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# ON STABILITY AND INSTABILITY OF THE ROOTS OF THE OSCILLATORY FUNCTION IN A CERTAIN NONLINEAR DIFFERENTIAL EQUATION OF THE THIRD ORDER

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## 1.

Our article deals with the analysis of (in)stability of the roots $\bar{x}$ of the oscillatory function $h(x)$ in the equation

$$(1) \qquad x''' + ax'' + g(x)\,x' + h(x) = 0\,,$$

where $a > 0$ is a constant, $g(x), h(x) \in \mathscr{C}^1(-\infty, \infty)$ and the function $h(x)$ has infinite number of isolated zero points $\bar{x}$ on the interval $(-\infty, \infty)$.

We will apply the well-known *Liapunov's second method* represented by the following

**Theorem 0.** a) *If there exists such a continuous positive definite function* $V(X)$ *that the relation*

$$V' := (\text{grad } V(X), F(X)) = \frac{\mathrm{d}V(X)}{\mathrm{d}t_{(0)}} \leqq 0$$

*holds with respect to the system*

$$(0) \qquad X' = F(X) \quad (F(0) = 0)$$

*and the set* $V' = 0$ *contains no trajectory except* 0, *then the trivial solution of the system* (0) *is asymptotically stable.*

b) *If there exists such a continuous function* $V(X)$ *which is not negative definite in any neighbourhood of the origin,* $V' > 0$ *and the set* $V' = 0$ *contains no trajectory except* 0, *then the trivial solution of the system* (0) *is unstable.*

For the proof see e.g. [1, pp. 19–23].

## 2.

**Theorem 1.** *If there exists such an h-neighbourhood of the root $\bar{x}$ of $h(x)$ in* (1) *that the conditions*

1) $h'(x) > 0$,
2) $a\, g(x) - h'(x) \geqq \delta > 0$ ($\delta$-*const.*),
3) $g'(\bar{x}) = 0$

*are satisfied for* $0 < |x - \bar{x}| < h$, *then $\bar{x}$ is asymptotically stable.*

Proof. It is obvious that the root $\bar{x}$ of $h(x)$ in (1) is asymptotically stable if and only if the same is true for the trivial solution of the equation

$$x''' + ax'' + g^*(x)\, x' + h^*(x) = 0\,, \tag{2}$$

where $g^*(x) := g(x + \bar{x})$, $h^*(x) := h(x + \bar{x})$.

Let us transform (2) to the equivalent system

$$x' = y\,, \quad y' = z\,, \quad z' = -h^*(x) - g^*(x)\, y - az\,. \tag{2'}$$

Since it is a well-known fact (see e.g. [2]) that the asymptotic stability of the trivial solution of (2′) can be reached under the assumptions 1), 2) and 3′): $g^{*\prime}(x) \operatorname{sgn} x \geqq 0$ holding for the functions $g^*$, $h^*$ in an $h$-neighbourhood of the solution mentioned, we have to prove that the same holds also under a more general assumption than 3′), namely $g^{*\prime}(0) = 0$.

Applying the Liapunov function (cf. [2])

$$V(x, y, z) = a \int_0^x h^*(s)\, \mathrm{d}s + h^*(x)\, y + \tfrac{1}{2}[g^*(x)\, y^2 + (ay + z)^2]\,,$$

which for $x \neq 0$ can be rewritten as

$$V(x, y, z) = W(x) + \frac{1}{2}\left\{g^*(x)\left[\frac{h^*(x)}{g^*(x)} + y\right]^2 + (ay + z)^2\right\},$$

where

$$W(x) = \int_0^x \frac{h^*(s)}{g^*(s)}\left[\frac{1}{2}\frac{h^*(s)}{g^*(s)}\, g^{*\prime}(s) + a\, g^*(s) - h^{*\prime}(s)\right] \mathrm{d}s \geqq$$

$$\geqq \int_0^x \frac{h^*(s)}{g^*(s)}\left[\frac{1}{2}\frac{h^*(s)}{g^*(s)}\, g^{*\prime}(s) + \delta\right] \mathrm{d}s\,,$$

we can see that such a number $h \geqq h_1 > 0$ must exist that the relation

$$\lim_{x \to 0}\left[\frac{h^*(x)}{g^*(x)}\, g^{*\prime}(x)\right] = 0 \quad |x| \leqq h_1$$

implied by 1)–3) yields

$$\frac{1}{2}\frac{h^*(x)}{g^*(x)}\, g^{*\prime}(x) + \delta > 0\,,$$

and consequently $V(x, y, z)$ is positive definite.

Since the derivative of $V(x, y, z)$ with respect to (2') satisfies

$$\frac{dV(x, y, z)}{dt_{(2')}} = -[a\, g^*(x) - h^{*\prime}(x) - \tfrac{1}{2}g^{*\prime}(x)\, y]\, y^2 \leqq -\tfrac{1}{2}\delta y^2$$

for $|g^{*\prime}(x)\, y| \leqq \delta$ when $|x| \leqq h_2 \leqq h_1$, $|y| \leqq k$ ($h_2$, $k$-suitable constants), and the set $V'_{(2')} = 0$ contains no trajectory except $(0, 0, 0)$, the trivial solution of (2') (and consequently also the root $\bar{x}$ of $h(x)$ in (1)) is, according to Theorem 0, a), asymptotically stable. Q.E.D.

## 3.

Let us proceed to the examination of unstable roots.

**Lemma 1.** *If there exists such an h-neighbourhood of the origin that the condition*

$$1')\ \ h(x) \operatorname{sgn} x < 0$$

*is satisfied for* $0 < |x| < h$, *then the trivial solution of* (1) *is unstable.*

Proof. Let us transform (1) to the equivalent system

$$(1') \qquad x' = y\,, \quad y' = z\,, \quad z' = -h(x) - g(x)\, y - az\,.$$

Using the function $V(x, y, z)$ with suitable parameters $\alpha$, $\beta$:

$$V(x, y, z) = -\alpha \int_0^x g(s)\, s\, ds + \beta \int_0^x h(s)\, ds - \alpha axy + \tfrac{1}{2}(\beta a + \alpha)\, y^2 - \alpha xz + \beta yz\,,$$

we obtain the following identity:

$$\frac{dV(x, y, z)}{dt_{(1')}} = \alpha\, h(x)\, x - y^2(\alpha a + \beta\, g(x)) + \beta z^2\,.$$

Thus, denoting $K := \max_{|x| \leqq h} |g(x)|$ and choosing $\beta := 1$, $\alpha :< -K/a$, we conclude that

$$\frac{dV}{dt_{(1')}} > 0 \quad \text{for} \quad |x| \leqq h \quad ((x, y, z) \neq (0, 0, 0))\,.$$

Since the function $V(x, y, z)$ is evidently idenfinite, the trivial solution of (1) is, according to Theorem 0, b), unstable. Q.E.D.

**Lemma 2.** *If there exists such an h-neighbourhood of the origin that the conditions*

$$1)\ \ h(x) \operatorname{sgn} x > 0\,,$$

$$2')\ \ g(x) \leqq -\delta < 0 \quad (\delta\text{-const.})$$

*are satisfied for* $0 < |x| < h$, *then the trivial solution of* (1) *is unstable.*

Proof. Employing the same Liapunov function as in the proof of Lemma 1 and choosing the parameters $\alpha$, $\beta$ as follows:

$$\alpha := 1\,, \quad \beta := \frac{a}{K}\,, \quad \text{where} \inf_{0<|x|<h} |g(x)| := K \geqq \delta\,,$$

we find that the relation

$$\frac{\mathrm{d}V}{\mathrm{d}t_{(1')}} > 0$$

is satisfied for $|x| \leqq h$ $((x, y, z) \neq (0, 0, 0))$. Therefore the same argument as that used in Lemma 1 confirms the above assertion.

**Lemma 3.** *If there exists such an h-neighbourhood of the origin that the conditions*

$$\begin{aligned} &1)\ \ h(0) = 0\,,\\ &2')\ \ h'(x) - a\,g(x) \geqq \delta > 0 \quad (\delta\text{-const.})\,,\\ &3')\ \ g(x) > 0 \end{aligned}$$

*are satisfied for* $0 < |x| < h$, *then the trivial solution of* (1) *is unstable.*

Proof. It is useful to notice that the assumption 1)–3') imply the existence of such a constant $0 < h_1 \leqq h$ that the relations

$$\frac{h(x)}{x} - a\,g(x) \geqq \frac{\delta}{2}\,, \quad h(x)\,\mathrm{sgn}\,x > 0 \tag{3}$$

hold for $0 < |x| < h_1$, because, by virtue of the L'Hospital rule,

$$\lim_{x\to 0}\left[\frac{h(x)}{x} - a\,g(x)\right] = h'(0) - a\,g(0) \geqq \delta \tag{4}$$

is satisfied in view of 1), 2') and the function $(h(x)/x - a\,g(x))$ is assumed to be continuous for $0 < |x| < h_1$; the second relation follows immediately from 3').

If we transform (1) to (1') again and employ the Liapunov function of the form

$$-V(x, y, z) = a\int_0^x g(s)\,s\,\mathrm{d}s - \alpha\int_0^x h(s)\,\mathrm{d}s - a^2xy + axz - (1 + \alpha)\,ay^2/2 - \alpha yz$$

with a positive parameter $\alpha$, we come to

$$\frac{\mathrm{d}V(x, y, z)}{\mathrm{d}t_{(1')}} = -\frac{ax}{h(x)}\,[h(x) + az]^2 - y^2[a^2 - \alpha\,g(x)] - z^2\left[\alpha - \frac{a^3x}{h(x)}\right].$$

Hence, in order to satisfy

$$\frac{\mathrm{d}V}{\mathrm{d}t_{(1')}} > 0 \quad ((x, y, z) \neq (0, 0, 0))\,,$$

the inequalities

$$a^2 > \alpha\, g(x)\,, \quad \alpha > \frac{a^3 x}{h(x)}\,, \tag{5}$$

i.e. $\dfrac{h(x)}{x} > \dfrac{a^3}{\alpha} > a\, g(x)$, must be fulfilled.

However, taking $\alpha := a^2/(g(0) + \delta a/4)$, (5) can be satisfied for $0 < |x| < h_2 \leqq \leqq h_1$, where $h_2$ is a certain suitable constant, as it follows immediately from (3), (4).

Since the remaining assumptions of Theorem 0, b) can be easily verified and $V(x, y, z)$ is indefinite, the trivial solution of (1) is again unstable.

**Theorem 2.** *If there exists such an h-neighbourhood of the root $\bar{x}$ of $h(x)$ in (1) that at least one of the following conditions is satisfied:*

1) $h'(x) < 0$,
2) $h'(x) > 0$, $g(x) \leqq -\delta < 0$,
3) $h(\bar{x}) = 0$, $g(x) > 0$, $h'(x) - a\, g(x) \geqq \delta > 0$

*for $0 < |x - \bar{x}| < h$, then the root $\bar{x}$ is unstable.*

Proof follows immediately from Lemmas 1, 2, 3 by the same arguments as those used in the proof of Theorem 1.

## 4.

Although we have succeeded in obtaining information about unstable roots of $h(x)$, by reversing the conditions 1), 2) of Theorem 1, neither of the two conditions can be said to be a necessary one.

### *References*


[1] *E. A. Barbashin:* Liapunov Functions (in Russian). Nauka, Moscow 1970.

[2] *V. Haas:* A stability result for a third order nonlinear differential equation. J. London Math. Soc. *40* (1965), 31–33.



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