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Časopis pro pěstování matematiky, Vol. 106 (1981), No. 4, 373--375

Persistent URL: `http://dml.cz/dmlcz/108482`

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SOME REMARKS ON DOMATIC NUMBERS OF GRAPHS

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(Received July 26, 1979)

E. J. Cockayne and S. T. Hedetniemi in the papers [1] and [2] define the domatic number of an undirected graph. Here we shall present some results concerning this concept. We shall investigate finite undirected graphs without loops and multiple edges.

A dominating set in a graph G is a subset D of the vertex set $V(G)$ of G with the property that each vertex of $V(G) - D$ is adjacent to at least one vertex of D. A partition of $V(G)$ into dominating sets is called *a domatic partition of* G. The maximal number of classes of a domatic partition of a graph G is called *the domatic number of* G and is denoted by $d(G)$.

In [2] it is suggested to relate the domatic number of a graph G to the connectivity of this graph. In this paper we shall prove some results concerning this topic.

The vertex (or edge) connectivity degree of a graph G is the minimal cardinality of a subset of the vertex set (or the edge set, respectively) of G with the property that by deleting this set from G a disconnected graph is obtained. (To delete a subset of the vertex set of G means to delete all vertices of this set and all edges which are incident to these vertices. To delete a subset of the edge set of G means to delete only all edges of this set.) The vertex connectivity degree of G will be denoted by $\omega(G)$, its edge connectivity degree by $\sigma(G)$.

Theorem 1. *Let p and q be non-negative integers, $p < q$. Then there exists a graph G such that $\omega(G) = p$, $d(G) = q$.*

Proof. Take two copies G', G'' of the complete graph K_q with q vertices. If $p = 0$, then G is the graph whose connected components are G' and G''. If $p \neq 0$, we choose pairwise distinct vertices $u_1, \ldots, u_p$ in G' and $v_1, \ldots, v_p$ in G'' and identify u_i with v_i for each $i = 1, \ldots, p$. In the following we shall denote the vertex obtained by identifying u_i with v_i by w_i for $i = 1, \ldots, p$. The remaining vertices of G' (or G'') will be denoted by $u_{p+1}, \ldots, u_q$ (or $v_{p+1}, \ldots, v_q$, respectively). In the case $p = 0$ we denote the vertices of G' by $u_1, \ldots, u_q$ and the vertices of G'' by $v_1, \ldots, v_q$. If we delete the set $\{w_1, \ldots, w_p\}$ from G, we obtain a disconnected graph. As each of the vertices $w_1, \ldots, w_p$ is adjacent to all the other vertices of G, after deleting less than p vertices the graph G remains connected; therefore $\omega(G) = p$. Let $D_i = \{w_i\}$ for $i = 1, \ldots, p$ and $D_i = \{u_i, v_i\}$ for $i = p + 1, \ldots, q$. Evidently $\{D_1, \ldots, D_q\}$ is a domatic partition of G and $d(G) \geqq q$. In [1] it was proved that $d(G) \leqq \delta(G) + 1$, where $\delta(G)$ is the minimal degree of a vertex of G. Here evidently $\delta(G) = q - 1$, hence $d(G) = q$.

Theorem 2. *Let p and q be non-negative integers, $p < q$. Then there exists a graph G such that $\sigma(G) = p$, $d(G) = q$.*

Proof. We take again two copies G' and G'' of K_q. Let the vertices of G' (or G'') be $u_1, \ldots, u_q$ (or $v_1, \ldots, v_q$, respectively). If $p = 0$, the graph G is the same as in the proof of Theorem 1. If $p \neq 0$, we join u_i with v_i by an edge for each $i = 1, \ldots, p$. Evidently $\sigma(G) = p$, where G is the graph thus obtained. Taking $D_i = \{u_i, v_i\}$ for $i = 1, \ldots, q$ we obtain a domatic partition $\{D_1, \ldots, D_q\}$ and, as $\delta(G) = q - 1$, we have $d(G) = q$.

Theorem 3. *Let h be a positive integer. Then there exists a graph G such that*

$$\omega(G) - d(G) = \sigma(G) - d(G) = h\,.$$

Proof. Let $n = 2h + 4$ and consider the complete graph K_n. As n is even, there exists a linear factor F of K_n. Let the edges of F be $e_1, \ldots, e_{h+2}$, let u_i, v_i be the end vertices of the edge e_i for $i = 1, \ldots, h + 2$. Let G be the graph obtained from K_n by deleting all edges of F. Evidently each subset of $V(G)$ which induces a disconnected subgraph of G is of the form $\{u_i, v_i\}$ for some i. Therefore $\omega(G) = n - 2 = 2h + 2$. It is easy to prove that also $\sigma(G) = n - 2 = 2h + 2$. No vertex of G is adjacent to all the other vertices, therefore each dominating set of G has at least two vertices. This implies $d(G) \leqq n/2$. Putting $D_i = \{u_i, v_i\}$ for $i = 1, \ldots, h + 2$ we obtain a domatic partition of G and hence $d(G) = n/2 = h + 2$. We have

$$\omega(G) - d(G) = \sigma(G) - d(G) = h\,.$$

The graph from the proof of Theorem 3 also has the property that $d(G) = \frac{1}{2}\,\delta(G) + 1$. We express a conjecture.

Conjecture. *For each graph G we have*

$$d(G) \geqq \tfrac{1}{2}\,\delta(G) + 1\,.$$

At the end we turn to another problem suggested in [2] – to characterize the uniquely domatic graphs.

A graph G is called *uniquely domatic*, if there exists exactly one domatic partition of G with $d(G)$ classes.

We shall characterize the uniquely domatic graphs whose domatic number is 2. First we prove a lemma.

Lemma. *Each uniquely domatic graph with a domatic number at least* 2 *is connected.*

Proof. Let G be a disconnected graph with $d(G) \geqq 2$. Then each connected component of G has at least two vertices; otherwise the domatic number of G would be 1. Let $d(G) = d$, let $\{D_1, \ldots, D_d\}$ be a domatic partition of G. Let C be a connected component of G, let $V(C)$ be its vertex set. As each vertex of C can be adjacent only to vertices of C, we have $D_i \cap V(C) \neq \emptyset$ for each $i = 1, \ldots, d$ and $\{D_1 \cap V(C), \ldots \ldots, D_d \cap V(C)\}$ is a domatic partion of C. Put $D'_1 = (D_1 - V(C)) \cup (D_2 \cap V(C))$, $D'_2 = (D_2 - V(C)) \cup (D_1 \cap V(C))$, $D'_i = D_i$ for $i = 3, \ldots, d$. It is easy to prove that $\{D'_1, \ldots, D'_d)$ is a domatic partition of G different from $\{D_1, \ldots, D_d\}$ and hence G is not uniquely domatic.

Theorem 4. *A graph with the domatic number* 2 *is uniquely domatic, if and only if it is a star or a complete graph* K_2.

Proof. Let G be a uniquely domatic graph with the domatic number 2. By Lemma the graph G must be connected. If G is neither a star nor K_2, then there exists a spanning tree T of G which is neither a star nor K_2. Therefore there exists an edge e of T which joins two non-terminal vertices of T. Let T' and T'' be the connected components of the forest obtained from T by deleting e. None of the graphs T', T'' is an isolated vertex, therefore $d(T') = d(T'') = 2$. Let $\{D'_1, D'_2\}$ (or $\{D''_1, D''_2\}$) be a domatic partition of T' (or T'', respectively). It is easy to see that $\{D'_1 \cup D''_1, D'_2 \cup D''_2\}$ and $\{D'_1 \cup D''_2, D'_2 \cup D''_1\}$ are domatic partitions of T and also of G. These partitions are evidently different, which is a contradiction with the assumption that G is uniquely domatic. Therefore G must be either a star or K_2. On the other hand, the unique domatic partition of a star into two classes is such that one class consists only of the center and the other consists of all other vertices, because if a terminal vertex of a star belonged to the same class as the center, it would not be adjacent to a vertex of the other class. An analogous situation occurs in the case of K_2.

References

[1] *Cockayne, E. J. - Hedetniemi, S. T.:* Towards a theory of domination in graphs. Networks 7 (1977), 247—261.

[2] *Cockayne, E. J.:* Domination of undirected graphs — a survey. In: Theory and Applications of Graphs, Proc. Michigan 1976, ed. by Y. Alavi and D. R. Lick. Springer-Verlag Berlin—Heidelberg—New York 1978, pp. 141—147.

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