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ON UPPER EMBEDDABILITY OF COMPLEMENTARY GRAPHS

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In this note, by a graph we shall mean a finite undirected graph with no loop or multiple edge. We shall prove that for every graph G, either G or its complement is upper embeddable.

The study of 2-cell embeddings of connected graphs in closed surfaces belongs to very fruitful branches of the graph theory (see books [7] and [10], or also Chapter 5 in [1] and surveys [6] and [9]). If there exists a 2-cell embedding of a connected graph G in the nonorientable closed surface of genus n, then $n \leqq |E(G)| - |V(G)| + 1$, where $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. If there exists a 2-cell embedding of a connected graph G in the orientable closed surface of genus n, then $n \leqq [(|E(G)| - |V(G)| + 1)/2]$. Ringel [8] proved that for every connected graph G, there exists a 2-cell embedding of G in the nonorientable closed surface of genus $|E(G)| - |V(G)| + 1$. That theorem has not a direct analogue for orientable closed surfaces. A graph G is said to be *upper embeddable* if it is connected and there exists a 2-cell embedding of G in the orientable closed surface of genus $[(|E(G)| - |V(G)| + 1)/2]$.

The following theorem gives two characterizations of upper embeddable graphs. Let H be a graph; we denote by $c(H)$ the number of components of H; moreover, we denote by $b(H)$ the number of components F of H with the property that $|E(F)| \equiv |V(F)| \pmod 2$.

Theorem 0. *Let G be a connected graph. The following statements are equivalent:*

(1) *G is upper embeddable;*

(2) *there exists a spanning tree T of G with the property that at most one component of $G - E(T)$ has an odd number of edges;*

(3) *for every $A \subseteq E(G)$, $b(G - A) + c(G - A) - 2 \leqq |A|$.*

The equivalence (1) $\Leftrightarrow$ (2) was proved independently in [2], [3], and [11]. The equivalence (2) $\Leftrightarrow$ (3) was proved in [4]. (Note that Theorem 0 is true even for pseudographs.)

In the present note we wish to prove that for every graph G, either G or its complement is upper embeddable. We shall first prove three lemmas.

Lemma 1. *Let G be a connected graph. Assume that there exists $A \subseteq E(G)$ such that $b(G - A) + c(G - A) - 2 > |A|$. Then $b(G - A) \geqq 2$.*

Proof. Since G is connected, $|A| \geqq c(G - A) - 1$. The desired result follows.

If G is a graph and $v \in V(G)$, then $\deg_G v$ denotes the degree of v in G.

Lemma 2. *Let G be a connected graph with $p \geqq 1$ vertices. Assume that there exists $v \in V(G)$ such that $\deg_G v \geqq p - 2$. Then G is upper embeddable.*

Proof. Assume that G is not upper embeddable. As follows from the equivalence $(1) \Leftrightarrow (3)$, there exists $A \subseteq E(G)$ such that $b(G - A) + c(G - A) - 2 > |A|$. According to Lemma 1, $b(G - A) \geqq 2$. Let F denote the component of $G - A$ with the property that $v \in V(F)$. Obviously, $|A| \geqq |V(G) - V(F)| - 1$. Since at least $b(G - A)$ components of $G - A$ contain cycles,

$$|V(G) - V(F)| - 1 \geqq (c(G - A) - 1) + 2(b(G - A) - 1) - 1 =$$
$$= b(G - A) + c(G - A) - 2 + (b(G - A) - 2) \geqq$$
$$\geqq b(G - A) + c(G - A) - 2\,,$$

which is a contradiction. Thus the lemma is proved.

Let G be a graph. If U_1 and U_2 are nonempty disjoint subsets of $V(G)$ such that $V(G) = U_1 \cup U_2$ and $|U_1| \geqq 3 \leqq |U_2|$, then we shall say that $\{U_1, U_2\}$ is a γ-partition of G. If $\{U_1, U_2\}$ is a γ-partition of G, then we denote by $E(G, \{U_1, U_2\})$ the set of edges $e \in E(G)$ with the property that one of the end-vertices of e belongs to U_1, and the other end-vertex belongs to U_2.

Lemma 3. *Let G be a connected graph with $p \geqq 1$ vertices. Then*

(i) *if for every γ-partition $\{U_1, U_2\}$ of G, $|E(G, \{U_1, U_2\})| \geqq 4$, then G is upper embeddable;*

(ii) *if $p \leqq 8$ and for every γ-partition $\{U_1, U_2\}$ of G, $|E(G, \{U_1, U_2\})| \geqq 2$, then G is upper embeddable.*

Proof. Assume that G is not upper embeddable. According to the equivalence $(1) \Leftrightarrow (3)$, there exists $A \subseteq E(G)$ such that $b(G - A) + c(G - A) - 2 > |A|$. As follows from Lemma 1, $c(G - A) \geqq b(G - A) \geqq 2$. Since G is connected, there exists $A' \subseteq A$ such that $b(G - A') = c(G - A')$ and $2(c(G - A') - 1) > |A'|$. Since $b(G - A') = c(G - A')$, every component of $G - A'$ contains at least three vertices. Hence, for every component F of $G - A'$, $\{V(F), V(G) - V(F)\}$ is a γ-partition of G.

If (i) holds, then $|A'| \geqq 2c(G - A')$, which is a contradiction. If (ii) holds, then $c(G - A') = 2$, and thus $|A'| \geqq 2$, which is a contradiction, too. Hence, the lemma is proved.

Remark. The statement (i) of Lemma 3 is an immediate consequence of the following theorem which is due to Payan and Xuong [5]: Every cyclically 4-edge connected graph is upper embeddable.

If G is a graph, then we denote by $\bar{G}$ the complement of G. We shall now prove the main result of this note.

Theorem 1. *If G is a graph, then either G or $\bar{G}$ is upper embeddable.*

Proof. Let G be a graph with $p \geqq 1$ vertices. According to Theorem 2.5 in [1], either G or $\bar{G}$ is connected. Without loos of generality we shall assume that G is connected.

Assume that $\bar{G}$ is not upper embeddable. We distinguish two cases:

Case 1. Assume that $\bar{G}$ is disconnected and it has a component with at most two vertices. Then G contains a vertex of degree $\geqq p - 2$. According to Lemma 2, G is upper embeddable.

Case 2. Assume that either (a) $\bar{G}$ is disconnected and every component of $\bar{G}$ contains at least three vertices, or (b) $\bar{G}$ is connected. It follows from the assumption (a) (if $\bar{G}$ is disconnected) or from Lemma 3 (if $\bar{G}$ is connected) that there exists a γ-partition $\{W_1, W_2\}$ of $\bar{G}$ such that $|E(\bar{G}, \{W_1, W_2\})| \leqq 3$, and if $p \leqq 8$, then $|E(\bar{G}, \{W_1, W_2\})| \leqq 1$. We wish to show that G is upper embeddable. On the contrary, we shall assume that G is not upper embeddable. Since G is connected, it follows from Lemma 3 that there exists a γ-partition $\{U_1, U_2\}$ of G such that $|E(G, \{U_1, U_2\})| \leqq 3$, and if $p \leqq 8$, then $|E(G, \{U_1, U_2\})| \leqq 1$. Denote $r = |E(G, \{U_1, U_2\}) \cup E(\bar{G}, \{W_1, W_2\})|$. Since $E(G, \{U_1, U_2\}) \cap E(\bar{G}, \{W_1, W_2\}| = \emptyset$, it holds that

$$r \leqq 6 , \quad \text{and if} \quad p \leqq 8 , \quad \text{then} \quad r \leqq 2 .$$

Denote $p_{ij} = |U_i \cap W_j|$ for $i, j = 1, 2$. Obviously, $p_{i1} + p_{i2} \geqq 3 \leqq p_{1j} + p_{2j}$ for $i, j = 1, 2$. Let v and v' be arbitrary distinct vertices of G; if either $v \in U_1 \cap W_1$, $v' \in U_2 \cap W_2$, or $v \in U_1 \cap W_2$, $v' \in U_2 \cap W_1$, then $vv' \in E(G, \{U_1, U_2\}) \cup E(\bar{G}, \{W_1, W_2\})$. Therefore,

$$p_{11}p_{22} + p_{12}p_{21} \leqq r .$$

Without loss of generality we shall assume that $p_{11} = \min \{p_{ij}; i, j = 1, 2\}$.

If $p_{11} = 0$, then $p_{12} \geqq 3 \leqq p_{21}$, and thus $r \geqq 9$, which is a contradiction. If $p_{11} \geqq 2$, then $r \geqq 8$, which is a contradiction, too. Let $p_{11} = 1$. Then $p_{12} \geqq 2 \leqq p_{21}$. If either $p_{12} \geqq 3$ or $p_{21} \geqq 3$, then $r \geqq 7$, which is a contradiction. Let $p_{12} = 2 = p_{21}$. Then $r \geqq 5$. This implies that $p \geqq 9$. Thus $p_{22} \geqq 4$. Therefore $r \geqq 8$, which is a contradiction.

This means that G is upper embeddable, which completes the proof.

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