Jiří Sedláček

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JIŘÍ SEDLÁČEK, Praha

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1. SOME AUXILIARY NOTIONS

In this contribution finite undirected graphs without loops and multiple edges are considered. We write $\mathcal{G} = [V, E]$ if the graph $\mathcal{G}$ has the vertex set V and the edge set E. The number of trees spanning the graph $\mathcal{G}$ will be denoted by $k(\mathcal{G})$. In this paper we need a well known notion which is reminiscent of the usual spanning tree. If $\mathcal{G} = [V, E]$ is a connected graph and x_1, x_2 are two vertices $(x_1 \neq x_2)$ then there exists a factor $\mathcal{F} = [V, E_0]$ of $\mathcal{G}$ with the following properties: $\mathcal{F}$ consists of two components $\mathcal{F}_1, \mathcal{F}_2$ which are trees and x_1, x_2 belong to $\mathcal{F}_1, \mathcal{F}_2$ respectively. Denote by $k_2(\mathcal{G}; x_1, x_2)$ the number of all the factors $\mathcal{F}$. Using a known determinant method the numbers $k(\mathcal{G})$ and $k_2(\mathcal{G}; x_1, x_2)$ can be determined.

Let n be a given natural number. In [3] we defined A_n to be the set of all the natural numbers m for which there exists a graph $\mathcal{G}$ on n vertices with $k(\mathcal{G}) = m$. In [4] we considered a restriction of the set A_n by requiring the graph $\mathcal{G}$ to be regular of degree t and the corresponding set of integers $m \in A_n$ was denoted by $B_n^{(t)}$. The cases $t = 0, 1$ and 2 are trivial and the investigation of the set $B_n^{(t)}$ begins to be interesting for $t = 3$. In [4] we showed that $|B_{2a}^{(3)}|$ tends to infinity with a (as one might expect by intuition)[1]. In this paper we obtain an analogous result for the set $B_n^{(t)}$ for arbitrary $t \geqq 3$.

One can see immediately that for any $t \geqq 3$ the set $B_n^{(t)}$ is empty for $n \leqq t$ and that $|B_{t+1}^{(t)}| = 1$. If t and n are odd then $B_n^{(t)} = \emptyset$. If t is even then $|B_{t+2}^{(t)}| = 1$.

2. REGULAR GRAPHS OF ODD DEGREE

In this section, $t \geqq 3$ denotes a fixed odd integer. We begin by giving two constructions which will be used later.

[1]) The cardinality of a finite set M will be denoted by $|M|$.

First auxiliary construction. For each even number $s \geqq t + 1$ let us construct the graph $\mathcal{G}_s = [V_s, E_s]$ by putting $V_s = \{w_1, w_2, \ldots, w_s\}$ where w_i is the residue class modulo s containing the number i. In order to define E_s we first let

$$E = \{w_i w_{i+1} \mid i = 1, 2, \ldots, s\} .$$

Further let h be the following mapping of V_s into the system of all subsets of V_s: For each $w_\alpha \in V_s$ let $w_\beta \in h(w_\alpha)$ if and only if there exists a number β' belonging to the residue class w_β and satisfying

$$\left|\beta' - \alpha - \frac{s}{2}\right| \leqq \tfrac{1}{2}(t - 3). \tag{1}$$

The assumption $w_\beta \in h(w_\alpha)$ implies $w_{\alpha+s} \in h(w_\beta)$. Thus $w_\alpha \in h(w_\beta)$. If one connects each vertex $w_\alpha \in V_s$ with each vertex of the set $h(w_\alpha)$ by an edge one obtains the edge set E^*. It is easy to see that E and E^* are disjoint.

We conclude the definition of the graph $\mathcal{G}_s$ by putting $E_s = E \cup E^*$. Obviously the degree of each vertex w_α equals t and the obtained graph $\mathcal{G}_s$ has s vertices.

Second auxiliary construction. Using the graph $\mathcal{G}_s$ we construct a new graph $\mathcal{G}^*_{s+1} = [V^*_{s+1}, E^*_{s+1}]$ as follows: Let v be a vertex not belonging to V_s and let $V^*_{s+1} = V_s \cup \{v\}$, $E^*_{s+1} = (E_s \cup \{vw_i \mid i = 1, 2, 3, \ldots, t - 1\}) - \{w_i w_{i+1} \mid i = 1, 3, 5, 7, \ldots \ldots, t - 2\}$. It is easy to see that $\mathcal{G}^*_{s+1}$ has $s + 1$ vertices one of which is of degree $t - 1$, the other of degree t.

The two preceding constructions can be used in the proof of Theorem 1. The proof is constructive and is based on schematic figures in which dotted lines show certain parts to be repeated. Also the entire subgraphs $\mathcal{A}_1$, $\mathcal{A}_2$, etc. are sketched in the figures whereby we intend to make convention that we are not going to emphasize the parts to be disjoint while describing particular subgraphs but assume that it is evident from the figures.

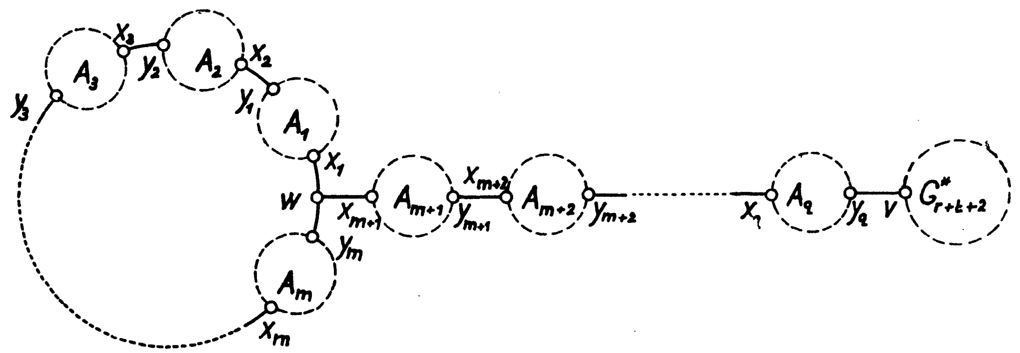

Fig. 1.

Theorem 1. *If $t \geqq 3$ is an odd integer then*

$$\lim_{a\to\infty} |B_{2a}^{(t)}| = +\infty .$$

Proof. Select a natural number a such that $2a \geqq \frac{1}{2}(t^2 + 4t + 3)$ and let

$$q = \left[\frac{2a - \frac{1}{2}(t^2 + 3)}{t + 1}\right].$$

It is easy to see that $q \geqq 1$. Further let m be a natural number and n a non-negative integer with $m + n = q$. In our proof we shall gradually construct the graph $\mathscr{G}^{(a,m)}$ which is outlined in Fig. 1 and 2. In order to describe it more closely, let us first construct auxiliary graphs $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3, \ldots$ each of which is isomorphic to the graph $[V_{t+1}, E_{t+1} - \{w_1 w_{t+1}\}]$ (see first auxiliary construction). Denote by x_i, y_i the two vertices of $\mathscr{A}_i$ having the local degrees $t - 1$. Fig. 1 shows the graph $\mathscr{G}^{(a,m)}$ for $t = 3$. Further one can see a vertex w and an auxiliary graph $\mathscr{G}^*_{r+t+2}$ the structure of which has been described by the second auxiliary construction. The number r is given by the formula

$$r = 2a - \tfrac{1}{2}(t^2 + 3) - q(t + 1).$$

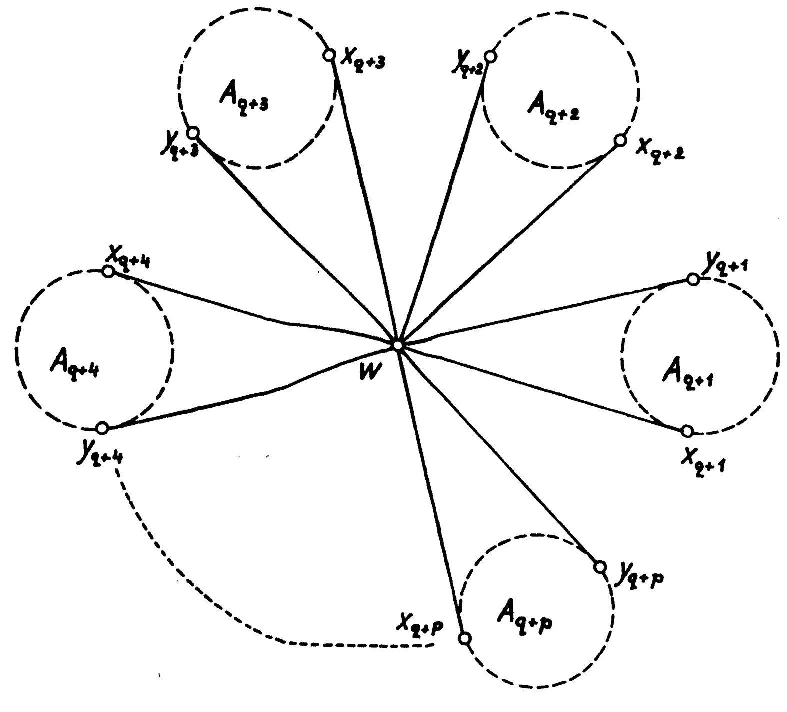

Fig. 2.

If $t > 3$ then the construction of $\mathcal{G}^{(a,m)}$ will be completed by an additional portion described in Fig. 2 in which $p = \frac{1}{2}(t - 3)$. In our next considerations the graph of Fig. 2 will be shortly denoted by $\mathcal{X}$. There is also a vertex w in Fig. 2. If we identify it with the vertex w of Fig. 1, the definition of $\mathcal{G}^{(a,m)}$ is finished.

The graph $\mathcal{G}^{(a,m)}$ which has been sketched above is obviously a connected regular graph of degree t, it has $2a$ vertices and w is its cut vertex. How many spanning trees does it have? The trees spanning the graph $\mathcal{G}^{(a,m)}$ can be divided into two disjoint types. If one denotes

$$(2) \qquad M = \{wx_1, y_1x_2, y_2x_3, \ldots, y_{m-1}x_m, y_mw\},$$

then the first type misses exactly one edge of the set M and every tree of the second type contains all the edges of M. One can find that there are[2])

$$\tau_1 = (m + 1)\, k(\mathcal{G}^*_{r+t+2})\, k(\mathcal{X}) \prod k(\mathcal{A}_i)$$

spanning trees of the first type. If we want to determine the number of spanning trees of the second type it is first necessary to take into account the number $k_2(\mathcal{A}_\varrho; x_\varrho, y_\varrho)$ (for some $\varrho \in \langle 1, m \rangle$) which was defined in the preceding chapter. Then the number of spanning trees of the second type is

$$\tau_2 = mk_2(\mathcal{A}_\varrho; x_\varrho, y_\varrho)\, k(\mathcal{G}^*_{r+t+2})\, k(\mathcal{X}) \prod_{i \neq \varrho \leqq m} k(\mathcal{A}_i).$$

Summarizing $k(\mathcal{G}^{(a,m)}) = \tau_1 + \tau_2$. One can see immediately that

$$k(\mathcal{G}^{(a,1)}) < k(\mathcal{G}^{(a,2)}) < k(\mathcal{G}^{(a,3)}) < \ldots < k(\mathcal{G}^{(a,q)}).$$

Thus $|B^{(t)}_{2a}| \geqq q$ and the theorem is proved.

3. REGULAR GRAPHS OF EVEN DEGREE

Let $t \geqq 4$ be a given even number. We shall be engaged in investigating the behaviour of $|B^{(t)}_b|$ for large values of b. For this investigation we need a further auxiliary construction.

Third auxiliary construction. For each natural number $s \geqq t + 1$ let us construct the graph $\mathcal{G}_s$ as follows: If s is odd we recall the first auxiliary construction and accept

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[2]) The next products will be a little complicated. To simplify the formulas we shortly write $\prod$ instead of $\prod_{i=1}^{q}$. Under the symbol $\prod$ only excluded values will be mentioned. A similar agreement is also made for the products in the proof of Theorem 2. Further let us put $k(\mathcal{X}) = 1$ for $t = 3$.

it for $\mathscr{G}_s$ (on the understanding that s is odd and t is even now). For an even s let us modify the first auxiliary construction replacing (1) by

$$0 < \left|\beta' - \alpha - \frac{s}{2}\right| \leqq \tfrac{1}{2}(t - 2) .$$

We see that $\mathscr{G}_s$ is a regular graph of degree t with s vertices.

Theorem 2. *If $t \geqq 4$ is an even integer then*

$$\lim_{b \to \infty} |B_b^{(t)}| = +\infty .$$

Proof. Choose a natural number b such that $b \geqq \frac{1}{2}(t^2 + 3t + 2)$ and put

$$q = \left[\frac{b - \frac{1}{2}(t^2 - t)}{t + 1}\right].$$

We see again that $q \geqq 1$. Let us select a natural number m and a non-negative n so that $m + n = q$. Our aim is to construct the graph $\mathscr{H}^{(b,m)}$ which is sketched in Fig. 3 and 2. To describe $\mathscr{H}^{(b,m)}$ more closely we again need the auxiliary graphs $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$, ... They were introduced in the proof of Theorem 1 (under the condition that t was odd). Fig. 3 shows the graph $\mathscr{H}^{(b,m)}$ for $t = 4$. There is a vertex w and an auxiliary graph $[V_{r+t+1}, E_{r+t+1} - \{w_1 w_{r+t+1}\}]$ which is shortly denoted by $\mathscr{B}$ in Fig. 3. We have

$$r = b - \tfrac{1}{2}(t^2 - t) - q(t + 1) .$$

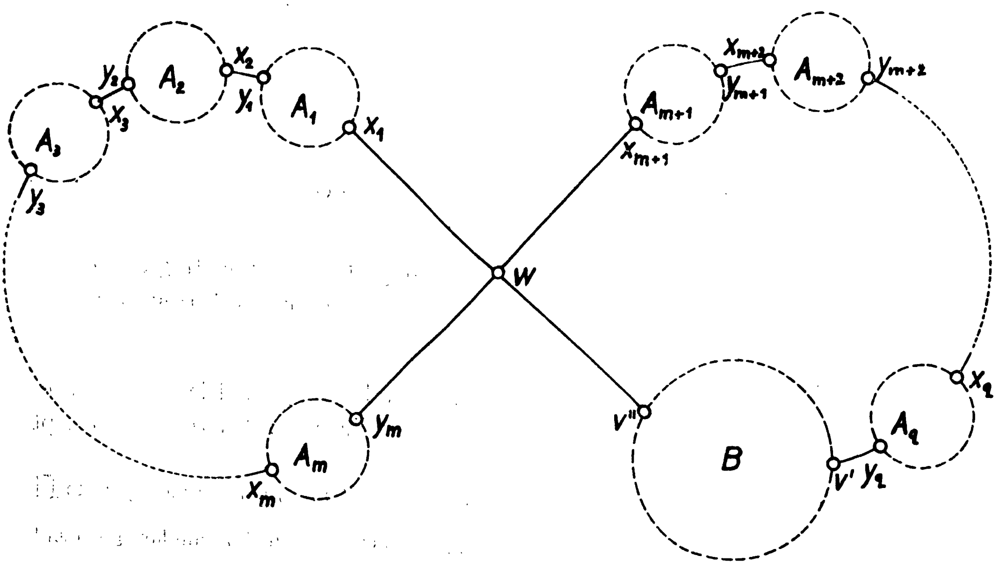

Fig. 3.

The vertices v' and v'' have the degree $t - 1$ with respect to $\mathscr{B}$. For $t > 4$ the construction of $\mathscr{H}^{(b,m)}$ will be extended by adding the graph $\mathscr{X}$ (see Fig. 2) for $p = \frac{1}{2}(t - 4)$. We again identify vertices denoted by w in Fig. 3 and 2.

Obviously $\mathscr{H}^{(b,m)}$ is a connected regular graph of degree t on b vertices and w is a cut vertex of $\mathscr{H}^{(b,m)}$. As in the proof of Theorem 1 we can determine the number of trees spanning the graph $\mathscr{H}^{(b,m)}$.

We again define the set M by (2) and put

$$N = \{wx_{m+1}, y_{m+1}x_{m+2}, y_{m+2}x_{m+3}, \ldots, y_{q-1}x_q, y_qv', v''w\} .$$

The trees spanning the graph $\mathscr{H}^{(b,m)}$ can be divided into three classes: The first type is short of an edge of M and one of N, the second type is missing one edge of $M \cup N$ and a tree of the third type contains all the edges of $M \cup N$. There are

$$\tau'_1 = (m + 1)(n + 2)\, k(\mathscr{B})\, k(\mathscr{X}) \prod k(\mathscr{A}_i)$$

trees of the first type[3]). The absent edge of the second type tree is taken either from the set M-there are

$$(m + 1)\, nk_2(\mathscr{A}_\varrho; x_\varrho, y_\varrho)\, k(\mathscr{B})\, k(\mathscr{X}) \prod_{i \neq \varrho > m} k(\mathscr{A}_i) + (m + 1)\, k_2(\mathscr{B}; v', v'')\, k(\mathscr{X}) \prod k(\mathscr{A}_i)$$

trees of that kind – or from the set N – there are

$$m(n + 2)\, k_2(\mathscr{A}_\varrho; x_\varrho, y_\varrho)\, k(\mathscr{B})\, k(\mathscr{X}) \prod_{i \neq \varrho \leq m} k(\mathscr{A}_i)$$

possibilities. Thus the number of the second type trees is

$$\tau'_2 = (2mn + m + q)\, k_2(\mathscr{A}_\varrho; x_\varrho, y_\varrho)\, k(\mathscr{B})\, k(\mathscr{X}) \prod_{i \neq \varrho} k(\mathscr{A}_i) +$$
$$+ (m + 1)\, k_2(\mathscr{B}; v', v'')\, k(\mathscr{X}) \prod k(\mathscr{A}_i) .$$

For the number of the trees of the third type one can easily find

$$\tau'_3 = mnk_2(\mathscr{A}_\varrho; x_\varrho, y_\varrho)\, k_2(\mathscr{A}_\sigma; x_\sigma, y_\sigma)\, k(\mathscr{B})\, k(\mathscr{X}) \prod_{\substack{i \neq \varrho \leq m \\ i \neq \sigma > m}} k(\mathscr{A}_i) +$$
$$+ mk_2(\mathscr{A}_\varrho; x_\varrho, y_\varrho)\, k_2(\mathscr{B}; v', v'')\, k(\mathscr{X}) \prod_{i \neq \varrho \leq m} k(\mathscr{A}_i) .$$

Consequently $k(\mathscr{H}^{(b,m)}) = \tau'_1 + \tau'_2 + \tau'_3$. Obviously

$$k(\mathscr{H}^{(b,1)}) < k(\mathscr{H}^{(b,2)}) < k(\mathscr{H}^{(b,3)}) < \ldots < k(\mathscr{H}^{(b,Q)})$$

where $Q = [q/2]$. Therefore $|B_b^{(t)}| \geq Q$; this completes the proof.[4])

[3]) For $t = 4$ we put $k(\mathscr{X}) = 1$.

[4]) First version of this paper was established as a research paper of the University of Calgary, Alberta, Canada.

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Author's address: Praha 1, Žitná 25 (Matematický ústav ČSAV v Praze).