Valter Šeda

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ON A BOUNDARY VALUE PROBLEM OF THE FOURTH ORDER

VALTER ŠEDA, Bratislava

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In the paper a sufficient condition for the existence of a unique solution to an arbitrary interpolation problem of the fourth order is given. The n-parameter families theory is used in the proof.

Consider a differential equation

$$x^{(4)} = f(t, x, x', x'', x''') \tag{1}$$

where $f : (a, b) \times R^4 \to R(-\infty < a < b < \infty)$ satisfies the assumptions

(A) *f is continuous on $(a, b) \times R^4$;*

(B) *all solutions of (1) can be extended to (a, b);*

(C) *for any $a < t_1 < t_2 < t_3 < t_4 < b$ and any $A_k \in R$ $(k = 1, 2, 3, 4)$ there exists at most one solution of (1),*

$$x(t_k) = A_k \quad (k = 1, 2, 3, 4)\,; \tag{2}$$

(D) *there exists a $K > 0$ such that*

$$f(t, x, x', x'', x''') \geqq 0 \quad (f(t, x, x', x'', x''') \leqq 0)$$

for all $(t, x, x', x'', x''') \in (a, b) \times R^4$ such that $x'' \geqq K$, $x''' \geqq K$ ($x'' \leqq -K$, $x''' \leqq -K$).

Under these hypotheses the following existence statement will be proved.

Theorem 1. *Assume that (1) satisfies conditions (A)–(D). Then given any $a < t_1 < t_2 < t_3 < t_4 < b$ and any A_k $(k = 1, \ldots, 4)$, the BVP (1), (2) has a unique solution.*

The proof will be based on the n-parameter families theory developed by Hartman in [2] as well as on a result by Klaasen [5]. The results obtained have been put together in [3] by Jackson and in [6] by the author. For the special case $n = 4$ they will be stated here as

Lemma 1 (HARTMAN, KLAASEN). *Suppose that* (1) *satisfies conditions* (A)–(C) *and the compactness condition*

(E) *if* $[c, d]$ *is a compact subinterval of* (a, b) *and* $\{x_p\}$ *is a sequence of solutions of* (1) *which is uniformly bounded on* $[c, d]$, *then there is a subsequence* $\{x_{p(r)}\}$ *such that* $\{x_{p(r)}^{(i)}\}$ *converges uniformly on* $[c, d]$ *for* $0 \leqq i \leqq 3$.

Then given any $a < t_1 < \dots t_4 < b$ *and any* A_k $(k = 1, \dots, 4)$, *there exists unique solution* x *of the problem* (1), (2).

Our aim is to show that assumptions (A)–(D) imply the hypotheses of Lemma 1 which proves Theorem 1. The proof will consist of a chain of lemmas. The first of them gives a result in the special case $n = 4$ proved by Jackson for arbitrary n in [3, p. 90].

Lemma 2 (JACKSON). *Assume that the differential equation* (1) *satisfies hypotheses* (A)–(C). *Then, if* $[c, d]$ *is a compact subinterval of* (a, b) *and* $\{x_p\}$ *is a sequence of solutions of* (1) *which is uniformly bounded on* $[c, d]$, *it follows that the sequence* $\{V_c^d(x_p)\}$ *of total variations of the functions* x_p *on* $[c, d]$ *is bounded.*

Lemma 3. *Suppose that* (1) *satisfies conditions* (A) *and* (C). *Then to any* $M > 0$, $a_0 > 0$ *and* $[c, d] \subset (a, b)$ *there exists a* $\delta > 0$, $\delta = \delta(M, a_0, [c, d])$ *such that for any solution* x *of* (1) *existing on* $[c, d]$ *with* $|x(t)| \leqq M$ *for each* $t \in [c, d]$ *the following implication holds*:

If there are four points $c \leqq t_1 < t_2 < t_3 < t_4 \leqq d$ *at which*

$$x(t_k) = a_1 t_k + b_1 \quad (k = 1, \dots, 4) \tag{3}$$

and $t_4 - t_1 < \delta$, $|a_1| \leqq a_0$, b_1 *is arbitrary, then*

$$|x'(t)| \leqq a_0 + 1\,, \quad |x''(t)| \leqq 1\,, \quad |x'''(t)| \leqq 1 \quad on \quad [t_1, t_4]\,. \tag{4}$$

Proof. By (C), there exists at most one solution of the BVP (1), (3). Using the Schauder Fixed Point Theorem, a solution y of (1), (3) will be found which satisfies (4) when $t_4 - t_1 < \delta$ with a suitable $\delta > 0$.

Let $Q = [c, d] \times [-M - 1, M + 1] \times [-a_0 - 1, a_0 + 1] \times [-1,1] \times \times [-1,1]$ and let $K = \max |f(t, x, x', x'', x''')|$ on Q. Clearly K depends on M, a_0 and $[c, d]$. The solution x of (1), (3) can be written in the form

$$x(t) = a_1 t + b_1 + \int_{t_1}^{t_4} G(t, s)\, f[s, x(s), \dots, x'''(s)]\, ds \quad (t \in [t_1, t_4])\,, \tag{5}$$

where G is the Green function of the problem $x^{(4)} = 0$,

$$x(t_k) = 0\,, \quad k = 1, \dots, 4\,. \tag{2'}$$

Consider the space $C^{(3)}([t_1, t_4])$ endowed with the norm $\|x\| = = \max_{k=0,1,2,3} \{ \max_{t\in[t_1,t_4]} |x^{(k)}(t)|\}$ and a closed, convex and bounded subset $S = \{x \in \in C^{(3)}([t_1, t_4]) : |x(t)| \leqq M + 1, |x'(t)| \leqq a_0 + 1, |x''(t)| \leqq 1, |x'''(t)| \leqq 1\}$. In view of Lemma 2, [7], the operator $T: C^{(3)}([t_1, t_4]) \to C^{(3)}([t_1, t_4])$ determined by the right-hand side of (5) is continuous and compact. Let $x \in S$. Then $T(x)(t) = a_1 t + + b_1 + \int_{t_1}^{t_4} G(t, s) f[s, x(s), \ldots, x'''(s)]\, ds = a_1 t + b_1 + u(t)$. Since $u \in C^{(4)}([t_1, t_4])$ and satisfies (2'), Lemma 8.7 [4, p. 145] implies $|u^{(k)}(t)| \leqq (t_4 - t_1)^{(4-k)} K/(4 - k)!$ $(t \in [t_1, t_4],\ k = 0, 1, 2, 3)$. Hence if $\delta = \min(1, 1/K)$, $t_4 - t_1 \leqq \delta$, then $T(S) \subset S$ and thus there exists a solution y of the problem (1), (3) which lies in S and hence satisfies (4). By (C), $x(t) = y(t)$ in $[t_1, t_4]$ which completes the proof.

Lemma 4. *Let k, $1 \leqq k \leqq 3$, be a natural number, $K > 0$ a real number, $x \in \in C^{(k)}([c, d])$ such that $|x^{(k)}(t)| \geqq K$ for all $t \in [c, d]$.*
Then the total variation $V_c^d(x)$ of x in $[c, d]$ satisfies the relations

(6) $$V_c^d(x) \geqq K(d - c) \quad \textit{if} \quad k = 1$$

(7) $$V_c^d(x) \geqq \frac{K}{4}(d - c)^2 \quad \textit{for} \quad k = 2$$

and

$$V_c^d(x) \geqq \frac{K}{216}(d - c)^3 \quad \textit{for} \quad k = 3\,.$$

Proof. Since $V_c^d(-x) = V_c^d(x)$, only the case

(8) $$x^{(k)}(t) \geqq K \quad \text{in} \quad [c, d]$$

will be considered. (6) is clear. If x satisfies (8) for $k = 2$, then x' can have at most one zero in $[c, d]$. If $x'(t_0) = 0$, then $x'(t) \geqq K(t - t_0)$ for $t_0 \leqq t \leqq d$ as well as $x'(t) \leqq K(t - t_0)$ for $c \leqq t \leqq t_0$ which gives $|x'(t)| \geqq K|t - t_0|$ in $[c, d]$ and thus $V_c^d(x) = \int_c^d |x'(t)|\, dt \geqq K[(t_0 - c)^2 + (d - t_0)^2]/2 \geqq K(d - c)^2/4$. If $x'(t) > 0$ in $[c, d]$, then $x'(t) \geqq x'(c) + K(t - c)$ and hence $V_c^d(x) \geqq K(d - c)^2/2$. In the case $x'(t) < 0$ in $[c, d]$ the inequality $x'(t) \leqq x'(d) + K(t - d)$ implies $V_c^d(x) \geqq \geqq K(d - c)^2/2$. Thus (7) is proved to be true.

Consider now the case $k = 3$. Suppose first that there is a $t_0 \in [c, d]$ such that $x''(t_0) = 0$. Then, in view of (8),

(9) $$x''(t) \geqq K(t - t_0) \quad \text{for} \quad t_0 \leqq t \leqq d \quad \text{and} \quad x''(t) \leqq K(t - t_0) \quad \text{if} \quad c \leqq t \leqq t_0\,.$$

The following subcases may arise:

a) $x'(t_0) \geqq 0$. Then $x'(t) \geqq K(t - t_0)^2/2$ in $[t_0, d]$ and $V_{t_0}^d(x) \geqq K(d - t_0)^3/6$ while $x'(t) \geqq K(t - t_0)^2/2$ in $[c, t_0]$ which implies $V_c^{t_0}(x) \geqq K(t_0 - c)^3/6$. Thus

$V_c^d(x) \geqq K[\frac{1}{2}(d - t_0)^3 + \frac{1}{2}(t_0 - c)^3]/3$ which is, in virtue of the property of $M_t(x, \alpha)$ [1, p. 30], greater or equal to $K(d - c)^3/24$.

b) $x'(t) < 0$ in $[c, d]$. Then using (9), we get $x'(t) \leqq x'(c) + K(t - c)(t + c - 2t_0)/2 \leqq K(t - c)(t + c - 2t_0)/2$ in $[c, t_0]$ and $V_c^{t_0}(x) \geqq K(t_0 - c)^3/3$. In $[t_0, d]$ we have $x'(t) \leqq K(t - d)(t + d - 2t_0)/2$ and $V_{t_0}^d(x) \geqq K(d - t_0)^3/3$. Then $V_c^d(x) = \frac{2}{3}K[\frac{1}{2}(d - t_0)^3 + \frac{1}{2}(t_0 - c)^3] \geqq K(d - c)^3/12$.

c) $x'(t_0) < 0$ and there exist c_1 and d_1, $c \leqq c_1 < t_0 < d_1 \leqq d$, such that $x'(t) < 0$ in (c_1, d_1), $x'(c_1) = x'(d_1) = 0$ and $x'(t) > 0$ in $[c, c_1)$ and $(d_1, d]$. Then, by the result of the case b),

$$V_{c_1}^{d_1}(x) \geqq \frac{K}{3}[(d_1 - t_0)^3 + (t_0 - c_1)^3] \tag{10}$$

is true. In $[c, c_1]$ we have $x'(t) = x'(c_1) + \int_{c_1}^t x''(s)\,ds \geqq K[(t - t_0)^2 - (c_1 - t_0)^2]/2 = K(t - c_1)(t + c_1 - 2t_0)/2$. Therefore $V_c^{c_1}(x) \geqq K(c_1 - c)^3/6$.

In $[d_1, d]$ we come to the inequality $x'(t) \geqq K[(t - t_0)^2 - (d_1 - t_0)^2]/2$ which gives $V_{d_1}^d(x) \geqq K(d - d_1)^3/6$. The last inequalities together with (10) lead to the result

$$\begin{aligned} V_c^d(x) &\geqq K[\tfrac{1}{6}(c_1 - c)^3 + \tfrac{1}{3}(d_1 - t_0)^3 + \tfrac{1}{3}(t_0 - c_1)^3 + \tfrac{1}{6}(d - d_1)^3] \geqq \\ &\geqq K[\tfrac{1}{6}(c_1 - c) + \tfrac{1}{3}(d_1 - t_0) + \tfrac{1}{3}(t_0 - c_1) + \tfrac{1}{6}(d - d_1)]^3 \geqq \\ &\geqq \frac{K}{216}(d - c)^3 . \end{aligned}$$

If $x''(t) > 0$ in $[c, d]$, then instead of (9) we have $x''(t) \geqq K(t - c)$ for all $t \in [c, d]$ and again we have three cases a), b), c), where t_0 is replaced by c. Thus in the case a) we come to the inequality $V_c^d(x) \geqq K(d - c)^3/6$, in the case b) we have $V_c^d(x) \geqq K(d - c)^3/3$. The case c) implies that $V_c^d(x) \geqq K(d_1 - c)^3/3 + K(d - d_1)^3/6 = \frac{1}{2}K[\frac{2}{3}(d_1 - c)^3 + \frac{1}{3}(d - d_1)^3]$. When $x''(t) < 0$, then t_0 is replaced by d and, in view of the symmetry of the results obtained, we come to the same inequalities. Thus the lemma is proved.

Proof of Theorem 1. Suppose $\{x_p\}$ is a sequence of solutions of (1) which is uniformly bounded on $[c, d]$, say by a constant M. Then, by Lemma 2, the sequence $\{V_c^d(x_p)\}$ is bounded.

Two cases may occur. Either $\lim_{p\to\infty} \sum_{k=0}^{3} |x_p^{(k)}(t)| = \infty$ uniformly on $[c, d]$ is not true and then the Kamke Convergence Theorem can be applied in order to complete the proof of the theorem, or $\lim_{p\to\infty} \sum_{k=0}^{3} |x_p^{(k)}(t)| = \infty$ uniformly on $[c, d]$. This is equivalent to

$$\lim_{p\to\infty} \max_{k=0,1,2,3} |x_p^{(k)}(t)| = \infty \quad \text{uniformly on } [c, d] . \tag{11}$$

We shall show that (11) leads to contradiction with the boundedness of $\{V_c^d(x_p)\}$.

Consider $K_1 \geqq 2K$ where K is given in hypothesis (D). Then, by (11), there exists a $P > 0$ such that for all $p > P$ and all $t \in [c, d]$,

$$\max_{k=1,2,3} |x_p^{(k)}(t)| > K_1 . \tag{12}$$

Fix a $p > P$ and consider the set $S_1 = \{t \in [c, d] : |x_p'(t)| > K_1\}$. If $S_1 \neq \emptyset$, then the components of S_1 are intervals which are open with the possible exception of those containing c or d. If there existed infinitely many components of S_1, then there would exist a point $c_1 \in [c, d]$ which is a limit point of the sequence of endpoints of the components considered and at the same time of local minimizers and local maximizers of x_p' which gives $|x_p'(c_1)| = K_1$, $x_p''(c_1) = 0$, $x_p'''(c_1) = 0$. This contradicts (12) and hence there exists only a finite number of intervals of S_1.

S_1 is open in $[c, d]$, thus $[c, d] - S_1$ is closed. We add to S_1 all one-point components of $[c, d] - S_1$. Then S_1 remains open. Consider the set $S_2 = \{t \in [c, d] - - S_1 : |x_p''(t)| > K_1\}$. S_2 is open in the closed set $[c, d] - S_1$. Suppose there are infinitely many components of S_2. Then there exists a limit point c_2 of the endpoints of the components of S_2 such that $|x_p''(c_2)| = K_1$, $x_p'''(c_2) = 0$ and hence (12) implies that $|x_p'(c_2)| > K_1$ which contradicts the fact that S_1 is open. Therefore there exist only finitely many components of S_2 . S_2 is open in $[c, d] - S_1$ and hence $[c, d] - - S_1 - S_2$ is closed. It will remain closed when all one-point components of this set are added to S_2. Then (12) gives that $S_3 = \{t \in [c, d] - S_1 - S_2 : |x_p'''(t)| > > K_1\} = [c, d] - S_1 - S_2$. Since S_1, S_2 consist of finitely many intervals, the same is true about S_3.

The consecutive intervals (components) of S_1, S_2 and S_3 are displaced by the following rules:

1. If an interval $i_1(i_2)$ from $S_1(S_2)$ is followed by an interval $i_2(i_3)$ from $S_2(S_3)$, then the sign of $x_p''(x_p''')$ in $i_2(i_3)$ is different from the sign of $x_p'(x_p'')$ in $i_1(i_2)$.

2. If an interval $i_2(i_3)$ from $S_2(S_3)$ is followed by an interval $i_1(i_2)$ from $S_1(S_2)$, then the sign of $x_p'(x_p'')$ in $i_1(i_2)$ is the same as that of $x_p''(x_p''')$ in $i_2(i_3)$.

These two rules are based on the meaning of the sign of the derivative.

3. If $i_1 \subset S_1$ is neither the first nor the last interval (briefly i_1 is an ordinary interval) of the system of all components of S_1, S_2, S_3, then x_p' attains its local extremum in i_1.

The proof follows from the fact that x_p' has the same value at both end points of i_1.

4. If an ordinary interval $i_1 \subset S_1$ is followed by an interval $i_3 \subset S_3$ and the sign of x_p''' in i_3 is different from the sign of x_p' in i_1, then i_3 is followed by an $i_2 \subset S_2$ if there exists an interval following i_3.

The proof is based on the monotonicity of the integral.

Assumption (D) implies

5. If an interval $i_3 \subset S_3$ is followed by an interval $i_2 \subset S_2$, then the latter can be followed only by an interval $i_1 \subset S_1$ which is then the last interval in the system of components of S_1, S_2, S_3.

6. If an interval $i_1 \subset S_1$ is followed by an interval $i_3 \subset S_3$ and this in turn is followed by an interval $i_1^* \subset S_1$ and the sign ε of x_p' in i_1, i_1^* is the same as the sign of x_p''' in i_3, then in the case $\varepsilon = 1$ $(\varepsilon = -1)$ x_p' must possess a unique local minimum at t_0 in i_3 (a unique local maximum at t_0 in i_3). Denote by t_1 the endpoint of i_3. Hence $x_p'''(t_1) > K_1$, $x_p'(t_1) = K_1$ if $\varepsilon = 1$ and $x_p'''(t_1) < -K_1$, $x_p'(t_1) = -K_1$ if $\varepsilon = -1$. Three cases may occur:

(a) $x_p'(t_0) \leqq 0$ $(x_p'(t_0) \geqq 0)$ if $\varepsilon = 1$ $(\varepsilon = -1)$.

(b) $0 < x_p'(t_0) < K_1/2$ $(0 > x_p'(t_0) > -K_1/2)$ when $\varepsilon = 1$ $(\varepsilon = -1)$. Suppose now that $x_p''(t_1) \leqq K_1/2$ $(x_p''(t_1) \geqq -K_1/2)$. Since $x_p'''(t) > K_1$ in i_3 and $x_p''(t_0) = 0$, it is $0 \leqq x_p''(t) \leqq K_1/2$ in $[t_0, t_1]$. In the case $\varepsilon = -1$ we come to $0 \geqq x_p''(t) \geqq \geqq -K_1/2$. Therefore

$$K_1/2 < x_p'(t_1) - x_p'(t_0) \leqq K_1(t_1 - t_0)/2$$

$$(-K_1/2 > x_p'(t_1) - x_p'(t_0) \geqq -K_1(t_1 - t_0)/2$$

and hence $t_1 - t_0 > 1$ in both cases $\varepsilon = \pm 1$.

On the other hand,

$$K_1/2 \geqq x_p''(t_1) - x_p''(t_0) \geqq K_1(t_1 - t_0)$$

$$(-K_1/2 \leqq x_p''(t_1) - x_p''(t_0) \leqq -K_1(t_1 - t_0))$$

which gives $t_1 - t_0 \leqq \frac{1}{2}$ which is a contradiction.

Thus, if $0 < x_p'(t_0) < K_1/2$ $(0 > x_p'(t_0) > -K_1/2)$, then $x_p''(t_1) > K_1/2$ $(x_p''(t_1) < < -K_1/2)$ and since $K_1/2 \geqq K$ $(-K_1/2 \leqq -K)$ and $x_p'''(t_1) > K_1$ $(x_p'''(t_1) < -K_1)$, assumption (D) implies that i_1^* is the last interval in the system of all components of S_1, S_2, S_3.

(c) $K_1/2 \leqq x_p'(t_0)$ $(-K_1/2 \geqq x_p'(t_0))$ implies that the contribution of the set $i_1 \cup i_3 \cup i_1^*$ to $V_c^d(x_p)$ is

$$V_{i_1 \cup i_3 \cup i_1^*}(x_p) \geqq \frac{K_1}{2}\,\mu(i_1 \cup i_3 \cup i_1^*),$$

where $\mu(j)$ means the length of the interval j.

7. If the intervals $i_1 \subset S_1$, $i_2 \subset S_2$, $i_3 \subset S_3$, $i_1^* \subset S_1$ follow in this order and the sign ε of x_p' in i_1^* is the same as the sign of x_p''' in i_3, then x_p' attains its unique local minimum for $\varepsilon = 1$ (a unique local maximum for $\varepsilon = -1$) in $i_2 \cup i_3$ at a point $t_0 \in i_3$. With respect to monotonicity of the integral, the case (a) from 6 cannot occur (otherwise i_3 would be followed by i_2). The case (b) remains in validity and in the case (c) we have $V_{i_1 \cup i_2 \cup i_3 \cup i_1^*}(x_p) \geqq K_1\ \mu(i_1 \cup i_2 \cup i_3 \cup i_1^*)/2$.

By the rule 5 we get

8. In a triple of any three consecutive intervals — components of S_1, S_2, S_3 — either there exists an interval from S_1 or the triple is the last one or it can be followed by an $i_1 \subset S_1$, which is the last component of S_1, S_2, S_3.

The rules 1, 2, 4, 5, 6, 7 and 8 imply

9. If i_1, i_1^* are two consecutive intervals from S_1, then either i_1^* is the last of all intervals from S_1, S_2, S_3 or $V_{i_1 \cup \ldots \cup i_1^*}(x_p) \geqq K_1\, \mu(i_1 \cup \ldots \cup i_1^*)/2$, or x_p' changes its sign in the triple or quadruple $i_1, \ldots, i_1^*$ at least once.

Lemma 4 guarantees that the contribution of S_1 to $V_c^d(x_p)$ is greater or equal to $K_1\, \mu(S_1)$ where $\mu(S_1)$ is the total length of S_1. This estimation does not depend on the number m_1 of components of S_1. On the other hand, if $m_2(m_3)$ is the number of components $[c_i, d_i]$ ($[\gamma_i, \delta_i]$) of $S_2(S_3)$, and $\mu(S_2)\,(\mu(S_3))$ is the total length of $S_2(S_3)$, then by Lemma 4 and using the fact that $M_t(x, \alpha)$ is a nondecreasing function of t ([1, p. 30]), we come to the inequalities

$$V_{S_2}(x_p) \geqq \frac{K_1}{4}\, m_2 \sum_{i=1}^{m_2} \frac{1}{m_2} (d_i - c_i)^2 \geqq \frac{1}{m_2} \frac{K_1}{4}\, \mu^2(S_2),$$

$$V_{S_3}(x_p) \geqq \frac{K_1}{216}\, m_3 \sum_{i=1}^{m_3} \frac{1}{m_3} (\delta_i - \gamma_i)^3 \geqq \frac{1}{m_3^2} \frac{K_1}{216}\, \mu^3(S_3).$$

Thus if m_2, m_3 remain bounded for $K_1 \to \infty$, then $V_c^d(x_p) \to \infty$ which contradicts the boundedness of $\{V_c^d(x_p)\}$ and proves the theorem. Hence we may suppose that one of the numbers m_2. m_3 is sufficiently great and by 8, so is m_1.

Put $\delta = \delta(M, a_0, [c, d])$ where $a_0 = 6M/(d - c)$. Without loss of generality we can assume that

$$(13) \qquad \delta < 1 \quad \text{and} \quad \delta < (d - c)/4\,.$$

Two cases may arise:

1. There exists a subinterval i of $[c, d)$ of the length δ in which x_p has at most two local minima (and at most 3 local maxima). Then the sign of x_p' shows at most 5 changes in i.

Consider first those intervals from S_1, S_2, S_3 which have nonempty intersection with i as well as with $[c, d] - i$. There are at most two of them and if their intersection π with i has the total length greater or equal to $\delta/4$, then Lemma 4 implies

$$(14) \qquad V_i(x_p) \geqq V_\pi(x_p) \geqq \frac{K_1}{216} \frac{1}{4} \frac{\delta^3}{4^3}\,.$$

The second subcase is that the total length of all intervals from S_1, S_2, S_3 which are contained in i is greater than $3\,\delta/4$. The following cases have to be considered. They exclude each other:

(a) The total length of all intervals $i_1 \subset S_1$ contained in i is greater or equal to $\delta/8$. Then, in view of Lemma 4,

$$(15) \qquad V_i(x_p) \geqq K_1 \frac{\delta}{8}\,.$$

(b) The mentioned total length from the case (a) is less than $\delta/8$. We consider the systems $i_1, \ldots, i_1^*$ of consecutive intervals from S_1, S_2, S_3 which start and end with an interval from S_1 and which are contained in i. Suppose that the total length of all those systems where x'_p does not change the sign is greater or equal to $\delta/2$. Using 9 and the fact that the intervals from S_1 can be counted twice, we conclude that

$$(16) \qquad V_i(x_p) \geqq \frac{K_1}{2}\left(\frac{\delta}{2} - \frac{2\delta}{8}\right) = \frac{K_1\delta}{8}.$$

(c) The total length of all systems $i_1, \ldots, i_1^*$ of consecutive intervals from S_1, S_2, S_3 which have similar properties as those in (b) except that x'_p changes its sign at least once in any such system is greater or equal to $\delta/8$. There are at most 5 such systems and hence at least one of them is greater or equal to $\delta/5.8 = \delta/40$. The contribution of that system to $V_i(x_p)$ is greater or equal to $K_1\delta^3/216 \,.\, 4^2 \,.\, 8^3$. Hence

$$(17) \qquad V_i(x_p) \geqq \frac{K_1}{216}\,\frac{1}{16}\,\frac{\delta^3}{512}.$$

(d) The total length of the systems $i_1, \ldots, i_1^*$ mentioned in the case (b) is less than $\delta/2$, and that of the systems $i_1, \ldots, i_1^*$ mentioned in the case (c) is less than $\delta/8$. Hence the remaining intervals lying in i which must belong to S_2 or S_3 have the total length greater than $\delta/8$. With respect to 8, there are at most four and one of them is longer than $\delta/32$. Its contribution to $V_i(x_p)$ is greater than $K_1\,\delta^3/216.32^3$, hence

$$(18) \qquad V_i(x_p) \geqq \frac{K_1}{216}\,\frac{\delta^3}{32^3}.$$

The inequalities (14)–(18) show that (11) implies that $\{V_c^d(x_p)\} \to \infty$ and hence (11) cannot occur.

In order to complete the proof of the theorem we have to prove that the second case which will be dealt with cannot arise when p is sufficiently great.

2. In each subinterval of $[c, d)$ of the length δ, x_p has at least 3 local minima (and thus at least 2 local maxima). Then the local minima of x_p in $[c, d)$ form a monotone sequence. Otherwise there would be a b_1 and four points $t_1 < t_2 < t_3 < t_4$ in an interval of the length δ such that $x_p(t_k) = b_1$. In virtue of Lemma 3, (11) implies that for sufficiently great p, (4) contradicts (12).

Suppose that the sequence of local minima of x_p in $[c, d)$ is nonincreasing. The case that this sequence is nondecreasing can be dealt with in a similar way. Consider any pair of consecutive minimizers $t_0 < t_1$ of x_p in (c, d). We have $t_1 - t_0 < \delta$. Furthermore if $|(x_p(t_1) - x_p(t_0))/(t_1 - t_0)| < a_0$, then there exists a straight line with 4 points of intersection with the graph of x_p in $(t_0 - \varepsilon, t_1 + \varepsilon)$ where $t_1 - t_0 + 2\varepsilon < \delta$ and the direction a of that line satisfies $|a| < a_0$. This again contradicts Lemma 3 for all p sufficiently great. If $|(x_p(t_1) - x_p(t_0))/(t_1 - t_0)| \geqq a_0$ for every pair of consecutive local minimizers $t_0 < t_1$ of x_p, i.e. $(x_p(t_1) - x_p(t_0))/(t_1 - t_0) \leqq -a_0$, then the same is true when t_0 is the first and t_1 the last local minimizer of x_p

in $[c, d]$. Their distance is $t_1 - t_0 \geqq d - c - 2\delta$ and, with respect to (13), $t_1 - t_0 \geqq \geqq (d - c)/2$. Hence $4M/(d - c) \geqq a_0$ which is a contradiction with the definition of a_0.

The next theorem describes the behaviour of solutions of (1) near the endpoints of (a, b).

Theorem 2. *If* (1) *satisfies conditions* (A)–(D), *then for each solution x of of* (1) *which is defined on (a, b) there exist (finite or infinite)*

$$\lim_{t \to a+} x^{(i)}(t)\,, \quad \lim_{t \to b-} x^{(i)}(t) \quad (i = 0, 1)\,.$$

Proof. Only the case $t \to a +$ will be investigated. The other case can be proved similarly. Suppose that for a solution x of (1) $\lim_{t \to a+} x(t)$ does not exist. Then there exist two real numbers $c_1 < c_2$ and two decreasing sequences $\{t_n\}, \{s_n\}$ tending to a with $a < t_n < s_n < b$ such that $x(s_n) \geqq c_2$, $x(t_n) \leqq c_1$ $(n = 1, 2, \ldots)$. Since $s_n - t_n \to \to 0$ as $n \to \infty$, by the mean value theorem there exist other two sequences $\{\tau_n\}, \{\sigma_n\}$ with similar properties as $\{t_n\}, \{s_n\}$ and such that $\lim_{n \to \infty} x'(\sigma_n) = \infty$, $\lim_{n \to \infty} x'(\tau_n) = -\infty$. Hence $x'(\sigma_n) \geqq c_2$, $x'(\tau_n) \leqq c_1$ for all sufficiently great n. The same situation arises when $\lim_{t \to a+} x'(t)$ does not exist. Repeating the considerations we obtain the existence of two decreasing sequences $\{\bar{t}_n\}, \{\bar{s}\}$ such that $\lim_{n \to \infty} \bar{t}_n = \lim_{n \to \infty} \bar{s}_n = a$, $a < \bar{t}_n < \bar{s}_n < b$, $\lim_{n \to \infty} x''(\bar{t}_n) = -\infty$, $\lim_{n \to \infty} x''(\bar{s}_n) = \infty$. Then there exist three points $\bar{\tau}_1 < \bar{\tau}_2 < \bar{\tau}_3$ with $x''(\bar{\tau}_1) = K$ (K has been taken from assumption (D)), $x''(\bar{\tau}_2) = 2K$, $K < x''(t) < < 2K(\bar{\tau}_1 < t < \bar{\tau}_2)$, $\bar{\tau}_2 - \bar{\tau}_1 < 1$ and $x''(\bar{\tau}_3) < 0$. By the mean value theorem there exists a $\bar{\sigma}_1, \bar{\tau}_1 < \bar{\sigma}_1 < \bar{\tau}_2$, such that $x'''(\bar{\sigma}_1) > K$. Assumption (D) means that $x''''(t) \geqq 0$ as far as $x''(t) \geqq K$, $x'''(t) \geqq K$. Hence the inequalities $x''(t) > K$, $x'''(t) > > K$ are true, first in a neighbourhood of $\bar{\sigma}_1$ from the right and then by (D) in the whole interval $[\bar{\sigma}_1, b)$, which contradicts the existence of $\bar{\tau}_3$. This completes the proof of Theorem 2.

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Author's address: 816 31 Bratislava, Mlynská dolina, Matematický pavilón (Matematicko-fyzikálna fakulta UK).