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ON A SET IN $\mathbb{R}^n$ UNDER COORDINATE TRANSFORMATIONS

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1. INTRODUCTION AND PRELIMINARIES

Let G be an open subset in $\mathbb{R}^n$. In this paper we consider vector-valued functions $T: G \to \mathbb{R}^n$. Such a function can be written in the form $T = (f_1, f_2, \ldots, f_n)$, where $f_i : G \to R$ for each $i = 1, 2, \ldots, n$. Following Apostol ([1], p. 417), if G is an open subset in $\mathbb{R}^n$, then a vector-valued function $T: G \to \mathbb{R}^n$ is called a coordinate transformation on G if it has the following three properties:

a) $T \in C'$ on G, that is $\partial f_i / \partial x_j$ (partial derivative) exists and is continuous on G for each $i, j \in \{1, 2, \ldots, n\}$.

b) T is one-to-one on G.

c) The Jacobian determinant $J_T(t) = \det [\partial f_i / \partial x_j(t)] \neq 0$ for all t in G.

The following notation will be used in this paper.

(1) $S[c, \varrho]$ will denote the closed ball in R^n with center c and radius ϱ and $S(c, \varrho)$ will denote the corresponding open ball.

(2) If A and B are two sets, then $A \setminus B$ will stand for the set of all elements of A which are not in B.

(3) For each Lebesgue measurable subset X of $\mathbb{R}^n$, $|X|$ will denote the Lebesgue measure of X.

(4) A subset B of a topological space Y is said to possess the Baire property (or to be a Baire set) if B can be written in the form $B = (G \setminus P) \cup Q$, where G is an open set and P, Q are sets of the first category (i.e. countable unions of nowhere dense sets).

(5) The set of points x in $\mathbb{R}^n$ for which $x + \lambda$ belongs to the set E, $E \subset \mathbb{R}^n$, is denoted by $E - \lambda$.

(6) By the difference set $D(A)$ of a set $A \subset \mathbb{R}^n$ we mean the set of all vectors $x - y$, where $x, y \in A$.

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(7) If $T: \mathbb{R}^n \to \mathbb{R}^n$ and C is a subset of $\mathbb{R}^n$, then $T(C)$ denotes the transformed set of C under T, i.e. $T(C) = \{T(c) : c \in C\}$, and $T(c)$ denotes the image of c under the transformation T.

(8) The zero vector in $\mathbb{R}^n$, i.e. $(0, 0, \ldots, 0)$, will be denoted by $\mathbf{0}$.

In this paper we prove several general theorems dealing with collections of transformations and their actions on subsets of R^n of positive measure or on second category subsets of R^n possessing the Baire property. As corollaries of our theorems we obtain results of H. Steinhaus [12], S. Piccard [11], and S. Kurepa [5]. In addition an example is given to show that Theorem 2′ in [7] cannot be improved.

2. RESULTS

Theorem 1. *Suppose A, $A \subset \mathbb{R}^n$, is a set of positive Lebesgue measure and a is a point of density of A. If $T_1, T_2, \ldots, T_p$ are p transformations satisfying*

(α) *$T_i : \mathbb{R}^n \to \mathbb{R}^n$ for each $i = 1, 2, \ldots, p$;*

(β) *T_i is continuous at $\mathbf{0}$ and $T_i(\mathbf{0}) = \mathbf{0}$ for each $i = 1, 2, \ldots, p$;*

(γ) *$T_i(a) = a$ for each $i = 1, 2, \ldots, p$;*

(δ) *there exists an $\bar{R} > 0$ such that T_i is a coordinate transformation on $S(a, \bar{R})$ for each $i = 1, 2, \ldots, p$.*

Then there exists a ball K with center at the origin so that for every $x \in K$ there are vectors $a(x)$, $a_k^{k'}(x) \in A$; $k, k' \in \{1, 2, \ldots, p\}$ such that

$$T_k(a_k^{k'}(x)) = a(x) + T_{k'}(x)\,; \quad k, k' \in \{1, 2, \ldots, p\}\,.$$

Proof. There exist positive numbers R_1 and R_2, $R_1 < \bar{R}$, such that for each s, $0 < s < 1$, we have

i) $S[a, sR_2] \subset T_k(S[a, sR_1])$ for each $k = 1, 2, \ldots, p$. The proof of this statement is given (separately) in Remark 2 at the end of this paper. This is done in order to make the proof of Theorem 1 more readable. Since a is a point of density of A we have

ii) $|A \cap S[a, R]|/|S[a, R]| = 1 - \varepsilon(R)$ for each $R > 0$, where $0 \leqq \varepsilon(R) \leqq 1$ and $\varepsilon(R) \to 0$ as $R \to 0^+$.

Because of i) we have

iii) $T_k(A \cap S[a, sR_1]) \cup T_k(A^c \cap S[a, sR_1]) \supset S[a, sR_2]$ for each s, $0 < s < 1$, and each $k = 1, 2, \ldots, p$, where $A^c = \mathbb{R}^n \setminus A$.

Using the transformation formula for multiple integrals (see [1], Theorem 15.11) and ii) we have

iv) $|S[a, sR_2]| - |T_k(A^c \cap S[a, sR_1])| = \pi_n(sR_4)^n - K_k(s) . \varepsilon(sR_1) . \pi_n(sR_1)^n$ for each s, $0 < s < 1$, and each $k = 1, 2, \ldots, p$; and $\lim_{s \to 0^+} K_k(s) = |J_{T_k}(a)|$.

Here π_n is the constant of proportionality in the formula for the n-dimensional volume of a ball in $\mathbb{R}^n$.

iii) and iv) taken together imply that there exists s_0 such that if $0 < s < s_0$, then

v) $|S[a, sR_2] \setminus T_k(A \cap S[a, sR_1])| < (2p^2)^{-1} \cdot |S[a, sR_2]|$ for each $k = 1, 2, \ldots, p$.

From v) and (β) it follows that there exists a ball K with center at the origin such that

vi) $|S[a, sR_2] \setminus \{T_k(A \cap S[a, sR_1]) - T_{k'}(x)\}| < 2/(3p^2) \cdot |S[a, sR_2]|$ for each s, $0 < s < s_0$, for each $k, k' \in \{1, 2, \ldots, p\}$ and each $x \in K$.

We now show that if $s < s_0'$ and $x \in K$, then

$$X(x) = A \cap \left(\bigcap_{k=1}^{p} \bigcap_{k'=1}^{p} C_k^{k'}\right)$$

is a set of positive Lebesgue measure, where $C_k^{k'} = T_k(A \cap S[a, sR_1]) - T_{k'}(x)$ and s_0' $(0 < s_0' < s_0)$ is sufficiently small. To see this, pick s_0' so that $0 < s_0' < s_0$ and such that

vii) $|A \cap S[a, sR_2]| > \frac{3}{4}|S[a, sR_2]|$ if $0 < s < s_0'$. Let $0 < s < s_0'$, then for each $x \in K$,

$$X(x) \supset (A \cap S[a, sR_2]) \cap \left(\bigcap_{k=1}^{p} \bigcap_{k'=1}^{p} C_k^{k'}\right)$$

and therefore

viii) $|X(x)| > |S[a, sR_2]| - (\frac{1}{4}) \cdot |S[a, sR_2]| - \sum_{k=1}^{p} \sum_{k'=1}^{p} |S[a, sR_2] \setminus$
$\setminus \{T_k(A \cap S[a, sR_1]) - T_{k'}(x)\}| > (\frac{3}{4}) |S[a, sR_2]| - \sum_{k=1}^{p} \sum_{k'=1}^{p} 2/(3p)^2 |S[a, sR_2]|$.

Hence $|X(x)| > (\frac{3}{4} - \frac{2}{3}) |S[a, sR_2]| > 0$.

So there exist vectors

$$a(x) \in A \quad \text{and} \quad a_k^{k'}(x) \in A\,; \quad k, k' = 1, 2, \ldots, p\,,$$

such that

$$a(x) = T_k(a_k^{k'}(x)) - T_{k'}(x)\,; \quad k, k' = 1, 2, \ldots, p\,.$$

If we consider linear transformations we get the following corollary of Theorem 1.

Corollary 1a. *Suppose A, $A \subset \mathbb{R}^n$, is a set of positive measure and a is a point of density of A. If $T_1, T_2, \ldots, T_p$ are non-singular linear transformations, each leaving a fixed, then there exists a ball K with center at the origin so that for every $x \in K$ there are vectors $a(x)$ and $a_k^{k'}(x)$ in A, $k, k' = 1, 2, \ldots, p$, such that*

$$T_k(a_k^{k'}(x)) = a(x) + T_{k'}(x)\,; \quad k, k' = 1, 2, \ldots, p\,.$$

If we take $p = 1$ and T_1 to be the identity transformation ($T_1(x) = x$ for each x in $\mathbb{R}^n$) then Corollary 1a yields the following result of Steinhaus [12].

Corollary 1b. *If $A \subset \mathbb{R}^n$ is a set of positive Lebesgue measure, then $D(A)$, the difference set of A, contains a ball K with center at the origin.*

We now present the Baire set analogue of Theorem 1.

Theorem 1'. *Suppose that A, $A \subset \mathbb{R}^n$, is a Baire set of the second category, i.e. $A = (G \setminus P) \cup Q$ (where G is open and non-empty and P and Q are sets of the first category), and $a \in G$. If $T_1, T_2, \ldots, T_p$ are p transformations satisfying conditions (α), (β), (γ), and (δ) in Theorem 1, then there exists a ball K with center at the origin so that for every $x \in K$ there are vectors $a(x)$, $a_k^{k'}(x) \in A$, $k, k' \in \{1, 2, \ldots, p\}$ such that*

$$T_k(a_k^{k'}(x)) = a(x) + T_{k'}(x)\,, \quad k, k' \in \{1, 2, \ldots, p\}\,.$$

Proof. There exists an R_1, $\bar{R} > R_1 > 0$, such that $S[a, R_1] \subset G$. Because of (δ) (see [1], Theorem 13.5 and Example 3 on p. 356) each T_k is a homeomorphism when restricted to $S(a, R_1)$. Therefore, by virtue of (γ), there exists $R_2 > 0$ such that $R_2 < R_1$ and

i) $S(a, R_2) \subset T_k(S(a, R_1)$ for each $k = 1, 2, \ldots, p$.

Furthermore, since each T_k is a homeomorphism on $S(a, R_1)$, we have:

ii) $T_k(A \cap S(a, R_1)) \supset S(a, R_2) \setminus P_k$ for each $k = 1, 2, \ldots, p$, where each P_k is a set of the first category.

By (β) and ii) it follows that there exists a ball K with center at the origin and a number R_3 $(0 < R_3 < R_2)$ such that

iii) $T_k(A \cap S(a, R_1)) - T_{k'}(x) \supset S(a, R_3) \setminus P_k^{k'}$ holds for each $k, k' \in \{1, 2, \ldots, \}$, where $P_k^{k'}$ is of the first category.

We now show that if $x \in K$, then

$$X(x) = A \cap \Big(\bigcap_{k=1}^{p} \bigcap_{k'=1}^{p} C_k^{k'}\Big)$$

is a set of the second category, where $C_k^{k'} = T_k(A \cap S(a, R_1)) - T_{k'}(x)$ for each $k, k' \in \{1, 2, \ldots, p\}$.

To see this note that

$$X(x) \supset (A \cap S(a, R_3)) \cap \Big(\bigcap_{k=1}^{p} \bigcap_{k'=1}^{p} C_k^{k'}\Big)$$

and therefore

iv) $X(x) \supset (S(a, R_3) \setminus P) \cap \Big(\bigcap_{k=1}^{p} \bigcap_{k'=1}^{p} (S(a, R_3) \setminus P_k^{k'})\Big)$.

If we consider linear transformations we get the following corollary of Theorem 1'.

Corollary 1'a. *Suppose A, $A \subset \mathbb{R}^n$, is a Baire set of the second category, i.e. $A = (G \setminus P) \cup Q$ (where G is open and non-empty and P and Q are sets of the first category), and $a \in G$. If $T_1, T_2, \ldots, T_p$ are non-singular linear transformations, each leaving a fixed, then there exists a ball with center at the origin so that for every $x \in K$ there are vectors $a(x)$ and $a_k^{k'}(x)$ in A; $k, k' = 1, 2, \ldots, p$ such that*

$$T_k(a_k^{k'}(x)) = a(x) + T_{k'}(x)\,; \quad k, k' = 1, 2, \dots, p\,.$$

If we take $p = 1$ and T_1 to be the identity transformation then Corollary 1'a yields the following result of Piccard [11].

Corollary 1'b. *If $A \subset \mathbb{R}^n$ is a Baire set of the second category, then $D(A)$, the difference set of A, contains a ball K with center at the origin.*

We now state a theorem which yields a result of S. Kurepa as a corollary.

Theorem 2. *Suppose A, $A \subset \mathbb{R}^n$, is a set of positive measure and a is a point of density of A. Suppose further that $T_1, T_2, \dots, T_p$ are p transformations satisfying conditions (α), (γ) and (δ) in Theorem 1. If $T_1', T_2', \dots, T_p'$ are p transformations satisfying conditions (α) and (β), then there exists a ball K with center at the origin such that for every $x \in K$ there exist vectors*

$$a(x) \in A\,, \quad a_k(x) \in A\,, \quad k = 1, 2, \dots, p\,,$$

such that

$$T_k(a_k(x)) = a(x) + T_k'(x)\,, \quad k = 1, 2, \dots, p\,.$$

Proof. The proof of Theorem 2 is very similar to that of Theorem 1 and will therefore be omitted.

If we consider linear transformations we get the following corollary of Theorem 2.

Corollary 2a. *Suppose A, $A \subset \mathbb{R}^n$, is a set of positive measure and a is a point of density of A. Suppose further that $T_1, T_2, \dots, T_p$ are non-singular linear transformations each leaving a fixed. If $T_1', T_2', \dots, T_p'$ are p linear transformations, then there exists a ball K with center at the origin such that for every $x \in K$ there exist vectors*

$$a(x) \in A\,, \quad a_k(x) \in A\,, \quad k = 1, 2, \dots, p$$

such that

$$T_k(a_k(x)) = a(x) + T_k'(x)\,, \quad k = 1, 2, \dots, p\,.$$

If $T_1, T_2, \dots, T_p$ are all taken to be the identity transformation and $T_k' = (b_{ij}^k)$ is the linear transformation given for each $k = 1, 2, \dots, p$ as follows: $b_{ij}^k = \alpha_k$ for $i = j$ and $b_{ij}^k = 0$ for $i \neq j$, then Corollary 2a yields Theorem 1 of S. Kurepa [5]. Namely, we have

Corollary 2b. Let *$A \subset \mathbb{R}^n$ be a set of positive measure. For any system of p real numbers $\alpha_1, \alpha_2, \dots, \alpha_p$ $(\alpha_k \neq 0)$ there exists a ball K with center at the origin, such that for any $x \in K$ there are vectors $a_0(x), a_1(x), \dots, a_p(x)$ in A such that*

$$x = (a_1(x) - a_0(x))/\alpha_1 = (a_2(x) - a_0(x))/\alpha_2 = \dots = (a_p(x) - a_0(x))/\alpha_p\,.$$

We now present the Baire set analogue of Theorem 2.

Theorem 2′. *Suppose $A \subset R^n$ is a Baire set of the second category, i.e. $A = (G \setminus P) \cup Q$ (where G is open and non-empty and P and Q are sets of the first category), and $a \in G$. Suppose further that $T_1, T_2, \ldots, T_p$ are p transformations satisfying conditions (α), (γ) and (δ) in Theorem 1. If $T_1', T_2', \ldots, T_p'$ are p transformations satisfying conditions (α) and (β), then there exists a ball K with center at the origin such that for every $x \in K$ there exist vectors*

$$a(x) \in A\,, \quad a_k(x) \in A\,, \quad k = 1, 2, \ldots, p\,,$$

such that

$$T_k(a_k(x)) = a(x) + T_k'(x)\,, \quad k = 1, 2, \ldots, p\,.$$

Proof. The proof of Theorem 2′ is very similar to that of Theorem 1′ and will therefore be omitted.

Clearly, if we consider linear transformations we can obtain Baire set analogues of Corollaries 2a and 2b. These analogues (their statements and proofs) are straightforward and will not be listed here.

We conclude this paper by showing that Theorem 2′ in [7] cannot be improved.

Let Ω be a metric space and let $\mathscr{L}^n$ denote the family of all Lebesgue measurable subsets of R^n. Suppose further that for each $\omega \in \Omega$, T_ω is a mapping of $\mathscr{L}^n$ into $\mathscr{L}^n$. In [7] families of transformations $(T_\omega)_{\omega\in\Omega}$ are considered that satisfy the following conditions. There exists $\omega_0 \in \Omega$ and $a, b \in R^n$ such that:

i)′ $\limsup\limits_{n\to\infty} \{|a - T_{\omega_n}(K)|\} = r$ for each closed ball $K = S[b, r]$ with center b. Here $|y - C|$ denotes the set $\{|y - c| : c \in C\}$, where $|y - c|$ denotes the ordinary Euclidean distance between y and c.

ii) $E, F \in \mathscr{L}^n$, $E \subset F$ implies $T_\omega(E) \subset T_\omega(F)$ for each $\omega \in \Omega$.

iii) If $E \in \mathscr{L}^n$ and $\omega_n \to \omega_0$ (in Ω), then $\lim\limits_{n\to\infty} |T_{\omega_n}(E)| = |T_{\omega_0}(E)| = |E|$.

Theorem 2′ in [7] states: Suppose A and B are two sets of positive measure in R^n and a is a point of density one in A, b is a point of density one in B and ω_0 is a point of Ω. Suppose $(T_\omega)_{\omega\in\Omega}$ is a family of transformations of $\mathscr{L}^n$ into $\mathscr{L}^n$ satisfying the conditions (i)′, (ii) and (iii) with respect to the points a, b and ω_0 mentioned above. Then, if $(\omega_n)_{n=1}^{\infty}$ is a sequence in Ω converging to ω_0 and p is a positive integer, there exists p strictly increasing integers $n_1, n_2, \ldots, n_p$ such that

$$A \cap T_{\omega_{n_1}}(B) \cap T_{\omega_{n_2}}(B) \cap \ldots \cap T_{\omega_{n_p}}(B)$$

is a set of positive Lebesgue measure.

Neubrunn and Šalat [9], M. Pal [10], and H. Miller [8] have papers related to [7]. We now proceed to show that the last mentioned theorem cannot be improved.

Let K denote the open ball in R^2 with center at the origin and radius 1 and let L $(\subset R^2)$ be given by the formula $L = \{(0, 0)\} \cup [0, 1) \times (0, 1)$. Let f denote the function defined on K with range L, defined as follows:

$$f(0,0) = (0,0) \quad \text{and} \quad f(x, y) = (\theta/2\pi, r) \quad \text{for each} \quad (x, y) \in K \setminus \{(0,0)\},$$
$$\text{where } 0 \leqq \theta < 2\pi, \quad x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Then f is a one-to-one mapping of K onto L. Furthermore, the partial derivatives of f and f^{-1} exist and are continuous on $K \setminus \{(x, 0) : 0 \leqq x < 1\}$ and $(0, 1) \times (0, 1)$, respectively.

In addition, J_f and $J_{f^{-1}}$ (the Jacobians of f and f^{-1}, respectively) are non-zero at each point of $K \setminus \{(x, 0) : 0 \leqq x < 1\}$ and $(0, 1) \times (0, 1)$, respectively.

Each $x \in [0, 1]$ has a unique binary expansion

$$x = \sum_{i=1}^{\infty} x_i(x)\, 2^{-i},$$

where $x_i(x) = 0$ or 1 and

$$\sum_{i=1}^{\infty} x_i(x) = \infty$$

except if $x = 0$. The functions $(x_i)_{i=1}^{\infty}$ are mutually independent identically distributed random variables ([4], p. 1) defined on the probability space $([0, 1], \mathscr{M}, m)$, where $\mathscr{M}$ is the collection of all Lebesgue measurable subsets of $[0, 1]$ and m is the Lebesgue measure. Let $N = \{1, 2, 3, \ldots\}$ and suppose h is a one-to-one mapping from $N \times N$ onto N. Then the random variables $(y_i)_{i=1}^{\infty}$ defined by

$$y_i(x) = \sum_{n=1}^{\infty} x_{h(n,i)}(x) \cdot 2^{-n}$$

for each $i \in N$ and each $x \in [0, 1]$, are mutually independent and each of them is equidistributed on $[0, 1]$ (see [4], p. 1), i.e., given any subinterval I of $[0, 1]$ we have

$$|(x \in [0, 1] : y_i(x) \in I| = |y_i^{-1}(I)| = |I|$$

for each i in N.

Consider the mappings $(T_i')_{i=1}^{\infty}$ on $\mathscr{P}(K)$ (the collection of all subsets of K) into $\mathscr{P}(K)$ given as follows. If $B \subset K$, then $B = \bigcup(B_r : 0 \leqq r < 1)$, where B_r is the set of all points of B whose distance from the origin is r. Then define:

$$T_i'(B) = f^{-1}(\bigcup(y_i^{-1}(C_r) \times \{r\} : 0 < r < 1))$$

if $B_0 = \emptyset$ and

$$T_i'(B) = f^{-1}(\bigcup(y_i^{-1}(C_r) \times \{r\} : 0 < r < 1)) \cup (0, 0)$$

if $B_0 = \{(0, 0)\}$, where $f(B_r) = C_r \times \{r\}$, $C_r \subset [0, 1)$. Notice that $y_i^{-1}(C) \subset [0, 1)$ if $C \subset [0, 1)$.

For each $i \in N$, let T_i be the extension of T_i' given as follows:

$$T_i(B) = T_i'(B \cap K) \cup (B \setminus K) \quad \text{for each subset } B \text{ of } \mathbb{R}^2.$$

It is not difficult to see that for each $i \in N$, $T_i(B)$ is a measurable subset of $\mathbb{R}^2$ whenever B is a measurable subset of $\mathbb{R}^2$ and that $|T_i(B)| = |B|$ for each measurable set B.

Consider the following figure.

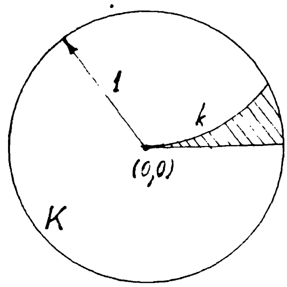

Fig. 1.

Let the curve k be chosen so that k is strictly monotonic and so that A, the open ball K less the shaded area (including the points on k, $(0, 0)$ and the points on the positive x-axis) has metric density 1 at the origin. Clearly A is an open subset of K.

We now show that if the mappings $(T_i)_{i=1}^{\infty}$ are given as above and $(T_{i_j})_{j=1}^{\infty}$ is any subsequence of $(T_i)_{i=1}^{\infty}$ then

$$|A \cap \bigcap_{j=1}^{\infty} T_{i_j}(A)| = 0 .$$

To see this notice that $f(A_r)$ is of the form $I_r \times \{r\}$, where I_r ($\subset [0, 1)$) is an interval of length strictly less than 1 for each r, $0 < r < 1$. Here, as before, A_r denotes the set of all points of A whose distance from the origin is r. For each r, $0 < r < 1$, the sets $\{(y_{i_j}^{-1}(I_r) : k \in \mathbb{N})\}$ are independent (in the sense of probability theory) and each set has the same Lebesgue measure, namely $|y_{i_j}^{-1}(I_r)|$ is equal to the length of I_r for each $j \in \mathbb{N}$. Therefore, since the length of I_r is less than one, the measure of

$$\bigcap_{j=1}^{\infty} y_{i_j}^{-1}(I_r)$$

is zero. This implies that

$$\bigcap_{j=1}^{\infty} T_{i_j}(A) = \bigcap_{j=1}^{\infty} (\bigcup_{0<r<1} T_{i_j}(A_r)) = \bigcup_{0<r<1} (\bigcap_{j=1}^{\infty} T_{i_j}(A_r)) =$$
$$= \bigcup_{0<r<1} (\bigcap_{j=1}^{\infty} f^{-1}(y_{i_j}^{-1}(I_r) \times \{r\}))$$

and therefore

$$\bigcap_{j=1}^{\infty} T_{i_j}(A) = \bigcup_{0<r<1} S_r ,$$

where S_r is a set of points, each having distance r from the origin, whose one-dimensional Lebesgue measure ([2], Example 1.5, p. 6) is zero an hence ([3], p. 193)

$$\left|\bigcap_{j=1}^{\infty} T_{i_j}(A)\right| = 0 .$$

Consider the metric space $\Omega = \mathbb{N} \cup \{0\}$, with the metric d, where $d(i, j) = |1/i - 1/j|$ for each $i, j \in \mathbb{N}$ and $d(i, 0) = 1/i$ for each $i \in \mathbb{N}$ and $d(0, 0) = 0$. Let $(T_i)_{i=1}^{\infty}$ be the sequence of transformations given above and let $T_0(B) = B$ for each Lebesgue measurable subset B of $\mathbb{R}^2$. Then T_n maps $\mathscr{L}^2$ into $\mathscr{L}^2$ (the collection of all Lebesgue measurable subsets of $\mathbb{R}^2$) for each $n \in \Omega$.

The collection of mappings $\{T_n\}_{n\in\Omega}$ has the following properties.

i′) If U is a closed ball in $\mathbb{R}^2$ with center at the origin and $n \in \Omega$, then $\sup \{|T_n(x)| : x \in U\} = r$, where r is the radius of U and $|y|$ is the distance between y and the origin.

ii) If $E, F \in \mathscr{L}^2$ and $E \subset F$, then $T_n(E) \subset T_n(F)$ for each $n \in \Omega$.

iii) If $E \in \mathscr{L}^2$, then $\lim_{n\to\infty} |T(E)| = |T_0(E)| = |E|$.

To see the last property observe that $E \in \mathscr{L}^2$ implies

$$T_n(E) = \bigcup_{0<r<1} T_n(E_r) \cup (E \setminus K) \quad \text{if} \quad E_0 = \emptyset$$

and

$$T_n(E) = \bigcup_{0<r<1} T_n(E_r) \cup (E \setminus K) \cup \{(0, 0)\}$$

if $E_0 = \{(0, 0)\}$, where E_r $(0 \leqq r < 1)$ is the set of all points in E whose distance from the origin is r. The one-dimensional Lebesgue measure of $T_n(E_r)$ and E_r are equal for each r, $0 \leqq r < 1$, and therefore $|T_n(E)| = |E|$ for each $n \in \Omega$ and hence iii) follows.

We have thus proved the following theorem.

Theorem 3. *There exist Lebesgue measurable subsets A and B of $\mathbb{R}^2$, each having positive measure, and points a and b having density one in A and B, respectively, and a point ω_0 in Ω (a metric space) and a family $(T_\omega)_{\omega\in\Omega}$ of transformations of $\mathscr{L}^2$ into $\mathscr{L}^2$ satisfying* i′), ii) *and* iii) *with respect to the points a, b and ω_0 such that for every sequence $(\omega_n)_{n=1}^{\infty}$ in Ω converging to ω_0,*

$$A \cap \bigcap_{k=1}^{\infty} T_{\omega_{n_k}}(B)$$

is a set of measure zero for each subsequence $(\omega_{n_k})_{k=1}^{\infty}$ of $(\omega_n)_{n=1}^{\infty}$.

Remark 1. It is not difficult to verify that Theorem 2 in [6] follows from Theorem 2 in this paper. It is also interesting to note that Theorem 1 in [6], whose proof is in some ways similar to the proof of Theorem 1 in this paper, can be generalized by omitting its first hypothesis (namely that $|T_i(x)| \leqq |x|$ $(i = 1, 2, \ldots, k)$ for any $x \in K[0, \varrho]$).

Remark 2. We now prove i) in the proof of Theorem 1 by showing the following lemma.

Lemma. *Suppose $T = (f_1, f_2, \ldots, f_n)$ is a transformation of $\mathbb{R}^n$ into $\mathbb{R}^n$. (i.e. $T: \mathbb{R}^n \to \mathbb{R}^n$ and $f_i: \mathbb{R}^n \to R$ for $i = 1, 2, \ldots, n$). Suppose further that:*

(a) $T(a) = b$ $(a, b \in \mathbb{R}^n)$;

(b) *there exists a neighborhood $\mathbb{N}$ of a such that $\partial f_i/\partial x_j$ exists and is continuous on $\mathbb{N}$ for each $i, j \in \{1, 2, \ldots, n\}$;*

(c) $J_T(a) \neq 0$.

Then there exist $R_1, R_2 > 0$ such that

(d) $S[b; sR_2] \subset T(S[a, sR_1])$ *for each s, $0 < s < 1$.*

Proof. By the above conditions (see [1], Theorems 13.4, 13.5) there exists a positive number $\bar{R}$ and an open subset K of $\mathbb{R}^n$ containing b such that T maps $S(a, \bar{R})$ homeomorphically onto K. Since T restricted to $S(a, \bar{R})$ is a closed mapping, it follows that each s, $0 < s < 1$, there exists a unique positive number $r(s)$ such that

$$T(S[a, s\bar{R}]) \supset S[a, r(s)]$$

and

$$T(S[a, s\bar{R}]) \not\supset S[a, r]$$

if $r > r(s)$.

If $\lim_{s \to 0^+} r(s)/s\bar{R} > 0$, it is easy to see that (d) follows.

If $\lim_{s \to 0^+} r(s)/s\bar{R} = 0$, then there exists a strictly decreasing null sequence $(s_n)_{n=1}^{\infty}$ with the property that $\lim_{n \to \infty} r(s_n)/s_n\bar{R} = 0$.

If $\varepsilon > 0$, then it follows that there exists two sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in $\mathbb{R}^n$ such that

α) $y_n = T(x_n)$ for each n,

β) $x_n \in S(a, \bar{R})$ for each n and $\lim_{n \to \infty} x_n = a$,

γ) $y_n \in K$ for each n and $\lim_{n \to \infty} y_n = b$,

δ) $\|y_n - b\|/\|x_n - a\| < (1 + \varepsilon)\, r(s_n)/s_n\bar{R}$,

where $\| \;\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. Property δ) follows from the definition of the function $r(s)$. Let $G = (g_1, g_2, \ldots, g_n)$ denote the inverse of T on K, i.e. for each $y \in K$, $G(y) = x$ if and only if $T(x) = y$. Of course G is a homeomorphism of K onto $S(a, \bar{R})$, $G(b) = a$ and all the partial derivatives $\partial g_i/\partial x_j$ exist and are continuous on a neighborhood of b (see [1], Theorem 13.6). This in turn implies (see [1], p. 356, example 3) that there exists a neighborhood $\bar{N}$ of b, $\bar{N} \subset K$, and a positive number A such that $\|G(y_1) - G(y_2)\| \leqq A\|y_1 - y_2\|$ whenever $y_1, y_2 \in \bar{N}$. The last inequality clearly contradicts δ) given above and hence $\lim_{s \to 0^+} r(s)/s\bar{R} = 0$ is impossible. This implies that $\lim_{s \to 0^+} r(s)/s\bar{R} > 0$ and (d) follows, completing the proof.

Notice that the ratio R_1/R_2 in our lemma can be taken to be A.

Remark 3. Various authors have generalized Corollary 1b. One such generalization can be found in the paper: M. E. Kuczma and M. Kuczma: An elementary proof and an extension of a theorem of Steinhaus. Glasnik Mat. *6* (*26*) (1971), 11—18.

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