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ON A RELATIONSHIP BETWEEN MONOIDS AND MONOUNARY ALGEBRAS

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1. INTRODUCTION

In the note [6], B. Zelinka assigned a monounary algebra to any finite semigroup and characterized the class of such algebras in terms of the graph theory. In this article, we omit the hypothesis that the semigroup is finite and characterize the class of algebras assigned to monoids in terms of monounary algebras. On the basis of this characterization, we derive Zelinka's result.

2. ALGEBRA ASSIGNED TO A MONOID

We recall (cf. [4] p. 1365) that a *monounary algebra* is an ordered pair (S, f) where S is a set and f a mapping of S into S. Let $\mathbb{N}$ be the set of all nonnegative integers. For any $p \in \mathbb{N}$, we denote by f^p the pth iteration of f. We put

$$\varrho = \{(x, y) \in S \times S;\ \text{there exist } p, q \in \mathbb{N} \text{ with } f^p(x) = f^q(y)\}\ .$$

Then ϱ is an equivalence on S whose blocks are called *components* of (S, f).

In what follows, we do not distinguish between algebras and their carriers; we also omit the adjective "monounary". If $t \in S$, we denote by $\boldsymbol{Sb}\ t$ the subalgebra of (S, f) generated by the set $\{t\}$ and by $\boldsymbol{Cp}\ t$ the component of (S, f) containing t. Clearly, $\boldsymbol{Sb}\ t = \{f^l(t);\ l \in \mathbb{N}\}$. It is easy to see that any component of (S, f) is also a subalgebra.

1. Definition. Let $(S, \cdot, e)$ be a monoid with the binary operation $\cdot$ and the unit e, $a \in S$ an arbitrary element. For any $x \in S$, we put $f(x) = x \,.\, a$. Then (S, f) is an algebra; it will be referred to as an algebra *assigned* to $(S, \cdot, e)$ *with respect to* a.

For any algebra (S, f), we denote by $\boldsymbol{End}(S, f)$ the monoid of all endomorphisms of (S, f) where the nullary operation is id_S and the binary operation is the composition $\circ$ of endomorphisms.

2. Theorem. *Let $(S, \cdot, e)$ be a monoid, $a \in S$ its arbitrary element, (S, f) the algebra assigned to $(S, \cdot, e)$ with respect to a. Then there exists a submonoid $(E, \circ, \mathrm{id}_S)$ of $\boldsymbol{End}(S, f)$ such that, for any $x \in S$, there exists exactly one $k_x \in E$ such that $k_x(e) = x$.*

Proof. For any $x \in S$, we put $k_x(t) = x \,.\, t$ where $t \in S$ is arbitrary. Thus, $k_x(f(t)) = x \,.\, (t \,.\, a) = (x \,.\, t) \,.\, a = f(k_x(t))$ for any $t \in S$ which entails that k_x is an endomorphism of (S, f) for any $x \in S$. Furthermore, $k_e = \mathrm{id}_S$ and $k_x(k_y(t)) = x \,.\, (y \,.\, t) = (x \,.\, y) \,.\, t = k_{x.y}(t)$ for any $x, y, t \in S$ and, thus, $(\{k_s;\ s \in S\}, \circ, \mathrm{id}_S)$ is a submonoid of $\boldsymbol{End}(S, f)$. Clearly, k_x is the only element k in $\{k_s;\ s \in S\}$ with $k(e) = x$. $\square$

3. Definition. Let (S, f) be an algebra, $e \in S$ its element, and $(E, \circ, \mathrm{id}_S)$ a submonoid of $\boldsymbol{End}(S, f)$ such that, for any $x \in S$, there exists exactly one $k_x \in E$ such that $k_x(e) = x$. Then e is said to be a *marked* element and $(E, \circ, \mathrm{id}_S)$ a *selective monoid corresponding to e*. An algebra having a marked element and a corresponding selective monoid is said to be *suitable.*

4. Corollary. *An algebra assigned to a monoid is suitable.* $\square$

5. Definition. Let (S, f) be an algebra, S_1, S_2 its components, h a homomorphism of S_1 into S_2. We put

$$k(t) = \begin{cases} h(t) & \text{for any} \quad t \in S_1 , \\ t & \text{for any} \quad t \in S - S_1 . \end{cases}$$

Clearly, k is an endomorphism of (S, f); it will be called the *normal extension of h and denoted by* $\boldsymbol{nl}\, h$.

6. Definition. An algebra (S, f) is said to be *very suitable* if there exists an element $e \in S$ and for any $x \in S$ a homomorphism h_x of $\boldsymbol{Cp}\ e$ into $\boldsymbol{Cp}\ x$ such that $h_x(e) = x$ and that $(\{\boldsymbol{nl}\, h_x;\ x \in S\}, \circ, \mathrm{id}_S)$ is a selective monoid corresponding to e.

7. Theorem. *An algebra is suitable if and only if it is very suitable.*

Proof. By 6 and 3, any very suitable algebra is also suitable.

Let (S, f) be a suitable algebra, e its marked element, $(E, \circ, \mathrm{id}_S)$ its selective monoid corresponding to e. For any $x \in S$, there exists exactly one $k_x \in E$ such that $k_x(e) = x$. We put $h_x = k_x \restriction \boldsymbol{Cp}\ e$. Then h_x is a homomorphism of $\boldsymbol{Cp}\ e$ into $\boldsymbol{Cp}\ x$ such that $h_x(e) = x$ for any $x \in S$.

Clearly, $\boldsymbol{nl}\, h_x(e) = h_x(e) = k_x(e) = x$ for any $x \in S$.

Let $x, y, t \in S$ be arbitrary. Two cases may occur.

(a) If $\boldsymbol{Cp}\ y \neq \boldsymbol{Cp}\ e$, then $\boldsymbol{nl}\, h_y[S] \subseteq S - \boldsymbol{Cp}\ e$ and $\boldsymbol{nl}\, h_x \restriction (S - \boldsymbol{Cp}\ e) = \mathrm{id}_{S - \boldsymbol{Cp} e}$. This implies that $\boldsymbol{nl}\, h_x \circ \boldsymbol{nl}\, h_y = \boldsymbol{nl}\, h_y$.

(b) Suppose $\boldsymbol{Cp}\ y = \boldsymbol{Cp}\ e$.

If $t \in S - \boldsymbol{Cp}\, e$, then $\boldsymbol{nl}\, h_y(t) = t$ which implies that $(\boldsymbol{nl}\, h_x \circ \boldsymbol{nl}\, h_y)(t) = t$.

If $t \in \boldsymbol{Cp}\, e$, then $\boldsymbol{nl}\, h_y(t) = k_y(t)$. Since $e, t \in \boldsymbol{Cp}\, e$, we obtain $k_y(e) = y \in \boldsymbol{Cp}\, e$, $k_y(t) \in \boldsymbol{Cp}\, e$. Since $(E, \circ, \mathrm{id}_S)$ is selective, there exists $z \in S$ such that $k_x \circ k_y = k_z$. This implies that $k_z(t) = k_x(k_y(t)) = k_x(\boldsymbol{nl}\, h_y(t)) = (\boldsymbol{nl}\, h_x \circ \boldsymbol{nl}\, h_y)(t)$. Thus, $\boldsymbol{nl}\, h_z(t) = (\boldsymbol{nl}\, h_x \circ \boldsymbol{nl}\, h_y)(t)$.

Since $\boldsymbol{nl}\, h_z(t) = t$ for any $t \in S - \boldsymbol{Cp}\, e$, we obtain $\boldsymbol{nl}\, h_z(t) = (\boldsymbol{nl}\, h_x \circ \boldsymbol{nl}\, h_y)(t)$ for any $t \in S$. Thus, $\boldsymbol{nl}\, h_x \circ \boldsymbol{nl}\, h_y = \boldsymbol{nl}\, h_z$.

We have proved that $(\{\boldsymbol{nl}\, h_x;\ x \in S\}, \circ)$ is a semigroup. Clearly, $\boldsymbol{nl}\, h_e = \mathrm{id}_S$. Thus $(\{\boldsymbol{nl}\, h_x;\ x \in S\}, \circ, \mathrm{id}_S)$ is a monoid; its selectivity is obvious.

Hence, (S, f) is very suitable. $\square$

3. CHARACTERIZATION OF ALGEBRAS ASSIGNED TO MONOIDS

1. Definition. Let (S, f) be a suitable algebra, e its marked element, $(E, \circ, \mathrm{id}_S)$ its selective monoid corresponding to e. For any $x \in S$ and any $t \in S$, we put $x \,.\, t = k_x(t)$ where k_x is the only element $k \in E$ with $k(e) = x$. Then $(S, \cdot)$ is said to be the *groupoid associated with (S, f) with respect to e and $(E, \circ, \mathrm{id}_S)$*.

2. Theorem. *Let (S, f) be a suitable algebra, $(S, \cdot)$ the groupoid associated with (S, f) with respect to a marked element $e \in S$ and a corresponding selective monoid $(E, \circ, \mathrm{id}_S)$. Then $(S, \cdot, e)$ is a monoid and (S, f) is assigned to $(S, \cdot, e)$ with respect to $f(e)$.*

Proof. (1) For any $x \in S$, let k_x denote the only element $k \in E$ with $k(e) = x$. Then $x \,.\, e = k_x(e) = x$. Furthermore, $\mathrm{id}_S = k_z$ for some $z \in S$ because id_S is the unit in $\boldsymbol{End}(S, f)$ and, thus, $\mathrm{id}_S \in E$. We have $z = k_z(e) = \mathrm{id}_S(e) = e$ which implies that $e \,.\, x = k_e(x) = \mathrm{id}_S(x) = x$ for any $x \in S$. Thus, e is the unit in S.

(2) Let $x, y, z \in S$ be arbitrary. Then $x \,.\, (y \,.\, z) = k_x(k_y(z)) = k_t(z)$ for some $t \in S$. Clearly, $t = k_t(e) = k_x(k_y(e)) = k_x(y)$. Thus, $x \,.\, (y \,.\, z) = k_{k_x(y)}(z)$. On the other hand, we have $(x \,.\, y) \,.\, z = k_{x.y}(z) = k_{k_x(y)}(z)$. Hence, the associative law is satisfied in $(S, \cdot)$.

(3) Let us put $a = f(e)$, $g(x) = x \,.\, a$ for any $x \in S$. Then $g(x) = x \,.\, a = k_x(a) = k_x(f(e)) = f(k_x(e)) = f(x)$ for any $x \in S$. Hence, $g = f$ and (S, f) is the algebra assigned to $(S, \cdot, e)$ with respect to $f(e)$. $\square$

3. Characterization Theorem. *Let (S, f) be an algebra. Then the following assertions are equivalent.*

(i) *(S, f) is an algebra assigned to a monoid $(S, \cdot, e)$ with respect to an element $a \in S$.*

(ii) *(S, f) is suitable.*

(iii) *(S, f) is very suitable.*

Proof. (ii) and (iii) are equivalent by 2.7, (i) implies (ii) by 2.4, (ii) implies (i) by 2. $\square$

An algebra is said to be *connected* if it has exactly one component.

1. Definition (cf [3] Def. 4, [4] Def. 2, [5] p. 107, [1] 2.4, [2] p. 427). Let (S, f) be a finite connected algebra. We put

$$\begin{aligned} Z_f &= \{x \in S;\ \text{there exists } r(x) \in \mathbb{N} - \{0\} \text{ such that } f^{r(x)}(x) = x\},\\ R_f &= \operatorname{card} Z_f,\\ n_f(x) &= \min\{n \in \mathbb{N};\ f^n(x) \in Z_f\} \quad \text{for any} \quad x \in S,\\ n_f &= \max\{n_f(x);\ x \in S\}. \end{aligned}$$

It is easy to see (cf. l.c.) that $R_f > 0$ and that $f^{R_f}(x) = x$ is equivalent to $x \in Z_f$ for any $x \in S$. Furthermore, $x \in S$, $p > 0$, $f^p(x) = x$ imply that $R_f \mid p$ (i.e., R_f divides p).

The following lemma is well-known (cf. [3] Lemma 6, 7; [4] Hilfssatz 2.7, 2.8; [5] 3.4; [1] 3.1, [2] Theorem).

2. Lemma. *If (S, f), (T, g) are finite connected algebras and h a homomorphism of the former into the latter, then $R_g \mid R_f$ and $h[Z_f] \subseteq Z_g$.* $\square$

3. Definition. Let (S, f) be a finite connected algebra, let $e \in S$ be such that $n_f(e) = n_f$. For any $t \in S$, there exists the least possible integer $m \in \mathbb{N}$ such that $f^m(t) \in \boldsymbol{Sb}\, e$; we denote it by $m_e(t)$. Furthermore, there exists the least possible integer $p \in \mathbb{N}$ such that $f^{m_e(t)}(t) = f^p(e)$; we denote it by $p_e(t)$. We put $q_e(t) = p_e(t) - m_e(t)$ for any $t \in S$. If no confusion is possible, we write m, p, q for m_e, p_e, q_e, respectively.

The function q_e will be used to define certain homomorphisms. We shall need some properties of m_e, p_e, q_e.

4. Lemma. *Let (S, f) be a finite connected algebra, let $e \in S$ be such that $n_f(e) = n_f$. Then the following assertions hold.*

(i) *For any $t \in S$, we have $m_e(t) \leqq p_e(t)$, $q_e(t) \geqq 0$.*

(ii) *If $m_e(t) > 0$, then $m_e(f(t)) = m_e(t) - 1$, $p_e(f(t)) = p_e(t)$, $q_e(f(t)) = q_e(t) + 1$.*

(iii) *If $m_e(t) = 0$, then $m_e(f(t)) = 0$ and $p_e(f(t)) \leqq p_e(t) + 1$. If, moreover, $p_e(f(t)) \leqq p_e(t)$, then $t \in Z_f$ and $R_f \mid p_e(t) - p_e(f(t)) + 1$.*

Proof. (1) Let $t \in S$ be arbitrary.

If $f^{m(t)}(t) \in Z_f$, then $m(t) = n_f(t) \leqq n_f = n_f(e) \leqq p(t)$.

Suppose $f^{m(t)}(t) \in S - Z_f$. Then $f^{p(t)}(e) \in S - Z_f$ which implies that $p(t) < n_f(e) = n_f$ and thus $m(t) + n_f - p(t) > 0$. Hence $f^{m(t)+n_f-p(t)-1}(t) = f^{n_f-1-p(t)}(f^{m(t)}(t)) = f^{n_f-1-p(t)}(f^{p(t)}(e)) = f^{n_f-1}(e) = f^{n_f(e)-1}(e) \in S - Z_f$, which yields $m(t) + n_f - p(t) - 1 < n_f(t)$. Furthermore, $f^{m(t)+n_f-p(t)}(t) = f^{n_f(e)}(e) \in Z_f$ and, thus, $m(t) + n_f - p(t) \geqq n_f(t)$. Hence $m(t) + n_f - p(t) = n_f(t) \leqq n_f$, which implies that $m(t) \leqq p(t)$.

We have proved $m(t) \leqq p(t)$ and, therefore, $q(t) \geqq 0$ for any $t \in S$, which is (i).

(2) Let us have $t \in S$, $m(t) > 0$.

Then $f^{p(t)}(e) = f^{m(t)-1}(f(t))$ and thus $m(f(t)) = m(t) - 1$, $p(f(t)) = p(t)$, which yields $q(f(t)) = q(t) + 1$. We have proved (ii).

(3) Suppose $t \in S$, $m(t) = 0$.

Then $f^{p(t)}(e) = t$ and thus $f^{p(t)+1}(e) = f(t)$, which implies $m(f(t)) = 0$ and $p(f(t)) \leqq p(t) + 1$. By 3, we obtain $f^{p(f(t))}(e) = f(t)$.

If, moreover, $p(f(t)) \leqq p(t)$, we obtain $t = f^{p(t)}(e) = f^{p(t)-p(f(t))}(f^{p(f(t))}(e)) = f^{p(t)-p(f(t))}(f(t)) = f^{p(t)-p(f(t))+1}(t)$. Since $p(t) - p(f(t)) + 1 \geqq 1$, we obtain $t \in Z_f$ and $R_f \mid p(t) - p(f(t)) + 1$.

We have proved (iii). □

5. Lemma (cf. [5] 3.9, 3.10). *Let (S, f), (T, g) be finite connected algebras such that $R_g \mid R_f$ and $n_f \geqq n_g$. Let $e \in S$ be such that $n_f(e) = n_f$ and let $x \in T$ be arbitrary. For any $t \in S$, we put $h_x(t) = g^{q_e(t)}(x)$. Then the following assertions hold.*

(i) $h_x[Z_f] \subseteq Z_g$.

(ii) *h_x is a homomorphism of (S, f) into (T, g) such that $h_x(e) = x$.*

Proof. (1) By $4(i)$, h_x is a mapping of S into T.

Let us have $t \in Z_f$. Then $f^{R_f}(t) = t$ and $m(t) = m_e(t) = 0$, which yields $g^{R_f}(h_x(t)) = g^{R_f}(g^{p(t)}(x)) = g^{R_f + p(t)}(x)$ where $p = p_e$. Since $f^{p(t)}(e) = t$, we obtain $p(t) \geqq n_f(e) = n_f \geqq n_g \geqq n_g(x)$, which implies that $g^{p(t)}(x) \in Z_g$. Since $R_g \mid R_f$, we have $g^{R_f + p(t)}(x) = g^{p(t)}(x) = h_x(t)$. We have proved $g^{R_f}(h_x(t)) = h_x(t)$, which means $h_x(t) \in Z_g$. We have proved (i).

(2) Let $t \in S$ be arbitrary.

If $m(t) > 0$, we obtain $h_x(f(t)) = g^{q(f(t))}(x) = g^{q(t)+1}(x) = g(g^{q(t)}(x)) = g(h_x(t))$ by $4(ii)$.

Suppose that $m(t) = 0$. Then $m(f(t)) = 0$ by 4(iii) and the following cases can occur.

(a) $p(f(t)) = p(t) + 1$.

Then $q(f(t)) = q(t) + 1$ and we obtain $h_x(f(t)) = g(h_x(t))$ similarly as above.

(b) $p(f(t)) \leqq p(t)$.

Then $t \in Z_f$ and $R_f \mid p(t) - p(f(t)) + 1$ by 4(iii). Thus $f(t) \in Z_f$ and $h_x(f(t)) \in Z_g$ by (i). This yields $h_x(f(t)) = g^{p(t)-p(f(t))+1}(h_x(f(t))) = g^{p(t)-p(f(t))+1}(g^{p(f(t))}(x)) = g^{p(t)+1}(x) = g(g^{q(t)}(x)) = g(h_x(t))$.

We have proved that $h_x(f(t)) = g(h_x(t))$ for any $t \in S$. Thus we obtain (ii). □

6. Lemma. *Let (S, f), (T, g) be finite connected algebras such that $R_g \mid R_f$ and $n_f \geqq n_g$. Let $e \in S$ be such that $n_f(e) = n_f$. Let $x \in T$, $y \in S$ be arbitrary, put $z =$*

$= g^{q_e(y)}(x)$, $h_x(t) = g^{q_e(t)}(x)$, $h_y(t) = f^{q_e(t)}(y)$, $h_z(t) = g^{q_e(t)}(z)$ *for any* $t \in S$. *Then* $h_z = h_x \circ h_y$.

Proof. Clearly, $z \in T$; by 5(ii), h_y is an endomorphism of (S, f) and h_x, h_z are homomorphisms of (S, f) into (T, g). We have $h_z(t) = g^{q_e(t)}(z) = g^{q_e(t)}(h_x(y)) = h_x(f^{q_e(t)}(y)) = h_x(h_y(t))$ for any $t \in S$. $\square$

7. Theorem. *Let* (S, f) *be a finite algebra,* $((S_i, f_i))_{i \in I}$ *the family of all its components. Then the following two conditions are equivalent.*

(i) (S, f) *is suitable.*

(ii) *There exists* $i_0 \in I$ *such that* $R_{f_i} \mid R_{f_{i_0}}$ *and* $n_{f_i} \leqq n_{f_{i_0}}$ *for any* $i \in I$.

Proof. (1) If (S, f) is suitable, then there exist a marked element $e \in S$ and a selective monoid $(E, \circ, \mathrm{id}_S)$ corresponding to e, where $E = \{h_x;\ x \in S\}$, h_x being an endomorphism of (S, f) with $h_x(e) = x$ for any $x \in S$. Let $i_0 \in I$ be such an element that $e \in S_{i_0}$. We put $k_x = h_x \restriction S_{i_0}$. Since k_x is a homomorphism of (S_{i_0}, f_{i_0}) into (S_i, f_i) for any $i \in I$ and $x \in S_i$, we have $R_{f_i} \mid R_{f_{i_0}}$ for any $i \in I$ by 2. Let $i \in I$ and $x \in S_i$ be arbitrary. Put $n_0 = n_{f_{i_0}}(e)$. Then $f_{i_0}^{n_0}(e) \in Z_{f_{i_0}}$ and hence $f_i^{n_0}(x) = f_i^{n_0}(h_x(e)) = k_x(f_{i_0}^{n_0}(e)) \in Z_{f_i}$ by 2. Thus $n_{f_i}(x) \leqq n_0 = n_{f_{i_0}}(e)$. This implies that $n_{f_i} \leqq n_{f_{i_0}}(e)$ for any $i \in I$. Particularly, $n_{f_{i_0}} \leqq n_{f_{i_0}}(e)$ and hence $n_{f_{i_0}} = n_{f_{i_0}}(e)$. Thus $n_{f_i} \leqq n_{f_{i_0}}$ for any $i \in I$. We have proved that (i) implies (ii).

(2) Let (ii) hold. We take an element $e \in S_{i_0}$ such that $n_{f_{i_0}}(e) = n_{f_{i_0}}$.

Let $i \in I$ and $x \in S_i$, $x \neq e$, be arbitrary. For any $t \in S_{i_0}$, we put $h_x(t) = f_i^{q_e(t)}(x)$. By 5(ii), h_x is a homomorphism of (S_{i_0}, f_{i_0}) into (S_i, f_i) such that $h_x(e) = x$. We set $h_e = \mathrm{id}_{S_{i_0}}$, $E = \{\boldsymbol{nl}\, h_x;\ x \in S\}$. Clearly, $\boldsymbol{nl}\, h_e = \mathrm{id}_S$. Furthermore, for any $x \in S$, $\boldsymbol{nl}\, h_x$ is the only endomorphism in E that sends e to x.

Let $x, y \in S$. We have two possibilities.

(A) If either $x = e$ or $y = e$, then clearly $\boldsymbol{nl}\, h_x \circ \boldsymbol{nl}\, h_y$ coincides either with $\boldsymbol{nl}\, h_x$ or with $\boldsymbol{nl}\, h_y$ and, consequently, is in E.

(B) Suppose $x \neq e \neq y$. Two cases may occur.

(a) If $y \in S - S_{i_0}$, then $\boldsymbol{nl}\, h_y[S] \subseteq S - S_{i_0}$ and $\boldsymbol{nl}\, h_x \restriction (S - S_{i_0}) = \mathrm{id}_{S - S_{i_0}}$. Thus $\boldsymbol{nl}\, h_x \circ \boldsymbol{nl}\, h_y = \boldsymbol{nl}\, h_y$.

(b) If $y \in S_{i_0}$, then for any $t \in S_{i_0}$ we obtain $(\boldsymbol{nl}\, h_x \circ \boldsymbol{nl}\, h_y)(t) = \boldsymbol{nl}\, h_x(h_y(t)) = h_x(h_y(t)) = h_{h_x(y)}(t) = \boldsymbol{nl}\, h_{h_x(y)}(t)$ by 6. For any $t \in S - S_{i_0}$ we obtain $(\boldsymbol{nl}\, h_x \circ \boldsymbol{nl}\, h_y)(t) = \boldsymbol{nl}\, h_x(\boldsymbol{nl}\, h_y(t)) = \boldsymbol{nl}\, h_x(t) = t = \boldsymbol{nl}\, h_{h_x(y)}(t)$. We have proved that $\boldsymbol{nl}\, h_x \circ \boldsymbol{nl}\, h_y = \boldsymbol{nl}\, h_{h_x(y)}$.

Thus $x \neq e \neq y$ implies that $\boldsymbol{nl}\, h_x \circ \boldsymbol{nl}\, h_y$ is in E.

Hence $(E, \circ, \mathrm{id}_S)$ is a selective monoid of (S, f) corresponding to e. Thus (S, f) is very suitable and, therefore, suitable by 2.8. $\square$

By 3.3 and 7, we obtain

3. Characterization Theorem (cf. [6]). *Let (S, f) be a finite algebra, $((S, f_i))_{i \in I}$ the family of all its components. Then the following conditions are equivalent:*

(i) *(S, f) is an algebra assigned to a monoid $(S, \cdot, e)$ with respect to an element $a \in S$.*

(ii) *(S, f) is suitable.*

(iii) *(S, f) is very suitable.*

(iv) *There exists $i_0 \in I$ such that $R_{f_i} \mid R_{f_{i_0}}$ and $n_{f_i} \leqq n_{f_{i_0}}$ for any $i \in I$.* $\square$

5. EXAMPLES AND REMARKS

1. Example. Let G be a cyclic group of order 4, $G = \{1, a, a^2, a^3\}$. The diagrams of algebras assigned to G with respect to $1, a, a^2, a^3$, are, respectively, as follows.

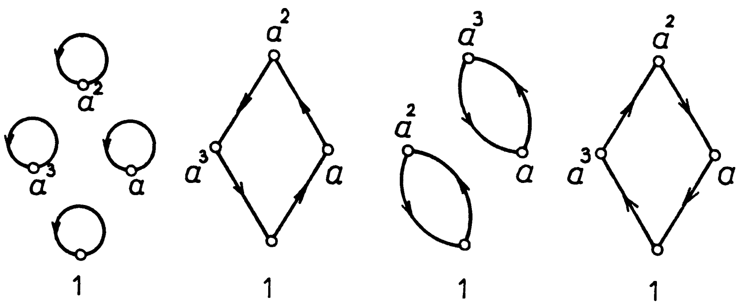

Fig. 1.

2. Example. Let (A, f) be an algebra with the following diagram.

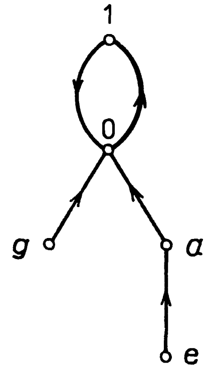

Fig. 2.

We assume e to be the marked element. Then the values of the functions m_e, p_n, q_e are in the first table. The second table contains the values of functions h_x, where $h_x(t) = f^{q_e(t)}(x)$ for any $t \in A$ and any $x \in A$ with $x \neq e$, $h_e = \mathrm{id}_A$.

	e	a	0	1	g
m_e	0	0	0	0	1
p_e	0	1	2	3	2
q_e	0	1	2	3	1

	e	a	0	1	g
h_e	e	a	0	1	g
h_a	a	0	1	0	0
h_0	0	1	0	1	1
h_1	1	0	1	0	0
h_g	g	0	1	0	0

We now present the table of the operation $\circ$ of the monoid $(E, \circ, \mathrm{id}_A)$, where $E = \{h_x;\ x \in A\}$. The fourth table gives the operation $\cdot$ of the monoid $(A, \cdot, e)$ associated with (A, f) with respect to e and $(E, \circ, \mathrm{id}_A)$.

	h_e	h_a	h_0	h_1	h_g
h_e	h_e	h_a	h_0	h_1	h_g
h_a	h_a	h_0	h_1	h_0	h_0
h_0	h_0	h_1	h_0	h_1	h_1
h_1	h_1	h_0	h_1	h_0	h_0
h_g	h_g	h_0	h_1	h_0	h_0

	e	a	0	1	g
e	e	a	0	1	g
a	a	0	1	0	0
0	0	1	0	1	1
1	1	0	1	0	0
g	g	0	1	0	0

Clearly, the algebra assigned to this monoid with respect to a coincides with (A, f).

3. Remarks. Let us cancel the element e in the algebra and in the monoid of 2. We obtain a subalgebra of (A, f) and a subsemigroup (not a submonoid!) of the constructed monoid. Clearly, this subalgebra is assigned to the subsemigroup in the sense of [6].

Let us start with this subalgebra and apply the construction described in the last paragraph of [6]. Then $tg = g$ for any t. On the other hand, $a^2 = 0$, $0a = 1$, $1a = 0$, $ga = 0$ according to the diagram of the algebra. Using the associative law we obtain $(1g)\,1 = g1 = g(0a) = (g0)\,a = (ga^2)\,a = ((ga)\,a)\,a = (0a)\,a = 1a = 0$. On the other hand, $1(g1) = 10$ as we have seen; furthermore, $10 = 1a^2 = (1a)\,a = 0a = 1$. Thus $(1g)\,1 = 0 \neq 1 = 1(g1)$ and hence the operation is not associative. Thus the simple construction of [6] gives only a groupoid associated with the algebra. The construction of a semigroup requires more complicated methods as we have seen in Section 4 of the present article. Nevertheless, the formulation of Theorem in [6] is correct.

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