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ON CERTAIN SPACES OF TRANSFORMATIONS OF INFINITE SERIES

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1. THE SYSTEMS $\mathfrak{M}_\infty$ AND $\mathfrak{M}$ OF TRANSFORMATIONS OF INFINITE SERIES

In this part of the paper two systems of transformations of series of a certain type are defined and their properties are studied.

Definition 1,1. $\mathfrak{A}$ denotes the system of all series $A = \sum_{n=1}^{\infty} t_n$, $t_n \geqq 0$ $(n = 1, 2, 3, \ldots)$, $\sum_{n=1}^{\infty} t_n \leqq 1$.

Definition 1,2. Let $c \geqq 0$. $\mathfrak{M}_c$ denotes the system of all functions φ defined on $\langle 0, 1 \rangle$ for which the following condition is fulfilled: If $A = \sum_{n=1}^{\infty} t_n \in \mathfrak{A}$, then $\sum_{n=1}^{\infty} \varphi(t_n)$ is convergent and $\left| \sum_{n=1}^{\infty} \varphi(t_n) \right| \leqq c$.

Definition 1,3. Let $\mathfrak{M}_\infty = \bigcup_{c \geqq 0} \mathfrak{M}_c$. Further, let $\mathfrak{M}$ denote the set of all such functions φ defined on $\langle 0, 1 \rangle$ for which the following is true: If $A = \sum_{n=1}^{\infty} t_n \in \mathfrak{A}$ then $\sum_{n=1}^{\infty} \varphi(t_n)$ is convergent.

Remark 1,1. To be short we shall write $\varphi\{A\}$ instead of $\sum_{n=1}^{\infty} \varphi(t_n)$ $(A = \sum_{n=1}^{\infty} t_n \in \mathfrak{A})$.

The set $\mathfrak{M}_\infty$ represents a system of functions preserving, in a way uniformly, the convergence of the series of the system $\mathfrak{A}$. In the paper [1] R. RADO showed that if a real function f defined for $x \in (-\infty, +\infty)$ preserves the convergence of all convergent series $\sum_{n=1}^{\infty} x_n$ (x_n are real numbers) then there exist positive numbers k, δ $(k = k(f), \delta = \delta(f))$ such that for $|x| < \delta$, $f(x) = kx$ (to be short, we say f is linear in a neighbourhood of the point 0). Evidently, the converse is also true. Let us note that if $\varphi \in \mathfrak{M}_\infty$, then φ may not be linear in a right neighbourhood of the point 0.

E.g., if we put $\varphi(0) = 0$ and $\varphi(x) = x \sin(1/x)$ for $x \in (0, 1\rangle$, then for each $A = \sum_{n=1}^{\infty} t_n \in \in \mathfrak{A}$, $|\varphi\{A\}| \leqq \sum_{n=1}^{\infty} |\varphi(t_n)| \leqq \sum_{n=1}^{\infty} t_n \leqq 1$, consequently $\varphi \in \mathfrak{M}_1 \subset \mathfrak{M}_\infty$ and evidently φ is not linear in any right neighbourhood of the point 0. But the following result ressembling in a way the result of Rado and characterising the elements of $\mathfrak{M}_\infty$ can be proved.

Theorem 1,1 *Let φ be defined on the interval $\langle 0, 1\rangle$. Then φ belongs to $\mathfrak{M}_\infty$ if and only if all the following conditions are satisfied:*

(1) $$\varphi(0) = 0$$

(2) $$\limsup_{t \to 0+} \frac{|\varphi(t)|}{t} < +\infty$$

(3) $$\varphi \text{ is bounded on } \langle 0, 1\rangle$$

Corollary. *If $\varphi \in \mathfrak{M}_\infty$, then* $\lim_{t \to 0+} \varphi(t) = \varphi(0) = 0$.

Proof. If φ is defined on $\langle 0, 1\rangle$ and has the properties (1), (2), (3) then obviously $\varphi \in \mathfrak{M}_\infty$. Let $\varphi \in \mathfrak{M}_\infty$. Then in view of the definition of the set $\mathfrak{M}_\infty$, there exists $c \geqq 0$ such that $\varphi \in \mathfrak{M}_c$. It is easily seen that the magnitudes of the function φ on the intervals $(2^{-n}, 2^{-n+1}\rangle$ $(n = 1, 2, \ldots)$ are not greater than $c \,.\, 2^{-n+1}$. In fact, if at a point t of the interval $(2^{-n}, 2^{-n+1}\rangle$ the value $\varphi(t)$ were either greater than $c \,.\, 2^{-n+1}$, or less than $-c \,.\, 2^{-n+1}$ then the series

$$A = \underbrace{t + t + \ldots + t}_{2^{n-1} \text{ times}} + 0 + 0 + \ldots + 0 + \ldots$$

would belong to $\mathfrak{A}$ and $|\varphi\{A\}| = |2^{n-1} \varphi(t)| > c$. Now, let $t > 0$, $t < 1$. Then $t \in (2^{-n}, 2^{-n+1}\rangle$ for a suitably chosen n and consequently, $|\varphi(t)| \leqq c \,.\, 2^{-n+1}$. Thus, we have

$$\frac{|\varphi(t)|}{t} \leqq \frac{c \,.\, 2^{-n+1}}{2^{-n}} = 2c < +\infty$$

and therefore $\limsup_{t \to 0+} |\varphi(t)|/t \leqq 2c < +\infty$, hence, (2) is valid. The validity of (3) is also easily seen. If $\varphi(0) \neq 0$ were the case, then the series $\sum_{n=1}^{\infty} \varphi(0)$ would not be convergent in spite of the fact that $A = \sum_{n=1}^{\infty} 0 \in \mathfrak{A}$. Hence (1) is also true and the proof is finished.

Remark 1,2. From the proof of the above theorem it follows that $\varphi(0) = 0$, for $\varphi \in \mathfrak{M}$.

Remark 1,3. If $\varphi \in \mathfrak{M}$, then $\limsup_{t\to 0+} |\varphi(t)|/t < +\infty$ may not be necessarily take true. We may define the function φ e.g. in this way: $\varphi(2^{-n}) = n \cdot 2^{-n}$ $(n = 1, 2, 3, \ldots)$ and $\varphi(t) = 0$ for each $t \in \langle 0, 1\rangle$, $t \neq 2^{-n}$ $(n = 1, 2, 3, \ldots)$. Evidently, $\varphi \in \mathfrak{M}$ and $\limsup_{t\to 0+} |\varphi(t)|/t = +\infty$. Moreover, from this example it follows that $\mathfrak{M} \neq \mathfrak{M}_\infty$, thus the set $\mathfrak{M}_\infty$ is a proper part of the set $\mathfrak{M}$. As we have seen each function of the system $\mathfrak{M}_\infty$ is bounded, but the system $\mathfrak{M}$ contains also unbounded functions. For example, the function φ defined by $\varphi(x) = (1 - x)^{-1}$ for $x \in \langle 2^{-1}, 1)$, $\varphi(1) = 0$, $\varphi(x) = x$ for $x \in \langle 0, 2^{-1})$ belongs to $\mathfrak{M}$ and it is not bounded on $\langle 0, 1\rangle$.

Theorem 1,2. *If $\varphi \in \mathfrak{M}$, then $\lim_{t\to 0+} \varphi(t) = \varphi(0) = 0$.*

Proof. Let $\varphi \in \mathfrak{M}$ and let the assertion of the theorem is not true. Then there exists an $\varepsilon_0 > 0$ and a sequence of positive numbers $\{\delta_n\}_1^\infty$ such that $\delta_n \leqq 2^{-n}$ and $|\varphi(\delta_n)| \geqq \varepsilon_0$. Let e.g. $\varphi(\delta_n) > 0$ for infinitely many n (for $n \in N'$). Then $\sum_{n\in N'} \delta_n \in \mathfrak{A}$, but the series $\sum_{n\in N'} \varphi(\delta_n)$ does not converge, thus $\varphi \notin \mathfrak{M}$.

Theorem 1,3. *The sets $\mathfrak{M}_\infty$ and $\mathfrak{M}$ are real vector spaces (under the operations of addition and multiplication by a real number defined in usual way).*

Proof. As for $\mathfrak{M}$ the assertion follows immediately from the fundamental theorems concerning the convergent series and for $\mathfrak{M}_\infty$ the assertion follows from the Theorem 1,1.

Now, let us define on $\mathfrak{M}_\infty$ a nonnegative real function $\|\varphi\|$ in this way:

Definition 1,4. $\|\varphi\| = \sup_{A\in\mathfrak{A}} |\varphi\{A\}|$.

It is not difficult to verify that the function $\|\varphi\|$ fulfills the axioms of homogeneous norm. In fact,

(a) $\|\varphi\| \geqq 0$ and if $\varphi(t) = 0$ for each $t \in \langle 0, 1\rangle$, then $\|\varphi\| = 0$. If there exists a number $t \in \langle 0, 1\rangle$ such that $\varphi(t) \neq 0$, then $A = t + 0 + 0 + \ldots + 0 + \ldots \in \mathfrak{A}$ and $|\varphi\{A\}| > 0$, consequently $\|\varphi\| > 0$, $\|\varphi\| \neq 0$.

(b) If k is a real number, then $\|k\varphi\| = \sup_{A\in\mathfrak{A}} |k\varphi\{A\}| = |k| \sup_{A\in\mathfrak{A}} |\varphi\{A\}| = |k| \cdot \|\varphi\|$.

(c) $\|\varphi_1 + \varphi_2\| = \sup_{A\in\mathfrak{A}} |(\varphi_1 + \varphi_2)\{A\}|$, but $|(\varphi_1 + \varphi_2)\{A\}| \leqq |\varphi_1\{A\}| + |\varphi_2\{A\}| \leqq$ $\leqq \|\varphi_1\| + \|\varphi_2\|$. Hence $\|\varphi_1 + \varphi_2\| \leqq \|\varphi_1\| + \|\varphi_2\|$.

Thus, the set $\mathfrak{M}_\infty$ with this norm is a linear normed space.

Theorem 1,4. *Let φ_n $(n = 1, 2, 3, \ldots)$, φ be functions belonging to $\mathfrak{M}_\infty$. Let $\|\varphi_n - \varphi\| \to 0$ as $n \to \infty$. Then the sequence $\{\varphi_n\}_1^\infty$ is uniformly convergent on $\langle 0, 1\rangle$ to φ.*

Proof. Let $\lim_{n\to\infty} \|\varphi_n - \varphi\| = 0$, $\varphi_n, \varphi \in \mathfrak{M}_\infty$. Let $\varepsilon > 0$. Then there exists $n_0(\varepsilon)$ such that for $n \geqq n_0(\varepsilon)$ $\|\varphi_n - \varphi\| < \varepsilon$. Let A traverse over all series of the form $t + 0 + 0 + \ldots + 0 + \ldots$, $t \in \langle 0, 1\rangle$. Evidently $A \in \mathfrak{A}$ and $|\varphi_n\{A\} - \varphi\{A\}| < \varepsilon$ for each A of this kind, hence for each $t \in \langle 0, 1\rangle$ $|\varphi_n(t) - \varphi(t)| < \varepsilon$ whenever $n \geqq n_0(\varepsilon)$, so $\{\varphi_n\}_1^\infty$ is uniformly convergent on $\langle 0, 1\rangle$ to φ.

Remark 1,4. If $\varphi_n, \varphi \in \mathfrak{M}_\infty$ and if $\{\varphi_n\}_1^\infty$ is uniformly convergent on $\langle 0, 1\rangle$ to φ, the convergence in the sense of the norm may not take place. We shall show this fact on the following example: Let us put $\varphi_n(t) = t/(1 + n^2t^2)$ for $t \in \langle 0, 1\rangle$ and $\varphi(t) = 0$ for each $t \in \langle 0, 1\rangle$. Evidently, $\varphi_n, \varphi \in \mathfrak{M}_\infty$, $0 \leqq \varphi_n(t) \leqq t$ for each t and $\varphi_n(t) = 2nt/[2n(1 + n^2t^2)] \leqq 2^{-n}$, hence $\{\varphi_n\}_1^\infty$ is uniformly convergent to φ on $\langle 0, 1\rangle$. Let us put for fixed n $t_1 = 1/n$ where $1 \leqq i \leqq n$ and $t_i = 0$ for $i > n$. Then $A = \sum_{i=1}^\infty t_i \in \mathfrak{A}$, $\varphi_n\{A\} = 1/2$, so that $\|\varphi_n\| \geqq 1/2$ $(n = 1, 2, 3, \ldots)$. Consequently $\{\|\varphi_n - \varphi\|\}_1^\infty = \{\|\varphi_n\|\}_1^\infty$ does not converge to zero.

A simple consequence of the foregoing theorem is the following one.

Theorem 1,5. *The system $\mathfrak{M}_\infty^{(c)}$ of all functions continuous on $\langle 0, 1\rangle$ which belong to $\mathfrak{M}_\infty$ is a closed linear subspace of the space $\mathfrak{M}_\infty$.*

Proof. Evidently it suffices to prove that $\mathfrak{M}_\infty^{(c)}$ is closed in $\mathfrak{M}_\infty$. Let $\varphi_n \in \mathfrak{M}_\infty^{(c)}$ $(n = 1, 2, 3, \ldots)$, $\varphi_n \to \varphi$ (in the norm-convergence), $\varphi \in \mathfrak{M}_\infty$. On the base of foregoing theorem $\{\varphi_n\}_1^\infty$ is uniformly convergent to φ on $\langle 0, 1\rangle$ and hence in view of the known fact from analysis the continuity of φ follows. Hence $\varphi \in \mathfrak{M}_\infty^{(c)}$.

Theorem 1,6. *Let $c \geqq 0$. Then $\mathfrak{M}_c$ is a symmetric, convex and closed set in $\mathfrak{M}_\infty$.*

Proof. The properties of the symmetry and convexity are evident. Now, if $\varphi_n \in \mathfrak{M}_c$ $(n = 1, 2, 3, \ldots)$, $\varphi \in \mathfrak{M}_\infty$ and $\|\varphi_n - \varphi\| \to 0$, then to each $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that for each $n > n_0(\varepsilon)$ and each $A = \sum_{k=1}^\infty t_k \in \mathfrak{A}$

$$\Big|\sum_{k=1}^\infty \varphi(t_k)\Big| - \Big|\sum_{k=1}^\infty \varphi_n(t_k)\Big| \leqq \Big|\sum_{k=1}^\infty \varphi_n(t_k) - \sum_{k=1}^\infty \varphi(t_k)\Big| < \varepsilon\,,$$

hence $\big|\sum_{k=1}^\infty \varphi(t_k)\big| < \|\varphi_n\| + \varepsilon \leqq c + \varepsilon$. Since the inequality holds for each $\varepsilon > 0$, we have $\|\varphi\| \leqq c$, $\varphi \in \mathfrak{M}_c$.

Theorem 1,7. *The space $\mathfrak{M}_\infty$ with the norm $\|\varphi\|$ (see definition 1,4) is a Banach linear normed space.*

Proof. Let $\{\varphi_n\}_1^\infty$ be a fundamental sequence of points of $\mathfrak{M}_\infty$. Let $\eta > 0$. Then in view of the assumption there exists $n_0(\eta)$ such that for $m, n \geqq n_0(\eta)$ $\|\varphi_n - \varphi_m\| < \eta/4$ holds. If A^* travers over the series of the form $t + 0 + 0 + \ldots + 0 + \ldots$,

$t \in \langle 0, 1\rangle$, then from the above mentioned facts, it follows that for each $t \in \langle 0, 1\rangle$ there exists $\varphi(t) = \lim_{n\to\infty} \varphi_n(t)$ and $\{\varphi_n\}_1^\infty$ is uniformly convergent to φ on $\langle 0, 1\rangle$. It will be shown that $\varphi \in \mathfrak{M}_\infty$ and $\|\varphi_n - \varphi\| \to 0$ holds. Let $A = \sum_{k=1}^\infty t_k \in \mathfrak{A}$. To any positive integer λ there exists, in view of the uniform convergence of $\{\varphi_n\}_1^\infty$, a positive integer $n(\lambda) \geqq n_0(\eta)$ such that

$$-\frac{\eta}{4\lambda} < \varphi(t_k) - \varphi_{n(\lambda)}(t_k) < \frac{\eta}{4\lambda} \quad (k = 1,2, \ldots, \lambda)$$

Hence

$$\sum_{k=1}^{\lambda} \varphi_{n(\lambda)}(t_k) - \frac{\eta}{4} < \sum_{k=1}^{\lambda} \varphi(t_k) < \sum_{k=1}^{\lambda} \varphi_{n(\lambda)}(t_k) + \frac{\eta}{4}. \tag{4}$$

If we take into the account that $\|\varphi_{n_0} - \varphi_{n(\lambda)}\| < \eta/4$, we get for the series $t_1 + t_2 + \ldots + t_\lambda + 0 + 0 + \ldots \in \mathfrak{A}$ the following:

$$\sum_{k=1}^{\lambda} \varphi_{n_0}(t_k) - \frac{\eta}{4} < \sum_{k=1}^{\lambda} \varphi_{n(\lambda)}(t_k) < \sum_{k=1}^{\lambda} \varphi_{n_0}(t_k) + \frac{\eta}{4}. \tag{5}$$

From (4) and (5) we get

$$\sum_{k=1}^{\lambda} \varphi_{n_0}(t_k) - \frac{\eta}{2} < \sum_{k=1}^{\lambda} \varphi(t_k) < \sum_{k=1}^{\lambda} \varphi_{n_0}(t_k) + \frac{\eta}{2}. \tag{6}$$

Since (6) is valid for each positive integer λ, we have

$$\left|\liminf_{\lambda\to\infty} \sum_{k=1}^{\lambda} \varphi(t_k) - \limsup_{\lambda\to\infty} \sum_{k=1}^{\lambda} \varphi(t_k)\right| \leqq \eta .$$

From the last inequality valid for each $\eta > 0$, we have

$$\liminf_{\lambda\to\infty} \sum_{k=1}^{\lambda} \varphi(t_k) = \limsup_{\lambda\to\infty} \sum_{k=1}^{\lambda} \varphi(t_k),$$

so that the series $\sum_{k=1}^{\infty} \varphi(t_k)$ is convergent. Simultaneously from (6) it follows that

$$|\varphi\{A\}| \leqq \|\varphi_{n_0}\| + \frac{\eta}{2} \leqq c + \frac{\eta}{2} \quad (\varphi_{n_0} \in \mathfrak{M}_c).$$

Since n_0 is independent of A, we get $\|\varphi\| = \sup_{A\in\mathfrak{A}} |\varphi\{A\}| \leqq c + \eta/2$, so that $\varphi \in \mathfrak{M}_\infty$.

Further from (6) evidently follows for each $n \geqq n_0(\eta)$,

$$\sum_{k=1}^{\infty} \varphi_n(t_k) - \eta < \sum_{k=1}^{\infty} \varphi(t_k) < \sum_{k=1}^{\infty} \varphi_n(t_k) + \eta$$

hence $\|\varphi_n - \varphi\| \leqq \eta$ for $n \geqq n_0(\eta)$, so that $\|\varphi_n - \varphi\| \to 0$. This finishes the proof.

In what follows sets $\mathfrak{M}_\infty$ and $\mathfrak{M}$ will be studied as subspaces of the topological space $\mathfrak{F}$ of all real functions defined on $\langle 0, 1\rangle$. The topology considered in this space will be that which is given by the uniform convergence. $\bar{B}$ will denote the closure of that B in the topology given by uniform convergence.

Theorem 1,8. *Let $\mathfrak{M}_\infty$, $\mathfrak{M}$ and $\mathfrak{F}$ have the previous meaning. Then*

(a) $\mathfrak{M}_\infty$ and $\mathfrak{M}$ are not closed sets.

(b) $\overline{\mathfrak{M}}_\infty \neq \mathfrak{M}$ and none of the relations $\overline{\mathfrak{M}}_\infty \subset \mathfrak{M}$, $\mathfrak{M} \subset \overline{\mathfrak{M}}_\infty$ takes place.

Proof. (a) Let $t_k = 2^{-k}$ $(k = 1, 2, 3, \ldots)$. Let $\{\varphi_n\}_1^\infty$ be defined by $\varphi_n(t) = k^{-1}$ for $t = 2^{-k}$, $k \leqq n$, $\varphi_n(t) = 0$ for the rest of t, $t \in \langle 0, 1\rangle$. Then the sequence $\{\varphi_n\}_1^\infty$ converges to the function defined by $\varphi(t) = k^{-1}$ for $t = 2^{-k}$ $(k = 1, 2, 3, \ldots)$ and $\varphi(t) = 0$ for $t \in \langle 0, 1\rangle, t \neq 2^{-k} (k = 1, 2, \ldots)$. Evidently $\varphi_n \in \mathfrak{M}_\infty \subset \mathfrak{M}$ $(n = 1, 2, \ldots)$ but $\varphi \notin \mathfrak{M}$, because $A = \sum_{k=1}^{\infty} 2^{-k} \in \mathfrak{A}$ and $\varphi\{A\} = +\infty$. This proves (a) and simultaneously the non-validity of $\overline{\mathfrak{M}}_\infty \subset \mathfrak{M}$ is shown.

For concluding the proof it suffices to prove the existence of a function $\varphi \in \mathfrak{M}$ not belonging to $\mathfrak{M}_\infty$. Let us define φ by $\varphi(k/(k + 1)) = k$ for $k = 1, 2, 3, \ldots$ and $\varphi(t) = 0$ for $t \in \langle 0, 1\rangle$, $t \neq k/(k + 1)$ $(k = 1, 2, 3, \ldots)$. Then $\varphi \in \mathfrak{M}$. Actually, if $A = \sum_{n=1}^{\infty} t_n \in \mathfrak{A}$, then at most for two indexes j, r $t_j, t_r \in \langle 2^{-1}, 1\rangle$, so that in the series $\sum_{n=1}^{\infty} \varphi(t_n)$ all but at most two terms are equal to zero. If there existed a sequence $\{\varphi_n\}_1^\infty$, $\varphi_n \in \mathfrak{M}_\infty$ uniformly converging to the function φ then to each $\varepsilon > 0$ there would exist n_0 such that for each $n \geqq n_0$ and for all $t \in \langle 0, 1\rangle$

$$\varphi(t) - \varepsilon < \varphi_n(t) < \varphi(t) + \varepsilon \tag{7}$$

would hold. Let $\varphi_{n_0} \in \mathfrak{M}_{c_0}$ $(c_0 \geqq 0)$. Let us choose $k > c_0 + \varepsilon$ and put $t = k/(k + 1)$, $n = n_0$, then we get from (7)

$$\varphi\left(\frac{k}{k+1}\right) - \varepsilon < \varphi_{n_0}\left(\frac{k}{k+1}\right).$$

Hence

$$c_0 < k - \varepsilon = \varphi\left(\frac{k}{k+1}\right) - \varepsilon < \varphi_{n_0}\left(\frac{k}{k+1}\right).$$

Which is a contradiction, since $\varphi_{n_0} \in \mathfrak{M}_{c_0}$.

Theorem 1,4 shows that the convergence in norm is stronger than the uniform convergence. The following theorem characterises the convergence in norm.

Theorem 1,9. *Let $\{\varphi_n\}_1^\infty$ be a sequence of functions belonging to $\mathfrak{M}_\infty$. Then the sequence $\{\varphi_n\}_1^\infty$ converges to $\varphi \in \mathfrak{M}_\infty$ in the convergence in norm if and only if there exists such right neighbourhood U of the point 0, that the sequence $\{\varphi_n \chi_U\}_{n=1}^\infty$ (χ_U denotes the characteristic function of the set U) converges in the convergence in norm to $\varphi \chi_U$ and $\{\varphi_n\}_1^\infty$ converges uniformly to φ on $\langle 0, 1 \rangle - U$.*

Proof. The necessity follows from the theorem 1,4. We prove the sufficiency. Let U be a right neighbourhood of the point 0, $U = \langle 0, \delta)$; let $\{\varphi_n \chi_U\}_1^\infty$ converges to $\varphi \chi_U$ in the convergence in norm and $\varphi_n \rightrightarrows \varphi$ ($\rightrightarrows$ is the symbol of uniform convergence) on $\langle 0, 1 \rangle - U$. Let $A = \sum_{k=1}^\infty t_k \in \mathfrak{A}$. Then interval $\langle \delta, 1 \rangle = \langle 0, 1 \rangle - U$ contains at most $[\delta^{-1}]$ terms of the sequence $\{t_k\}_1^\infty$. Let $t_{k_1}, t_{k_2}, \ldots, t_{k_s}$ ($k_1 < k_2 < \ldots < k_s$) are those terms. In view of the convergence in norm of $\{\varphi_n \chi_U\}_1^\infty$ we know that to each $\varepsilon > 0$ there exists $n_1(\varepsilon)$ such that $\left|\sum_{i=1}^{\infty}{}' (\varphi_n(t_i) - \varphi(t_i))\right| < \varepsilon/2$ for $n > n_1(\varepsilon)$. The dash appearing at the symbol $\sum$ denotes that the summation is made for $i \neq k_1, k_2, \ldots, k_s$. Further according to the uniform convergence of $\{\varphi_n\}_1^\infty$ on $\langle \delta, 1 \rangle$, the existence of $n_2(\varepsilon)$ follows such that for $n > n_2(\varepsilon)$,

$$|\varphi_n(t) - \varphi(t)| < \frac{\varepsilon}{2[1/\delta]}, \quad t \in \langle \delta, 1 \rangle .$$

Then for $n > \max(n_1/\varepsilon), n_2(\varepsilon))$ we have

$$\left|\sum_{i=1}^{\infty} (\varphi_n(t_i) - \varphi(t_i))\right| \leqq \left|\sum_{i=1}^{\infty}{}' (\varphi_n(t_i) - \varphi(t_i))\right| +$$

$$+ \sum_{j=1}^{s} |\varphi_n(t_{k_j}) - \varphi(t_{k_j})| < \frac{\varepsilon}{2} + s \frac{\varepsilon}{2[1/\delta]} \leqq \varepsilon .$$

From the last inequality the assertion of the theorem immediately follows.

The following theorem is motivated by the preceding one.

Theorem 1,10. *A function φ defined and bounded on $\langle 0, 1 \rangle$ belongs to $\mathfrak{M}_\infty$ if and only if there exists a right neighbourhood U of zero and a function $\varphi_1 \in \mathfrak{M}_\infty$ such that $\varphi(x) = \varphi_1(x)$ for $x \in U$.*

Proof. Evidently it is sufficient to show that if φ is bounded on $\langle 0, 1 \rangle$ and in some neighbourhood $U = \langle 0, \delta)$ of zero $\varphi(x) = \varphi_1(x)$ where $\varphi_1 \in \mathfrak{M}_\infty$, then $\varphi \in \mathfrak{M}_\infty$. Let $|\varphi(x)| \leqq K$ for $x \in \langle 0, 1 \rangle$ and $\varphi_1 \in \mathfrak{M}_c$. Let $A = \sum_{k=1}^\infty t_k \in \mathfrak{A}$. Then in the interval

$\langle\delta, 1\rangle$ is at most $[1/\delta]$ terms of the sequence $\{t_k\}_1^\infty$. Let us denote those terms $t_{k_1}, \ldots, t_{k_s}$, $s \leqq [1/\delta]$. Then

$$\left|\sum_{k=1}^{\infty} \varphi(t_k)\right| \leqq \left|\sum_{i=1}^{\infty}{}' \varphi_1(t_i)\right| + \sum_{j=1}^{s} \left|\varphi(t_{k_j})\right|,$$

where $\sum'$ denotes that i runs over all positive integers with the exception of $k_1, k_2, \ldots, k_s$. According to the fact that $\varphi_1 \in \mathfrak{M}_c$ and in view of boundedness of φ we get

$$\left|\varphi\{A\}\right| = \left|\sum_{k=1}^{\infty} \varphi(t_k)\right| \leqq c + sK$$

where c, s, K on the right side are independent on A. Hence $\varphi \in \mathfrak{M}_\infty$ and $\|\varphi\| \leqq$ $\leqq c + sK$.

If $M(0, 1)$ denotes the metric space of all bounded functions on $\langle 0, 1\rangle$ with the metric $\varrho(f, g) = \sup_{t \in \langle 0,1\rangle} |f(t) - g(t)|$ then according to the theorem 1,1 (see (3)) $\mathfrak{M}_\infty$ is included in $\mathfrak{M}(0, 1)$. The set $\mathfrak{M}_\infty$ is not a closed subset of $M(0, 1)$ (see theorem 1,8, the proof of (a)). But the following theorem may be proved.

Theorem 1,11. *Let $M_1(0, 1)$ be the set of all the functions $\varphi \in M(0, 1)$ for which $\lim_{t\to 0+} \varphi(t) = \varphi(0) = 0$. Then $\overline{\mathfrak{M}}_\infty = M_1(0,1)$ ($\overline{\mathfrak{M}}_\infty$ denotes the closure in the space $M(0, 1)$).*

Proof. We have $\mathfrak{M}_\infty \subset M_1(0, 1) = \overline{M_1(0, 1)}$ (see the corollary following theorem 1,1), so that $\overline{\mathfrak{M}}_\infty \subset M_1(0, 1)$. Now, let $\varphi \in M_1(0, 1)$. Let us define φ_n ($n =$ $= 1, 2, 3, \ldots$) by $\varphi_n(t) = 0$ for $t \in \langle 0, 1/n\rangle$ and $\varphi_n(t) = \varphi(t)$ for $t \in (1/n, 1\rangle$. In view of Theorem 1,10, $\varphi_n \in \mathfrak{M}_\infty$ ($n = 1, 2, 3, \ldots$) and evidently $\varphi_n \rightrightarrows \varphi$ on $\langle 0, 1\rangle$. Hence $M_1(0, 1) \subset \overline{\mathfrak{M}}_\infty$. Thus $\overline{\mathfrak{M}}_\infty = M_1(0, 1)$ is proved.

In connection with the above mentioned Rado's result let $\mathfrak{L}$ denote the set of all such functions φ which have the following property: $\varphi \in \mathfrak{M}_\infty$ and there exists $\delta =$ $= \delta(\varphi)$ such that φ is linear on the interval $\langle 0, \delta(\varphi)\rangle$. As we have seen $\mathfrak{L} \neq \mathfrak{M}_\infty$. But it is not difficult to prove the following theorem showing that as for the approximations in the space $M(0, 1)$ the sets $\mathfrak{L}(\subset \mathfrak{M}_\infty)$ and $\mathfrak{M}_\infty$ are of the same value.

Theorem 1,12. *Let $\mathfrak{L}$ and $\mathfrak{M}_\infty$ have the previous meaning. Then $\overline{\mathfrak{L}} = \overline{\mathfrak{M}}_\infty$ $(= M_1(0, 1))$ (in the sense of uniform convergence).*

Proof. Let $f \in \mathfrak{M}_\infty$. Then to each positive integer n, there exists an interval $\langle 0, \delta_n\rangle$ in which $|f(x)| < 1/n$. Let us define g_n ($n = 1, 2, 3, \ldots$) on $\langle 0, 1\rangle$ by $g_n(x) =$ $= f(x)$ if $x \notin \langle 0, \delta_n\rangle$ and $g_n(x) = x/n\delta_n$ if $x \in \langle 0, \delta_n\rangle$. Evidently $g_n \rightrightarrows f$ on $\langle 0, 1\rangle$ and $g_n \in \mathfrak{L}$ ($n = 1, 2, 3, \ldots$). Hence $\mathfrak{M}_\infty \subset \overline{\mathfrak{L}}$. But $\mathfrak{L} \subset M_1(0, 1)$ and so $M_1(0, 1) =$ $= \overline{\mathfrak{M}}_\infty \subset \overline{\mathfrak{L}} = M_1(0, 1)$ and we get $\overline{\mathfrak{M}}_\infty = \overline{\mathfrak{L}}$.

A question appears, if $\mathfrak{L} \subset \mathfrak{M}_\infty$ is dense in $\mathfrak{M}_\infty$ (if $\mathfrak{M}_\infty$ is considered as a linear normed space with the norm defined in definition 1,4). We shall show that the answer is negative. The following lemma will be useful.

Lemma 1,1. *Let φ be defined and bounded on $\langle 0, 1 \rangle$. Then $\varphi \in \mathfrak{M}_\infty$ if and only if $|\varphi| \in \mathfrak{M}_\infty$.*

Proof. If $|\varphi| \in \mathfrak{M}_\infty$, then evidently $\varphi \in \mathfrak{M}_\infty$. Now, if $\varphi \in \mathfrak{M}_\infty$, then in view of Theorem 1,1 and the boundedness of φ on $\langle 0, 1 \rangle$ we obtain the existence of $K > 0$ such, that $|\varphi(t)| \leqq Kt$ for all $t \in \langle 0, 1 \rangle$. From the last inequality it follows that $|\varphi| \in \mathfrak{M}_\infty$.

Definition 1,5. Let $\mathfrak{C} \subset \mathfrak{M}_\infty$. The closure of the set $\mathfrak{C}$ in the sense of the convergence in norm (see the definition 1,4) will be denoted by cl $\mathfrak{C}$.

Theorem 1,13. *Let $\mathfrak{M}_\infty$ denote the linear space with the norm given in the definition 1,4 and let $\mathfrak{L} \subset \mathfrak{M}_\infty$ have the previous meaning. Then* cl $\mathfrak{L} \neq \mathfrak{M}_\infty$.

Proof. Let us put $f^+(x) = \max(0, x \sin(1/x))$ for $x \neq 0$ and $f^+(0) = 0$. Since $f(x) = x \sin(1/x)$, $x \neq 0$, $f(0) = 0$ belongs to $\mathfrak{M}_\infty$, we have $f^+ \in \mathfrak{M}_\infty$ according to Lemma 1,1 and to the fact that $\mathfrak{M}_\infty$ is a vector space. It will be shown that f^+ is not a limit function (in the sense of the convergence in norm in $\mathfrak{M}_\infty$) of any sequence $\{f_n\}_1^\infty$ of functions belonging to $\mathfrak{L}$. Denote as $\langle 0, \delta_n \rangle$ the interval on which f_n is linear and let $f_n(x) = k_n x$ on that interval. Evidently it suffices to consider the case $k_n \geqq 0$ ($n = 1, 2, 3, \ldots$). Let us consider two cases

1) There exists $\varepsilon_0 > 0$ such that to each N there is $n \geqq N$ such that $k_n > \varepsilon_0$.

2) To each $\varepsilon > 0$ there exists $n_1(\varepsilon)$ such that $k_n \leqq \varepsilon$ for $n \geqq n_1(\varepsilon)$.

If the first case occurs, then there exists an infinite set of indexes n such that $k_n > \varepsilon_0$. These indices form a set N'. If $n \in N'$, choose in the interval $\langle 0, \delta_n \rangle$ points $t_1, t_2, \ldots, t_r$ ($r = r(n)$) such that $1 \geqq t_1 + t_2 + \ldots + t_r \geqq \varepsilon_0/k_n$ and $\sin(1/t_j) = 0$ ($j = 1, 2, \ldots, r$). Let us put $t_j = 0$ for $j > r$. Then

$$A = \sum_{j=1}^{\infty} t_j \in \mathfrak{A}, \quad \left|\sum_{j=1}^{\infty} f^+(t_j) - f_n(t_j)\right| = k_n \sum_{j=1}^{r} t_j \geqq \varepsilon_0 .$$

Hence $\|f^+ - f_n\| \geqq \varepsilon_0$ for an infinite set of indices n ($n \in N'$), so that $\{f_n\}_1^\infty$ does not converge to f^+ in the norm.

Next, consider the second case and let $\varepsilon \leqq 1/3$. Let $\langle 0, \delta_n \rangle$ ($n = 1, 2, 3, \ldots$) have the previous meaning. For $n \geqq n_1(\varepsilon)$ we have $k_n \leqq \varepsilon$. Choose in the interval $\langle 0, \delta_n \rangle$ ($n \geqq n_1(\varepsilon)$) points $t_1, t_2, \ldots, t_s$ ($s = s(n)$) such that

$$\sin \frac{1}{t_i} = 1 \quad (i = 1, 2, \ldots, s), \quad 1 \geqq t_1 + t_2 + \ldots + t_s \geqq \tfrac{3}{4}$$

and put $t_j = 0$ for $j > s$. Then for this n

$$\sum_{j=1}^{\infty} |f^+(t_j) - f_n(t_j)| \geqq \sum_{j=1}^{s} t_j - \varepsilon \sum_{j=1}^{s} t_j \geqq \tfrac{3}{4}(1 - \varepsilon) > \tfrac{1}{2},$$

$\|f^+ - f_n\| \geqq \frac{1}{2}$ for $n \geqq n_1(\varepsilon)$. Hence $\{f_n\}_1^\infty$ does not converge to f^+ in the norm. The proof is finished.

2. THE SPACE $\mathfrak{M}_\infty$ AS AN INTERSECTION OF CERTAIN QUOTIENT SPACES

We shall show, that $\mathfrak{M}_\infty$ may be considered as an intersection of certain quotient spaces. At first we prove some general theorems concerning these spaces.

Under a quotient space on a vector space $\mathfrak{X}$ we shall undestand the set of all the equivalence classes formed by means of a subspace of the space $\mathfrak{X}$. The subspace by means of which the quotient space will be formed will be called the zero class of the quotient space. Under the operations of forming sums and multiplying by a real number, which are assumed to be defined in the usual way on the equivalence classes, the quotient space is a vector space (see [2] p. 25). If a metric or a norm on such quotient space is given then the latter will be called a quotient space with metric or norm respectively.

Remark 2,1. If $T \neq \emptyset$ and $\mathfrak{X}_t$ is a quotient space for $t \in T$, whose zero class is O_t, then $O^* = \bigcap_{t \in T} O_t$ is a subspace of the space $\mathfrak{X}$ (see e.g. [2] p. 23).

Definition 2,1. The quotient space $\mathfrak{X}^*$ formed by means of the subspace O^* will be called the intersection of quotient spaces $\mathfrak{X}_t$.

Theorem 2,1. *A set $C \subset \mathfrak{X}$ belongs to the intersection of the quotient spaces $\mathfrak{X}_t$ if and only if for each $t \in T$ there exists $C_t \in \mathfrak{X}_t$ such that $\bigcap_{t \in T} C_t \neq \emptyset$ and $C = \bigcap_{t \in T} C_t$. The latter representation of the set C by means of C_t $(t \in T)$ is unique.*

Proof. Let $C \in \mathfrak{X}^*$. Then $C \neq \emptyset$. If x is an arbitrarily chosen element in C, then for each $t \in T$ there exists a class $C_t \in \mathfrak{X}_t$ such that $x \in C_t$. Such class there is just one. If $y \in C$, then y belongs to the same class C_t $(t \in T)$ because $y - x \in O^* \subset O_t$. Thus $C \subset C_t$ $(t \in T)$ and from this inclusion $C \subset \bigcap_{t \in T} C_t$ follows immediately. Now, if $x, y \in \bigcap_{t \in T} C_t$, then $y - x \in O_t$ for each $t \in T$, consequently $y - x \in O^*$. The intersection $\bigcap_{t \in T} C_t$ contains all mutually equivalent (modulo O^*) elements. The last fact implies that $\bigcap_{t \in T} C_t$ is included in some class of $\mathfrak{X}^*$. From the disjointness of these classes we have $C = \bigcap_{t \in T} C_t$. Each class $C \in \mathfrak{X}^*$ may be represented in the form $C = \bigcap_{t \in T} C_t$, $C_t \in \mathfrak{X}_t$ $(t \in T)$. The uniqueness is seen from the proof.

Now, let $C = \bigcap_{t\in T} C_t \neq \emptyset$, $C_t \in \mathfrak{X}_t$ $(t \in T)$. We shall show that $C \in \mathfrak{X}^*$. From the preceding part of the proof it follows that $C = \bigcap_{t\in T} C_t \subset C'$, where $C' \in \mathfrak{X}^*$. Let $x \in C$ and $y \in C'$. Then $y - x \in O^*$ in view of the definition of C'. Hence $y - x \in O_t$ for each $t \in T$. Since $x \in C_t$, then also $y \in C_t$, for each t and we have $y \in C = \bigcap_{t\in T} C_t$, so that $C' \subset C$, $C' = C$.

Theorem 2,2. *Let $(\mathfrak{X}_t, \varrho_t)$ for $t \in T$ be quotient spaces on $\mathfrak{X}$ with metric ϱ_t. Let $\varrho(C, D) = \sup_{t\in T} \{\varrho_t(C_t, D_t)\} < +\infty$ for any two elements $C, D \in \mathfrak{X}^*$ $(C_t, D_t, t \in T$ are those elements of the space $\mathfrak{X}_t$ for which $C \subset C_t, D \subset D_t)$. Then $(\mathfrak{X}^*, \varrho)$ is a metric space.*

Proof. We shall prove that ϱ is a metric.

(a) Evidently $\varrho(C, D) \geqq 0$. If $\varrho(C, D) = 0$, then $\varrho_t(C_t, D_t) = 0$ for each $t \in T$ and $C_t = D_t$ $(t \in T)$ follows. From Theorem 2,1 $C = D$. If $C = D$ then again in view of Theorem 2,1 $C_t = D_t$ for each $t \in T$ consequently $\varrho(C, D) = 0$.

(b) The symmetry of ϱ is evident.

(c) Let $C, D \in \mathfrak{X}^*$. Then $\varrho(C, E) = \sup_{t\in T} \{\varrho_t(C_t, E_t)\}$, where $C = \bigcap_{t\in T} C_t$, $D = \bigcap_{t\in T} D_t$, $E = \bigcap_{t\in T} E_t$. Since ϱ_t is a metric on $\mathfrak{X}_t$, we have for each $t \in T$

$$\varrho_t(C_t, E_t) \leqq \varrho_t(C_t, D_t) + \varrho_t(D_t, E_t) \leqq \varrho(C, D) + \varrho(D, E),$$

hence

$$\varrho(C, E) = \sup_{t\in T} \{\varrho_t(C_t, E_t)\} \leqq \varrho(C, D) + \varrho(D, E).$$

Theorem 2,3. *Let $(\mathfrak{X}_t, \varrho_t)$ $(t \in T)$ be linear normed spaces (the norm $\|Y\|_t$ on $\mathfrak{X}_t$ is given by $\|Y\|_t = \varrho_t(Y, O_t)$) and let $\varrho(C, D) = \sup_{t\in T} \{\varrho_t, (C_t, D_t)\}$ for any $C, D \in \mathfrak{X}^*$, $C = \bigcap_{t\in T} C_t$, $D = \bigcap_{t\in T} D_t$. Then $(\mathfrak{X}^*, \varrho)$ is a normed space (with the norm $\|Y\| = \varrho(Y, O^*)$).*

Proof. a) We shall show that for k complex $\varrho(kC, O^*) = |k|\, \varrho(C, O^*)$ holds. Let $C \in \mathfrak{X}^*$, then $C = \bigcap_{t\in T} C_t$, $C_t \in X_t$, $kC = \bigcap_{t\in T} kC_t$, $\varrho(kC, O^*) = \sup_{t\in T} \{\varrho_t(kC_t, O_t)\} = |k| \sup_{t\in T} \{\varrho_t(C_t, O_t)\} = |k|\, \varrho(C, O^*)$.

b) If $C, D, E \in \mathfrak{X}^*$, then $\varrho(C + E, D + E) = \varrho(C, D)$. In fact, $\varrho(C + E, D + E) = \sup_{t\in T} \{\varrho_t(C_t + E_t, D_t + E_t)\}$, where $C = \bigcap_{t\in T} C_t$, $D = \bigcap_{t\in T} D_t$, $E = \bigcap_{t\in T} E_t$. Since $\varrho_t(C_t + E_t, D_t + E_t) = \varrho_t(C_t, D_t)$, for each $t \in T$ we have $\varrho(C + E, D + E) = \sup_{t\in T} \{\varrho_t(C_t, D_t)\} = \varrho(C, D)$.

Remark 2,2. If $O^* = \{0\}$ in the previous considerations (0 is the zero element of the space $\mathfrak{X}$), then the corresponding space $\mathfrak{X}^*$ has the onepoint sets as elements. In such case we shall identify $\mathfrak{X}^*$ with $\mathfrak{X}$.

Now we show that $\mathfrak{M}_\infty$ may be considered as an intersection of certain quotient spaces. For the set T, the set of all sequences $t = \{t_k\}_{k=1}^\infty$ with $A = \sum_{k=1}^\infty t_k \in \mathfrak{A}$ will be taken. If $t = \{t_k\}_1^\infty \in T$ then O_t denotes the set of all those functions φ for which $\varphi(t_k) = 0$ $(k = 1, 2, \ldots)$. O_t is evidently a subspace in $\mathfrak{M}_\infty$ so that we may form by means of the last a quotient space $\mathfrak{M}_t$. To show that $\mathfrak{M}_\infty$ is the intersection of quotient spaces $\mathfrak{M}_t$ $(t \in T)$ it suffices (according to the last remark) to show that $O^* = \{0\}$, 0 denotes the function which is identically zero on $\langle 0, 1 \rangle$). Let $\varphi \in O^*$, then $\varphi(t) = 0$ for each $t \in \langle 0, 1 \rangle$. In fact, it suffices to form a sequence $\{t_k\}_1^\infty$, in which $t_1 = t$, $t_k = 0$ for $k > 1$ and to note that $\varphi \in O_t$.

The metric ϱ_t in $\mathfrak{M}_t$ will be given as follows: If $D_t, E_t \in \mathfrak{M}_t$ $(t = \{t_k\}_1^\infty, \sum_1^\infty t_k \in \mathfrak{A})$, then $\varrho_t(D_t, E_t) = \sum_{k=1}^\infty |\varphi(t_k) - \psi(t_k)|$, where $\varphi \in D_t$, $\psi \in E_t$. It is easily seen that ϱ_t is independent of the choire of φ and ψ. The condition $\sup_{t \in T} \{\varrho_t(D_t, E_t)\} < +\infty$ follows from the fact that $\varphi - \psi \in \mathfrak{M}_\infty$ and consequently $|\varphi - \psi| \in \mathfrak{M}_\infty$ (see Lemma 1,1). To prove that ϱ_t is a metric makes no difficulties. Evidently $\varrho_t(D_t, E_t) = 0$ if and only if $\varphi(t_k) = \psi(t_k)$ $(k = 1, 2, 3, \ldots)$ but the last equality holds exactly if φ and ψ are from the same class. The symmetry and the triangular inequality are evident. It is also easily seen that $(\mathfrak{X}_t, \varrho_t)$ are normed spaces with the norm $\|D_t\| = \varrho_t(D_t, O_t)$. Thus the space $\mathfrak{M}_\infty$ with the metric $\varrho(\varphi, \psi) = \sup_{t \in T} \{\varrho_t(D_t, E_t)\}$, where $\varphi = \bigcap_{t \in T} D_t$, $\psi = \bigcap_{t \in T} E_t$ (we identify f and $\{f\}$) is in view of Theorem 2,3 a linear normed space with the norm $\|\varphi\|_1 = \sup_{t \in T} \sum_{k=1}^\infty |\varphi(t_k)|$. The norm $\|\varphi\|_\infty$ was defined on the space $\mathfrak{M}_\infty$ by $\|\varphi\| = \sup_{t \in T} |\sum_{k=1}^\infty \varphi(t_k)|$. It is immediately seen that on the sets of all nonnegative and nonpositive functions respectively the norms $\|\varphi\|$ and $\|\varphi\|_1$ are identical. In general this is not the case. If we put e.g. $\varphi(3/4) = 2$, $\varphi(1/4) = -1$, $\varphi(t) = 0$ for the rest of $t \in \langle 0, 1 \rangle$, then $\varphi \in \mathfrak{M}_\infty$, $\|\varphi\| = 2$, $\|\varphi\|_1 = 3$. But the following theorem shows that from the topological point of view, there is no difference between these two norms.

Theorem 2,4. *The convergence induced by the norm $\|\varphi\|$ is equivalent with that induced by $\|\varphi\|_1$.*

Proof. It suffices to prove the equivalence of convergences for sequences which converge to zero. Since $\|\varphi\| \leq \|\varphi\|_1$ we see that the convergence in norm of $\{\varphi_n\}_1^\infty$ induced by $\|\varphi\|_1$ implies the convergence in norm $\|\varphi\|$. Now, let $\{\varphi_n\}_1^\infty$ converge to 0 in the sense of the norm $\|\varphi\|$ i.e. $\lim_{n \to \infty} \|\varphi_n\| = 0$. To each $\varepsilon > 0$ there exists n_0 such that for $n > n_0$ and each $t = \{t_k\}_1^\infty \in T$ we have

$$(8) \qquad \Big|\sum_{k=1}^{\infty} \varphi_n(t_k)\Big| < \frac{\varepsilon}{2}.$$

Let $t^0 = \{t_k^0\}_1^\infty \in T$. Let us form a sequence $\{t_{k_l}^0\}_l$ from those t_k^0 for which $\varphi_n(t_k^0) \geqq 0$ and a sequence $\{t_{m_j}^0\}_j$ from those ones for which $\varphi_n(t_k^0) < 0$. Both of these sequence (after completing by zero terms if some of them is finite) belong to T and (8) implies

$$\sum_l \varphi_n(t_{k_l}^0) < \frac{\varepsilon}{2}, \quad \Big|\sum_j \varphi_n(t_{m_j}^0)\Big| = \sum_j \big|\varphi_n(t_{m_j}^0)\big| < \frac{\varepsilon}{2}.$$

Hence $\sum_{k=1}^{\infty} \big|\varphi_n(t_k^0)\big| = \sum_l \varphi_n(t_{k_l}^0) + \sum_j \varphi_n(t_{m_j}^0)$ and we get $\|\varphi_n\|_1 \leqq \varepsilon$ for $n > n_0$, so $\|\varphi_n\|_1 \to 0$ as $n \to \infty$.

In Theorem 1,7 the completness of the space $\mathfrak{M}_\infty$ with the norm $\|\varphi\|$ was proved. From the last fact and from Theorem 2,4 the completeness of $\mathfrak{M}_\infty$ with the norm $\|\varphi\|_1$ follows.

Now we shall prove two theorems concerning the completeness and separability of general quotient spaces.

Definition 2,2. Let $\{C_t\}$, $t \in T$ be a system containing for each $t \in T$ just one element of $\mathfrak{X}_t$. The system $\{C_t\}$, $t \in T$ will be called uniformly attainable if to each $t \in T$ there exists a sequence $\{C_t^n\}_{n=1}^\infty$ of elements of $\mathfrak{X}_t$ such that $\lim_{n\to\infty} C_t^n = C_t$ uniformly with respect to $t \in T$ and $\bigcap_{t\in T} C_t^n \neq \emptyset$.

Theorem 2,5. *Let $(\mathfrak{X}_t, \varrho_t)$ for $t \in T$ be complete linear metric spaces. The intersection $(\mathfrak{X}^*, \varrho)$ of the spaces $(\mathfrak{X}_t, \varrho_t)$ is complete if and only if each uniformly attainable system $\{C_t\}$, $t \in T$ has a nonempty intersection ($\bigcap_{t\in T} C_t \neq \emptyset$).*

Proof. Let each uniformly attainable system $\{C_t\}$, $t \in T$ have a nonempty intersection. Let $\{C^n\}_{n=1}^\infty$ be a fundamental sequence of elements of $\mathfrak{X}^*$. To $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that for $m, n \geqq n_0(\varepsilon)$

$$(9) \qquad \varrho(C^m, C^n) = \sup_{t\in T} \{\varrho_t(C_t^m, C_t^n)\} < \frac{\varepsilon}{2}$$

holds, where $C^m = \bigcap_{t\in T} C_t^m$, $C^n = \bigcap_{t\in T} C_t^n$. (9) implies

$$(10) \qquad \varrho_t(C_t^m, C_t^n) < \frac{\varepsilon}{2} \ (m, n \geqq n_0(\varepsilon))$$

for each $t \in T$. Since $\mathfrak{X}_t$ is complete, there exists $C_t \in \mathfrak{X}_t$ such that $\lim_{n\to\infty} C_t^n = C_t$.

Simultaneously from (10) it is seen that the convergence is uniform with respect to $t \in T$, so that if $n \geqq n_0(\varepsilon)$ then (10) implies $\varrho_t(C_t^n, C_t) \leqq \varepsilon/2 < \varepsilon$ for each $t \in T$. The last inequality implies $\sup_{t \in T} \{\varrho_t(C_t^n, C_t)\} < \varepsilon$ for $n \geqq n_0(\varepsilon)$. According to the assumption $\bigcap_{t \in T} C_t \neq \emptyset$ holds. Let us put $C = \bigcap_{t \in T} C_t$. Then $\{C^n\}_{n=1}^{\infty}$ converges to C because $\varrho(C^n, C) = \sup_{t \in T} \{\varrho_t(C_t^n, C_t)\} < \varepsilon$ for $n \geqq n_0(\varepsilon)$. The completeness of the space $(\mathfrak{X}^*, \varrho)$ is proved.

Let $(\mathfrak{X}^*, \varrho)$ be complete and let $\{C_t\}$, $t \in T$ be a uniformly attainable system. By the definition of $\{C_t\}$, $t \in T$, for each $t \in T$ there exists a sequence $\{C_t^n\}_{n=1}^{\infty}$ of elements of X_t such that

$$\lim_{n \to \infty} \varrho_t(C_t^n, C_t) = 0 \tag{11}$$

uniformly with respect to $t \in T$ and $C^n = \bigcap_{t \in T} C_t^n \neq \emptyset$ $(n = 1, 2, \ldots)$. Since $\varrho(C^m, C^n) = \sup_{t \in T} \{\varrho_t(C_t^m, C_t^n)\}$ and since $\{C_t^n\}_{n=1}^{\infty}$ are uniformly fundamental with respect to $t \in T$, the sequence $\{C^n\}_{n=1}^{\infty}$ is fundamental in $\mathfrak{X}^*$. Due to the completness of the space $(\mathfrak{X}^*, \varrho)$, there exists $C \in \mathfrak{X}^*$ such that

$$\lim_{n \to \infty} C^n = C \in \mathfrak{X}^* \tag{12}$$

In view of Theorem 2,1, $C = \bigcap_{t \in T} C'_t$, $C'_t \in \mathfrak{X}_t$. We shall show that $C'_t = C_t$ for each $t \in T$ and consequently $C = \bigcap_{t \in T} C_t \neq \emptyset$. Thus, it suffices to prove $C'_t = C_t$ for each $t \in T$. According to the definition of the metric, (12) implies $\lim_{n \to \infty} \varrho_t(C_t^n, C_t) = 0$ for each $t \in T$ and using (11) we get $C_t = C'_t$ in view of the uniqueness of the limit.

Theorem 2,6. *A necessary condition for the separability of the space $(\mathfrak{X}^*, \varrho)$ is the separability of the spaces $(\mathfrak{X}_t, \varrho_t)$.*

Proof. Let $(\mathfrak{X}^*, \varrho)$ be separable and let $t_0 \in T$. Let $\mathfrak{M}$ be a countable and dense set in $\mathfrak{X}^*$. Let $\mathfrak{M}_{t_0}$ denote the set of all those $C_{t_0} \in \mathfrak{X}_{t_0}$ which appear in the intersections $C = \bigcap_{t \in T} C_t$, where $C \in \mathfrak{M}$. Evidently $\mathfrak{M}_{t_0}$ is countable. We shall show that $\mathfrak{M}_{t_0}$ is dense in $\mathfrak{X}_{t_0}$. Let $D_{t_0} \in \mathfrak{X}_{t_0}$, then there exists $D \in \mathfrak{X}^*$ such that $D \subset D_{t_0}$. To each $\varepsilon > 0$ there exists $C \in \mathfrak{M}$ such that $\varrho(C, D) < \varepsilon$. The existence of such C follows from the density of $\mathfrak{M}$ in $\mathfrak{X}^*$. Among the terms C_t appearing in the intersection $\bigcap_{t \in T} C_t$ the term $C_{t_0} \in \mathfrak{M}_{t_0}$ appears. According to the definition of the metric ϱ we have $\varrho_{t_0}(C_{t_0}, D_{t_0}) \leqq \varrho(C, D) < \varepsilon$. Hence $(\mathfrak{X}_{t_0}, \varrho_{t_0})$ is a separable space. The proof is finished.

Remark 2,3. The condition stated in the preceding theorem is not sufficient. We shall show it on an example. An example is furnished by the space $\mathfrak{M}_{\infty}$ serve. $\mathfrak{M}_{\infty}$

will be considered as the intersection of the quotient spaces $\mathfrak{M}_t$, $t \in T$, $t = \{t_k\}_{k=1}^{\infty}$, $\sum_{k=1}^{\infty} t_k \in \mathfrak{A}$, $\|C_t\| = \sum_{k=1}^{\infty} |\varphi(t_k)|$, where φ is some representative of the class C_t. It is immediately seen that if $t \in T$ is fixed then $\mathfrak{X}_t = \mathfrak{M}_t$ is isometric with the space l_1 of all sequences $x = \{\xi_k\}_{k=1}^{\infty}$ for which $\sum_{k=1}^{\infty} |\xi_k| < +\infty$ with the norm $\|x\| = \sum_{k=1}^{\infty} |\xi_k|$. It is well known that l_1 is separable and consequently $\mathfrak{M}_t = \mathfrak{X}_t$ is separable space. In spite of it $\mathfrak{M}_\infty$ is not separable. It suffices to consider in $\mathfrak{M}_\infty$ the subset $\mathfrak{M}^*$ of all such $\varphi \in \mathfrak{M}_\infty$ which assume values 0 and 1 at the point of the form $k/(k+1)$ and 0 at other points. Evidently $\mathfrak{M}^*$ is an uncountable subset of $\mathfrak{M}_\infty$ and the distance of any two points of $\mathfrak{M}^*$ is $\geqq 1$.

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Výťah

O ISTÝCH PRIESTOROCH TRANSFORMÁCIÍ NEKONEČNÝCH RADOV

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Označme znakom $\mathfrak{A}$ systém všetkých nekonečných radov $A = \sum_{n=1}^{\infty} t_n$, $0 \leqq t_n \leqq 1$ $(n = 1, 2, 3, \ldots)$, ktorých súčty neprevyšujú číslo 1. Nech pri $c \geqq 0$ $\mathfrak{M}_c$ značí systém všetkých reálnych funkcii φ definovaných na $\langle 0, 1\rangle$, ktoré majú tú vlastnosť, že pre každý rad $A = \sum_{n=1}^{\infty} t_n \in \mathfrak{A}$ je $|\varphi\{A\}| = \left|\sum_{n=1}^{\infty} \varphi(t_n)\right| \leqq c$. Položme $\mathfrak{M}_\infty = \bigcup_{c \geqq 0} \mathfrak{M}_c$ a na množine $\mathfrak{M}_\infty$ definujme reálnu funkciu $\|\varphi\|$ takto: $\|\varphi\| = \sup_{A \in \mathfrak{A}} |\varphi\{A\}|$. V práci sa dokazuje, že $\|\varphi\|$ je norma a $\mathfrak{M}_\infty$ s touto normou je Banachov lineárny priestor. V práci sa podrobne študujú vlastnosti tohoto priestoru a dokazuje sa, že $\mathfrak{M}_\infty$ možno chápať aj ako prenik istých faktorových priestorov.

Резюме

О НЕКОТОРЫХ ПРОСТРАНСТВАХ ПРЕОБРАЗОВАНИЙ БЕСКОНЕЧНЫХ РЯДОВ

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Пусть $\mathfrak{A}$ обозначает множество всех бесконечных рядов $A = \sum_1^\infty t_n$, $0 \leqq t_n \leqq 1$, $(n = 1, 2, \ldots)$, суммы которых небольше 1.

Пусть $\mathfrak{M}_c$ $(c \geqq 0)$ обозначет множество всех действительных функции φ, определенных на отрезке $\langle 0, 1\rangle$ и имеющих следующее свойство: для всякого $A = \sum_1^\infty t_n \in \mathfrak{A}$ имеет место неравенство $|\varphi\{A\}| = |\sum_1^\infty \varphi(t_n)| \leqq c$. Пусть $\mathfrak{M}_\infty = \bigcup_{c \geqq 0} \mathfrak{M}_c$. Если на множестве $\mathfrak{M}_\infty$ определим действительную функцию $\|\varphi\|$ следующим образом $\|\varphi\| = \sup_{A \in \mathfrak{A}} |\varphi\{A\}|$, то эта функция является нормой, и $\mathfrak{M}_\infty$ с этой нормой является линейным пространством Банаха. В работе детально изучаются свойства этого постранства и показывается что $\mathfrak{M}_\infty$ можно рассматривать как пересечение определенного вида фактор-пространств.