{x \in B_3} F^*(t, x) \neq F(t, x)\} \subset M_0$.

Remark. The multivalued mapping F^* can be equivalently defined as $F^*(t, x) = \bigcap_{\delta > 0} \bigcap_{\substack{N = M \times B_3 \\ \mu(M) = 0}} \text{cl}\, F(B_\delta(t, x) - N)$, which is similar to the definition of Filippov's cone, see Vrkoč [6].

Definition 2. Let the mapping $F : Q \to \mathcal{K}$ be u.s.c. and let $y(\cdot)$ be a g-solution of the relation $\dot{x} \in F^*(t, x)$, $x(0) = x_0 \in B_1$ on $[0, 1]$. Then $y(\cdot)$ is called an *rg-solution of* (1) *and the set* $\{y(\cdot) \mid y(0) = x_0,\ y(\cdot)$ is an rg-solution of (1)$\}$ *is called* Sol $F(x_0)$.

As a trivial consequence of Definition 2 and Lemma 1 we obtain that all "nice" properties of Sentis' g-solutions (see [5]) are preserved: there is always an rg-solution, any classic solution is also an rg-solution and any rg-solution of (1) is a classic solution of the relation $\dot{x} \in \text{conv}\, F(t, x)$. Moreover, Sol $F_1(x_0) =$ Sol $F_2(x_0)$ whenever $\mu\{t \in [0, 1] \mid \exists_{x \in B_3} F_1(t, x) \neq F_2(t, x)\} = 0$ since then $F_1^* = F_2^*$.

Example 2. Let F_1 and F_2 be the same as in Example 1. Then $F_1^* = F_2^* = F_1$, there are exactly two rg-solutions fulfilling the initial condition $x(0) = 0$ (namely $x^+(t) = t$ and $x^-(t) = -t$) and these solutions are the classic ones. Let $M \subset R^n$. Denote $-M = \{x \in R^n | -x \in M\}$. Then neither the equation $\dot{x} \in -F_1(t, x)$ nor $\dot{x} \in -F_2(t, x)$ has a classic solution fulfilling $x(0) = 0$ but the function $y(\cdot)$ identically equal to zero is an rg-solution of both $\dot{x} \in -F_1(t, x)$ and $\dot{x} \in -F_2(t, x)$, $x(0) = 0$. Moreover we have $\operatorname{conv}(-F_1(\cdot, o) = [-1, 1]$, hence $y(\cdot)$ is a classic solution of both $\dot{x} \in \operatorname{conv}(-F_1(t, x))$ and $\dot{x} \in \operatorname{conv}(-F_2(t, x))$, $x(0) = 0$.

4.

The rg-solutions can be obtained not only in terms of F^* but via a direct modification of Definition 1 as well.

Theorem. *A function $y(\cdot)$ is an rg-solution of (1) if and only if for every $M \subset [0, 1]$, $\mu(M) = 0$ there are sequences $\{y_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ such that all conditions (i), ..., (v) of Definition 1 are fulfilled and $\bigcup_{n=1}^{\infty} h_n \cap M = \emptyset$.*

To prove the theorem we will use the following trivial lemma.

Lemma 2. *Let us suppose $a \in F^*(t, x)$, $M \subset [0, 1]$, $\mu(M) = 0$. Then there are sequences $\{(t_n, x_n)\}_{n=1}^{\infty}$ and $\{a_n\}_{n=1}^{\infty}$ such that $a_n \in F^*(t_n, x_n)$, $t_n \notin M$, $\lim_{n\to\infty} (t_n, x_n, a_n) = (t, x, a)$.*

Proof. From $a \in F^*(t, x)$ we obtain as a consequence of the identity $GF^* = G^*F$ and of Lemma 1 that $(t, x, a) \in GF^* = \operatorname{cl} G_{M_0 \cup M}F$, $\mu(M_0 \cup M) = 0$. Hence there exists a sequence $\{t_n, x_n, a_n\} \to (t, x, a)$ such that $t_n \notin M_0 \cup M$ and $a_n \in F(t_n, x_n)$. Since $F^*(\tau, \xi) = F(\tau, \xi)$ for $\tau \notin M_0$ the proof is complete.

Proof of the theorem: Since $\{t \in [0, 1] \mid \exists_{x \in B_3} F^*(t, x) = F(t, x)\} \subset M_0$, $\mu(M_0) = 0$, the "only if" part of the theorem follows immediately. To prove the "if" part let $y(\cdot)$ be an rg-solution and $M \subset [0, 1]$, $\mu(M) = 0$. Then there is a sequence $\{y_n\} \to y$ and the sequence $\{h_n\}$ such that the conditions (i), ..., (v) from Definition 1 are fulfilled with F^* instead of F. Condition (iii) written explicitly has the following form:

$$y_n(h_n^{k+1}) = y_n(h_n^k) + a_n^k(h_n^{k+1} - h_n^k) + \varepsilon_n^k\,, \quad a_n^k \in F^*(h_n^k, y_n(h_n^k))\,.$$

As a consequence of Lemma 2 we obtain that y_n, h_n^k, a_n^k and ε_n^k can be replaced by $\bar{y}_n$, $\bar{h}_n^k$, $\bar{a}_n^k$, $\bar{\varepsilon}_n^k$ such that

$$\bar{h}_n = \{0 = \bar{h}_n^0 < \bar{h}_n^1 < \dots < \bar{h}_n^{\nu_n+1} = 1\} \cap M = \emptyset \tag{2}$$

for every $n = 1, 2, 3, \dots$, $\bar{h}_n^k < h_n^{k+1}$, $(\bar{h}_n^k - h_n^k) < 1/(n \,.\, \nu_n)$, $\sum_{K=1}^{\nu_n} \|\bar{\varepsilon}_n^k\| \to 0$ as $n \to \infty$

and

(3) $$\bar{y}_n(\bar{h}_n^{k+1}) = \bar{y}_n(\bar{h}_n^k) + \bar{a}_n^k(\bar{h}_n^{k+1} - \bar{h}_n^k) + \bar{\varepsilon}_n^k\,, \quad \bar{a}_n^k \in F^*(\bar{h}_n^k, \bar{y}_n(\bar{h}_n^k))$$

for $n = 1, 2, \ldots$ and $k = 0, 1, 2, \ldots, v_n$.

We can proceed for example as follows. For every $n = 1, 2, \ldots$ we set $\bar{h}_n^0 = h_n^0 = 0$, $\bar{y}_n(\bar{h}_n^0) = x_0$, $\bar{h}_n^{v_n+1} = 1$, $\bar{y}_n(1) = y_n(1)$, $\bar{a}_n^0 = a_n^0$. Let us denote $1/(nv_n)$ by ϱ. As a consequence of Lemma 2 we can choose $\bar{h}_n^k$, $\bar{a}_n^k$ and ψ_n^k such that (2) is fulfilled and $|\bar{h}_n^k - h_n^k| < \varrho$, $\psi_n^k \in B_\varrho(y_n(h_n^k)) \subset B_3$, $\bar{a}_n^k \in F^*(\bar{h}_n^k, \psi_n^k)$ $\bar{a}_n^k \in B(a_n^k, \varrho)$ holds for $k = 1, 2, \ldots, v_n$. We set $\bar{y}_n(h_n^k) = \psi_n^k$ and choose such $\bar{\varepsilon}_n^k$ that (3) is fulfilled. Then

$$\bar{\varepsilon}_n^k = \bar{y}_n(\bar{h}_n^{k+1}) - \bar{y}_n(\bar{h}_n^k) - \bar{a}_n^k(\bar{h}_n^{k+1} - \bar{h}_n^k)$$

and

$$\|\bar{\varepsilon}_n^k\| \leqq \|\bar{y}_n(\bar{h}_n^{k+1}) - y_n(h_n^{k+1})\| + \|y_n(h_n^k) - \bar{y}_n(\bar{h}_n^k)\| + \|\bar{a}_n^k - a_n^k\| \, \|\bar{h}_n^{k+1} - \bar{h}_n^k\| + \\ + \|a_n^k\| \left(|\bar{h}_n^{k+1} - h_n^{k+1}| + |\bar{h}_n^k - h_n^k|\right) + \\ + \|y_n(h_n^{k+1}) - y_n(h_n^k) - a_n^k(h_n^{k+1} - h_n^k)\| \leqq 3\varrho + 2\varrho + \|\varepsilon_n^k\|\,.$$

Hence $\lim_{n\to\infty} \sum_{k=1}^{v_n} \|\bar{\varepsilon}_n^k\| = 0$. Similarly we obtain $\lim_{n\to\infty} \bar{y}_n = y$ uniformly on $[0, 1]$ and the proof is complete.

Remark. We have supposed $F : Q = [0, 1] \times B_3 \to \mathscr{K}$, where $\mathscr{K}$ is the set consisting of all compact non empty subsets of B_1. The reasons for taking $[0, 1]$, B_1 and B_3 are of course purely technical.

References

[1] *H. Hermes:* Calculus of Set Valued Functions and Control. Journal of Mathematics and Mechanics, Vol. *18*, No. 1, 1968, pp. 47—59.

[2] *J. Jarník, J. Kurzweil:* On conditions on right hand sides of differential relations. Čas. pěst. mat., *102* (1977), pp. 334—348.

[3] *R. Kuratowski:* Topology. Vol *1, 2* (Russian), Moscow 1966.

[4] *T. Rzeżuchowski:* Scorza-Dragoni Type Theorem for Upper Semicontinuous Multivalued Functions. Preprint, Institute of Mathematics, Warsaw Technical University, Warsaw 1978.

[5] *R. Sentis:* Equations differentielles a second membre measurable. Bolletino U.M.I. (5) 15-B (1978), pp. 724—742.

[6] *I. Vrkoč:* A new definition and some modifications of Filippov cone. Equadiff IV, Proceedings, Prague 1977, pp. 433—441. Lecture Notes in Mathematics, *703* Springer-Verlag 1979.

Author's address: 199 05 Praha 9 - Letňany, Beranových 130 (Výzkumný a zkušební letecký ústav).