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ON CONVOLUTION OF k CONTINUOUS FUNCTIONS

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It was shown in [1] that there exist functions $x(t)$, $y(t)$ continuous in an interval I such that the convolution $x * y = \int_0^t x(\tau)\, y(t - \tau)\, d\tau$ does not possess the derivative at any interior point $t \in I$. The aim of the present note is to generalize this result to the case of k functions, $k \geqq 2$.

First we shall introduce two lemmas. Lemma 1 is obtained from Lemma 1 [1] in an easy way by mathematical induction. Lemma 2 is in a sense analogous to Lemma 2 [1], although a differentiable oscillating function is used instead of a piecewise linear one. Again mathematical induction is used to prove the assertion for any integer $k \geqq 2$. In both cases the proofs are elementary and therefore they are omitted.

Lemma 1. *Let $I = \langle 0, t_0 \rangle$, $t_0 > 0$ or $I = \langle 0, +\infty)$. Let $k \geqq 2$ be an integer, $x_1, x_2, \ldots, x_k \in C(I)$ (i.e. continuous functions on I),*

$$|x_i(t)| \leqq L_i\,, \quad |x_i(t + h) - x_i(t)| \leqq K_i h$$

for all $t, t + h \in I$, $h > 0$. Denote $x * y = (x * y)(t) = \int_0^t x(\tau)\, y(t - \tau)\, d\tau$ for any $x, y \in C(I)$, $t \in I$. Then

$$|z_k(t)| = |(x_1 * x_2 * \ldots * x_k)(t)| \leqq L_1 L_2 \ldots L_k \frac{t^{k-1}}{(k-1)!}\,,$$

$$|z_k(t + h) - z_k(t)| \leqq L_1 L_2 \ldots L_{k-1} \left[L_k \frac{(t + h)^{k-2}}{(k-2)!} + K_k \frac{t^{k-1}}{(k-1)!} \right] h$$

for all $t, t + h \in I$, $h > 0$.

Lemma 2. *Let $x(t) = x_{ab}(t) = \frac{1}{2} b \cos 2\pi a t$. Then for any integer $k \geqq 2$ and all t*

$$\underbrace{x * x * \ldots * x}_{k \text{ times}} = \frac{b^k t^{k-1}}{2^{2k-1}(k-1)!} \cos 2\pi a t + \frac{b^k}{a} f_k(t, a)$$

where $|f_k(t, a)| \leqq c_k(t)$, i.e. f_k is bounded for each k, t as a function of a.

The main result of the present note is

Theorem 1. *For any integer* $k \geqq 2$ *there exists a function* $x(t) = x(t, k) \in \in C(\langle 0, +\infty))$ *such that the convolution* $x * x * \ldots * x$ *(k times) does not possess the derivative for any* $t > 0$.

Proof. Let the integer $k \geqq 2$ be fixed. Put

$$x_n(t) = \tfrac{1}{2} b_n \cos 2\pi a_n t \,, \quad b_n = 2^{-(k+1)^n} \,, \quad a_n = b_n^{-(k+1)}$$

for all positive integers n,

$$x(t) = \sum_{n=1}^{\infty} x_n(t) \,. \tag{1}$$

Since $|x(t)| \leqq \frac{1}{2} \sum_{n=1}^{\infty} 2^{-(k+1)^n}$, the series (1) converges absolutely and uniformly on $\langle 0, +\infty)$. Hence

$$x * x * \ldots * x = \sum_{\substack{p_i = 1 \\ i = 1, \ldots, k}}^{\infty} x_{p_1} * x_{p_2} * \ldots * x_{p_k} \,.$$

The following inequalities are obvious:

$$|x_n(t)| \leqq \tfrac{1}{2} b_n \,, \quad |x_n(t + h) - x_n(t)| \leqq \pi a_n b_n h \tag{2}$$

for all $t \geqq 0$, $h > 0$.

For fixed t let us choose integers $p_n, q_n, r_n, s_n, v_n, w_n$ analogously to [1], i.e.

$$p_n \text{ even} \,, \quad \tfrac{1}{2} p_n \leqq t a_n < \tfrac{1}{2} p_n + 1 \,,$$
$$q_n \text{ odd} \,, \quad \tfrac{1}{2} q_n \leqq t a_n < \tfrac{1}{2} q_n + 1 \,,$$

$$r_n = \frac{p_n}{2a_n} \to t \,, \quad s_n = \frac{p_n + 3}{2a_n} \to t \,, \quad v_n = \frac{q_n}{2a_n} \to t \,, \quad w_n = \frac{q_n + 3}{2a_n} \to t \,,$$

so that

$$s_n - r_n = w_n - v_n = \frac{3}{2a_n} \,. \tag{3}$$

Denoting $z = x * x * \ldots * x$, $z_{p_1, p_2 \ldots p_k} = x_{p_1} * x_{p_2} * \ldots * x_{p_k}$ let us estimate the difference

$$\left| \frac{z(s_n) - z(r_n)}{s_n - r_n} - \frac{z_{n,\ldots,n}(s_n) - z_{n,\ldots,n}(r_n)}{s_n - r_n} \right| = \left| \sum \frac{z_{p_1,\ldots,p_k}(s_n) - z_{p_1,\ldots,p_k}(r_n)}{s_n - r_n} \right| \tag{4}$$

where the sum is taken over all k-tuples of positive integers except $n, n, \ldots, n$. To this purpose, let us estimate the expression

$$\varDelta = \varDelta_{p_1,\ldots,p_k} = \left| \frac{z_{p_1,\ldots,p_k}(s_n) - z_{p_1,\ldots,p_k}(r_n)}{s_n - r_n} \right| .$$

(a) Let $p_k < n$. Then according to Lemma 1 and (2)

$$\varDelta \leqq \frac{1}{2^{k-1}} b_{p_1} b_{p_2} \ldots b_{p_{k-1}} \left(\tfrac{1}{2} b_{p_k} \frac{s_n^{k-2}}{(k-2)!} + \pi a_{p_k} b_{p_k} \frac{r_n^{k-1}}{(k-1)!} \right) \leqq$$

$$\leqq \frac{1}{2^{k-1}(k-2)!} b_{p_1} b_{p_2} \ldots b_{p_k} \left(\frac{1}{2} + \pi a_{p_k} \frac{t}{k-1} \right) (t+1)^{k-2} .$$

(b) Let $p_k > n$. Then again according to Lemma 1 and (2), (3)

$$\varDelta \leqq \frac{1}{2^k} b_{p_1} b_{p_2} \ldots b_{p_k} \frac{r_n^{k-1} + s_n^{k-1}}{(k-1)!} \cdot \frac{2a_n}{3} \leqq \frac{1}{3(k-1)!} b_{p_1} b_{p_2} \ldots b_{p_k} a_n (t+1)^{k-1} .$$

The following inequalities are needed in the sequel:

(5) $$\sum_{j=N}^{\infty} b_j \leqq 2b_N ,$$

(6) $$\sum_{j=1}^{N} a_j b_j \leqq 2b_{N+1}^{-k/(k+1)}$$

for any positive integer N.

In fact, it holds $b_{j+1} = b_j^{k+1}$; hence $b_{j+1}/b_j = b_j^k \leqq b_j \leqq 2^{-j}$ and consequently

$$\sum_{j=N}^{\infty} b_j \leqq b_N + \sum_{j=N+1}^{\infty} b_j \left(\frac{b_N}{b_{j-1}} \right) = b_N \left(1 + \sum_{j=N+1}^{\infty} \frac{b_j}{b_{j-1}} \right) \leqq 2b_N ,$$

$$\sum_{j=1}^{N} a_j b_j = \sum_{j=1}^{N} b_j^{-k} \leqq b_N^{-k} + \sum_{j=1}^{N-1} b_j^{-k} \left(\frac{b_{j+1}}{b_N} \right)^k = b_N^{-k} \left(1 + \sum_{j=1}^{N-1} \left(\frac{b_{j+1}}{b_j} \right)^k \right) \leqq$$

$$\leqq 2b_N^{-k} = 2b_{N+1}^{-k/(k+1)}$$

which proves inequalities (5), (6).

Now let us estimate the sum of all $\varDelta = \varDelta_{p_1,\ldots,p_k}$ such that $p_k \neq n$: According to (a), (b) and (5), (6) it holds

$$\sum_{p_k \neq n} \varDelta_{p_1,p_2,\ldots,p_k} = \sum_{p_1,\ldots,p_{k-1}=1}^{\infty} \left(\sum_{p_k=1}^{n-1} \varDelta + \sum_{p_k=n+1}^{\infty} \varDelta \right) \leqq$$

$$\leqq \sum_{p_1,\ldots,p_{k-1}=1}^{\infty} b_{p_1} b_{p_2} \ldots b_{p_{k-1}} \left\{ \frac{(t+1)^{k-2}}{2^k (k-2)!} \left[\sum_{p_k=1}^{n-1} b_{p_k} + \frac{\pi t}{k-1} \sum_{p_k=1}^{n-1} a_{p_k} b_{p_k} \right] + \right.$$

$$\left. + \frac{a_n(t+1)}{3(k-1)!} \sum_{p_k=n+1}^{\infty} b_{p_k} \right\} \leqq$$

$$\leqq \sum_{p_1,\ldots,p_{k-1}}^{\infty} b_{p_1} b_{p_2} \ldots b_{p_{k-1}} \{ A_k(t) + B_k(t)\, b_n^{-k/(k+1)} + C_k(t) \}$$

where A_k, B_k, C_k are some functions of t. (The equality $a_n = b_n^{-(k+1)} = b_{n+1}$ is used for the evaluation of the last term.) Finally, the inequality $\sum_{j=1}^{\infty} b_j \leqq 2$ (which is a special case of (5)) yields

$$\sum_{p_k \neq n} \Delta \leqq 2^{k-1} (A_k^*(t) + B_k(t)\, b_n^{-k/(k+1)}) \, .$$

Evidently, the same estimate is obtained for the sums $\sum_{p_j \neq n} \Delta$ where we assume successively $j = 1, 2, \ldots, k-1$. Summarizing all these estimates, we obtain obviously an estimate for the expression (4):

$$\left| \frac{z(s_n) - z(r_n)}{s_n - r_n} - \frac{z_{n,\ldots,n}(s_n) - z_{n,\ldots,n}(r_n)}{s_n - r_n} \right| \leqq$$

$$\leqq k \, . \, 2^{k-1} (A_k^*(t) + B_k(t)\, b_n^{-k/(k+1)}) \, .$$

On the other hand, Lemma 2 together with (3) yields

$$\left| \frac{z_{n,\ldots,n}(s_n) - z_{n,\ldots,n}(r_n)}{s_n - r_n} \right| = \left[\frac{2 b_n^k t^{k-1}}{2^{2k-1}(k-1)!} + \frac{b_n^k}{a_n} g_k(t, a_n) \right] \frac{2a_n}{3} =$$

$$= D_k(t)\, b_n^k a_n + \tfrac{2}{3} g_k(t; a_n)\, b_n^k = D_k(t)\, b_n^{-1} + \tfrac{2}{3} g_k(t, a_n)\, b_n^k \, ,$$

where $|g_k(t, a_n)| \leqq E_k(t)$ and the functions D_k, E_k are independent of n. Comparing the two last relations, we obtain immediately

$$\lim_{n \to \infty} \frac{z(s_n) - z(r_n)}{s_n - r_n} = +\infty \, .$$

Quite analogously it may be shown that

$$\lim_{n \to \infty} \frac{z(w_n) - z(v_n)}{w_n - v_n} = -\infty$$

and the assertion of the theorem follows immediately (cf. Lemma 3 [1]).

In an analogous way as in [1] it is possible to prove

Theorem 2. *There is a set $M \subset C^k = C \times C \times \ldots \times C$ (k times), $C = C(I)$ such that*

(a) *$C^k - M$ is of the 1st category in C^k,*

(b) *if $x = (x_1, x_2, \ldots, x_k) \in M$, then the convolution $x_1 * x_2 * \ldots * x_k$ does not possess the derivative at any interior point $t \in I$.*

However, we prefer to introduce the proof of a similar theorem which is perhaps more interesting since it deals (for any integer $k \geqq 2$) with the space C and not C^k. For the sake of brevity let us suppose $I = \langle 0, 1 \rangle$.

Theorem 3. *Let V be a set of all functions $x(t) \in C = C(\langle 0, 1 \rangle)$ such that the convolution $z = x * x * \ldots * x$ (k times) does not possess the derivative at any t, $0 < t < 1$. Then the complement $W = C - V$ is a set of the 1st category in C (with the usual uniform metric).*

Proof. For any positive integer n let G_n^+, G_n^- be the sets of functions from C with the following property:

To any $t \in I_n = \langle 1/n, 1 - 1/n \rangle$ there are number r, s, $r < s$ such that

$$t - \frac{1}{n} \leqq r \leqq t \leqq s \leqq t + \frac{1}{n}, \tag{7}$$

$$\frac{z(s) - z(r)}{s - r} > n \quad (< -n \text{ respectively}). \tag{8}$$

Put $M_1 = \bigcap_{n=1}^{\infty} G_n^+$, $M_2 = \bigcap_{n=1}^{\infty} G_n^-$, $M = M_1 \cap M_2$. Obviously $M \subset V$ and hence $W \subset C - M$. We shall show (for any positive integer n):

(i) G_n^+ as well as G_n^- are open sets, i.e. $F_n^+ = C - G_n^+$, $F_n^- = C - G_n^-$ are closed sets;

(ii) G_n^+ as well as G_n^- are dense sets in C.

This will prove Theorem 3 since then $W \subset C - M = \bigcup_{n=1}^{\infty} (F_n^+ \cup F_n^-)$ where F_n^+, F_n^- are nowhere dense in C.

(i) Let $x_\nu(t) \in F_n^+$ for all positive integers ν, $\lim_{\nu \to \infty} x_\nu(t) = x(t)$ in C, i.e. uniformly. Then Lemma 4 [1] implies $\lim_{\nu \to \infty} z_\nu(t) = z(t)$, $z_\nu = x_\nu * x_\nu * \ldots * x_\nu$ (by mathematical induction). As $x \notin G_n^+$, there is $t_\nu \in I_n$ such that for all $r_\nu < s_\nu$ satisfying

$$t_\nu - \frac{1}{n} \leqq r_\nu \leqq t_\nu \leqq s_\nu \leqq t_\nu + \frac{1}{n} \tag{9}$$

it holds

$$\frac{z_\nu(s_\nu) - z_\nu(r_\nu)}{s_\nu - r_\nu} \leqq n . \tag{10}$$

We can choose a convergent subsequence from the sequence t_ν; let us assume at once $\lim_{\nu\to\infty} t_\nu = t \in I_n$. For any $r < s$ satisfying (7) let us define $r_\nu = r + t_\nu - t$, $s_\nu = s + t_\nu - t$ so that $r_\nu \to r$, $s_\nu \to s$ and (10) holds. Hence also

$$\frac{z(s) - z(r)}{s - r} = \lim_{\nu\to\infty} \frac{z_\nu(s_\nu) - z_\nu(r_\nu)}{s_\nu - r_\nu} \leqq n$$

which proves (i).

(ii) Let $y \in C$, $\varepsilon > 0$, n a positive integer. We shall show that there exists $x \in G_n^+$, $\varrho(x, y) < \varepsilon$, ϱ being the uniform metric in C. The function y can be approximated by a continuous piecewise linear function η so that $\varrho(y, \eta) < \frac{1}{2}\varepsilon$. There is a constant A such that

$$|\eta(t)| \leqq A, \quad |\eta(t_1) - \eta(t_2)| \leqq A|t_1 - t_2| \tag{11}$$

for $0 < t_i < 1$, $i = 1, 2$.

Put $x = \eta + x_{a\varepsilon}$, $x_{a\varepsilon}(t) = \frac{1}{2}\varepsilon \cos 2\pi at$ (cf. Lemma 2). Evidently $\varrho(x, y) \leqq \varrho(x, \eta) + \varrho(y, \eta) < \varepsilon$. Hence it is sufficient to prove $x \in G_n^+$.

It holds

$$z = x * x * \ldots * x = x_{a\varepsilon} * x_{a\varepsilon} * \ldots * x_{a\varepsilon} + \sum_{j=1}^{k} \binom{k}{j} \underbrace{\eta * \ldots * \eta}_{j \text{ times}} * \underbrace{x_{a\varepsilon} * \ldots * x_{a\varepsilon}}_{(k-j) \text{ times}} . \tag{12}$$

Let $t \in I_n$. Put $r = (2m + 1)/2a$, $s = (2m + 6)/2a$ where m is an integer, $2m + 1 \leqq 2at \leqq 2m + 3$. Then obviously $\max(s - t,\ t - r) \leqq s - r = \frac{5}{2}a$; hence (7) holds for a large enough. Our aim is to estimate the expression $(z(s) - z(r))/(s - r)$; let us first consider the analogous expression for any term of the sum on the right hand side of (12). Denoting

$$z_j = \underbrace{\eta * \ldots * \eta}_{j \text{ times}} * \underbrace{x_{a\varepsilon} * x_{a\varepsilon} * \ldots * x_{a\varepsilon}}_{(k-j) \text{ times}}$$

it holds according to Lemma 1 and (11)

$$\left|\frac{z_j(s) - z_j(r)}{s - r}\right| \leqq \left(\frac{\varepsilon}{2}\right)^{k-j} A^j \left(\frac{s^{k-2}}{(k-2)!} + \frac{r^{k-1}}{(k-1)!}\right).$$

The right hand side of the inequality is bounded independently of a (since (7) holds for r, s). On the other hand, the first term on the right hand side of the identity (12), i.e. $z_0(t) = x_{a\varepsilon} * x_{a\varepsilon} * \ldots * x_{a\varepsilon}$ (k times) fulfils (again according to Lemma 1)

$$\frac{z_0(s) - z_0(r)}{s - r} = \frac{2\varepsilon^k t^{k-1}}{2^{2k-1}(k-1)!} \cdot \frac{2a}{5} + \varepsilon^k g_k(t, a)$$

where $g_k(t, a)$ is bounded as a function of a for any k, t. It is evident that if a is chosen sufficiently large then (8) holds and hence $x \in G_n^+$. The proof of (ii) for the sets G_n^- being quite analogous, we may consider the proof of Theorem 3 complete.

Remark. Let $\xi \in V$ (see Theorem 3). Take the set $V(\xi)$ of all functions from C with the following property: If $x_i \in V(\xi)$, $i = 1, 2, \ldots, k$, then the convolution $x_1 * x_2 * \ldots * x_k$ does not possess the derivative at any point $0 < t < 1$. It would be interesting to obtain some information on the structure of the sets $V(\xi)$ and their mutual relations. (If $\mathscr{V} = \bigcup_{\xi \in V} V(\xi)$, then evidently $V \subset \mathscr{V}$ and hence the complement of $\mathscr{V}$ in C is of the 1st category.)

References

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