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SOLUTION OF NONLINEAR FUNCTIONAL EQUATIONS IN LINEAR NORMED SPACES

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In this paper some conditions for the existence and uniqueness of solution of nonlinear functional equations in linear normed spaces are given. These theorems are based on local approximation of the nonlinear mappings by linear continuous mappings and on some open mapping theorems. First of all we introduce some well-known notations and definitions.

Let X, Y be linear normed spaces and let $f : X \to Y$. We define $m(f)$ on $V \subset X$ as the infimum, and $M(f)$ on $V \subset X$ as the supremum of the quantity $\|f(u_1) - f(u_2)\| \|u_1 - u_2\|^{-1}$ taken over all $u_1, u_2 \in V$ with $u_1 \neq u_2$. We shall say that Y is complete for $f : X \to Y$ if for each Cauchy sequence $\{x_n\} \in X$, the sequence $\{f(x_n)\}$ has a limit in Y. If a nonlinear mapping $f : X \to Y$ is continuous and compact, then f is said to be completely continuous. We shall say that the mapping $f : X \to Y$ is closed, if for each $\{x_n\} \in X$, $x_n \to x$ and $f(x_n) \to y \Rightarrow f(x) = y$.

Definition. *We shall say that the mapping $\varphi : X \to X_1$, where X, X_1 are linear normed spaces, is open, if $\varphi(G)$ is open in $\varphi(X)$ for each open set $G \subset X$.*

Lemma 1. *Let X, X_1 be linear normed spaces. Let $\varphi : X \to X_1$ be a linear mapping. Then φ is open if and only if there exists a positive constant M with the following property: If $y \in \varphi(X)$, then there exist $x \in X$ such that $\varphi(x) = y$ and $\|x\| \leqq M\|y\|$.*

Lemma 2. (*Open mapping Theorem.*) *Let X, X_1 be linear normed spaces, X complete. Let $\varphi : X \to X_1$ be a linear continuous mapping. Let $\varphi(X)$ be a set of the second category in X_1. Then φ is open and $\varphi(X) = X_1$.*

Let us consider the equation

$$F(x) = 0 . \tag{1}$$

We shall prove the following

Theorem 1. *Let F be a mapping of X into Y, where X, Y are linear normed spaces. Let Z be a Banach space and f, g mappings such that $f : Y \to Z$, $g : Z \to X$. Let φ be a linear continuous mapping of Z onto Z and E a closed subset of Z. Furthermore,* *let the following conditions be fulfilled*: 1) *For every* $u, v \in E$ *the inequality*

$$\|f\,F(g(u)) - f\,F(g(v)) - \varphi(u - v)\| \leqq \alpha\|u - v\| \tag{2}$$

holds, where the mapping f *is such that* $m(f) = a > 0$ *on the set* $F(g(E)) \subset Y$, $f(0) = 0$. 2) *The closed ball* $D = \{z \in Z : \|z - z_1\| \leqq r\}$ *is contained in* E, *where* z_1 *is defined by the equality* $z^{(0)} = \varphi(z_1 - z_0)$, z_0 *being an arbitrary element of* E, $z^{(0)}$ *being defined by* $z^{(0)} = f\,F(g(z_0))$, $r \geqq \beta(1 - \beta)^{-1}\,\|z_1 - z_0\|$, $\beta = \alpha M < 1$ (M *being a constant from lemma* 1). *If* $M(g) < +\infty$ *on* E *then the equation* (1) *has at least one solution* x^* *in* $g(D) \subset X$. *The sequence* $\{x_n\}$ *defined by*

$$x_n = g(z_n)\,, \quad z^{(n-1)} = \varphi(z_n - z_{n-1})\,, \quad z^{(n)} = f\,F(g(z_n)) \tag{3}$$

converges in the norm topology of X *to* x^* *and*

$$\|x^* - x_n\| \leqq \beta^n(1 - \beta)^{-1}\,M(g)\,\|z_1 - z_0\|\,. \tag{4}$$

Proof. According to Baire's theorem and lemma 2, φ is an open mapping from Z onto Z. Since $z^{(0)} \in Z$, it follows from lemma 1 that there exist $z_1 \in Z$ such that $z^{(0)} = \varphi(z_1 - z_0)$. Now, from the third equality in (3) we have $z^{(1)} = f\,F(g(z_1))$. Again, according to lemma 1 there exists a $z_2 \in Z$ such that $z^{(1)} = \varphi(z_2 - z_1)$ and $\|z_2 - z_1\| \leqq M\|z^{(1)}\|$. In view of $z^{(1)} = f\,F(g(z_1)) = z^{(0)} - f\,F(g(z_0)) + f\,F(g(z_1)) = \varphi(z_1 - z_0) - f\,F(g(z_0)) + f\,F(g(z_1))$. We obtain using (2) $(z_0, z_1 \in E)$ that $\|z^{(1)}\| \leqq \alpha\|z_1 - z_0\|$. Thus we have

$$\|z_2 - z_1\| \leqq M\|z^{(1)}\| \leqq M\alpha\|z_1 - z_0\| = \beta\|z_1 - z_0\| < r\,.$$

Hence $z_2 \in D$. Let us suppose that $z_k \in D$, $(1 \leqq k \leqq n - 1)$. We shall prove that $z_n \in D$. Since $z^{(n-1)} \in Z$ and φ is an open mapping from Z onto Z, according to lemma 1 there exist $z_n \in Z$ such that $z^{(n-1)} = \varphi(z_n - z_{n-1})$ and $\|z_n - z_{n-1}\| \leqq M\|z^{(n-1)}\|$. The equality $z^{(n-1)} = \varphi(z_{n-1} - z_{n-2}) - f\,F(g(z_{n-2})) + f\,F(g(z_{n-1}))$ together with (2) implies $\|z^{(n-1)}\| \leqq \alpha\|z_{n-1} - z_{n-2}\|$. Hence

$$\|z_n - z_{n-1}\| \leqq \beta\|z_{n-1} - z_{n-2}\| \leqq \ldots \leqq \beta^n\|z_1 - z_0\|\,.$$

Since

$$\|z_n - z_1\| \leqq \sum_{k=1}^{n} \beta^k\|z_1 - z_0\| < \beta(1 - \beta)^{-1}\,\|z_1 - z_0\| \leqq r\,,$$

we have $z_n \in D$. Thus $z_n \in D$ for every n $(n \geqq 1)$. Furthermore,

$$\|z_{n+k} - z_n\| \leqq \sum_{i=1}^{k} \|z_{n+i} - z_{n+i-1}\| \leqq \sum_{i=1}^{k} \beta^i\|z_n - z_{n-1}\| < \beta(1 - \beta)^{-1}\,\|z_n - z_{n-1}\| \leqq \beta^n(1 - \beta)^{-1}\,\|z_1 - z_0\|\,;$$

$$\|x_{n+k} - x_n\| = \|g(z_{n+k}) - g(z_n)\| \leqq M(g)\,\|z_{n+k} - z_n\| \leqq M(g)\,\beta^n(1 - \beta)^{-1}\,\|z_1 - z_0\|\,. \tag{5}$$

We see that the sequences $\{x_n\}$, $\{z_n\}$ are Cauchy in X, Z respectively. There exists $\lim_{n\to\infty} z_n = z^*$ and $z^* \in D$. Setting $x^* = g(z^*)$ we obtain

$$\|x^* - x_n\| = \|g(z^*) - g(z_n)\| \leqq M(g) \|z^* - z_n\| \to 0 \quad \text{as} \quad n \to \infty .$$

Hence $x_n \to x^*$ and $x^* \in g(D) \subset X$. It follows from (2) that the mapping $f\,F(g)$ is continuous on E. Employing $\|z^{(n)}\| \leqq \alpha\beta^{n-1}\|z_1 - z_0\|$ we conclude that $z^{(n)} \to 0$ and $f\,F(g(z_n)) \to 0$ as $n \to \infty$. In view of the continuity of $f\,F(g)$ we have $f\,F(g(z_n)) \to \to f\,F(g(z^*))$, because $z_n \to z^*$. Hence $f\,F(g(z^*)) = f\,F(x^*) = 0$. Since $m(f) = = a > 0$ on $F(g(E))$ and $f(0) = 0$, we obtain that $0 = \|f\,F(x^*) - f(0)\| \geqq m(f) . . \|F(x^*)\|$. Hence $F(x^*) = 0$. From (5) we obtain immediately (4). This concludes the proof.

Theorem 2. *Under the assumptions of theorem 1 (with the exception of the assumption $M(g) < \infty$ on E which is omitted here) let F be such that $m(F) = b > 0$ on $g(E) \subset X$. Then the equation (1) has a unique solution x^* in $g(D) \subset X$. The errors $\|x^* - x_n\|$ satisfy*

$$\|x^* - x_n\| \leqq \beta^n(\|\varphi\| + \alpha)\,[ab(1 - \beta)]^{-1}\,\|z_1 - z_0\| . \tag{6}$$

Proof. We shall prove that $M(g) < +\infty$ on E. Let $u, v \in E$, then

$$\|f\,F(g(u)) - f\,F(g(v))\| \leqq \|\varphi(u - v) - f\,F(g(u)) + f\,F(g(v))\| + \\ + \|\varphi\|\,\|u - v\| \leqq (\alpha + \|\varphi\|)\,\|u - v\| .$$

Since $m(f) = a > 0$ on $F(g(E)) \subset Y$ and $m(F) = b > 0$ on $g(E) \subset X$, we have $\|f\,F(g(u)) - f\,F(g(v))\| \geqq ab\|g(u) - g(v)\|$. Hence $\|g(u) - g(v)\| \leqq (ab)^{-1} . . (\alpha + \|\varphi\|)\,\|u - v\|$ for every $u, v \in E$. Thus

$$M(g) \leqq (\alpha + \|\varphi\|)\,(ab)^{-1} \quad \text{on} \quad E . \tag{7}$$

According to theorem 1, there exists at least one point $x^* \in g(D)$ such that $F(x^*) = 0$. Assume that $F(x_1^*) = 0$, $F(x_2^*) = 0$, $x_1^* \in g(D)$, $x_2^* \in g(D)$, $x_1^* \neq x_2^*$. Then $0 = = \|F(x_1^*) - F(x_2^*)\| \geqq b\|x_1^* - x_2^*\|$. Thus $x_1^* = x_2^*$ on $g(D)$. (6) follows from (4) and (7).

Now we state a theorem establishing the global solutions of nonlinear problems. This result is related close to [4].

Theorem 3. *Let F be a mapping of X into Y, where X, Y are linear normed spaces. Let Z be a Banach space and f, g mappings such that $f : Y \to Z$, $g : Z \to X$. Let φ be a linear continuous mapping of Z onto Z. Furthermore, let the following conditions be fulfilled: 1) For every $u, v \in Z$ the inequality (2) holds. 2) The mappings F, f are such that $m(F) = b > 0$ on X, $m(f) = a > 0$ on Y, $f(0) = 0$ and $\beta = = \alpha M < 1$ (M is a constant from lemma 1). Then the equation (1) has a unique* *solution x^* in X. Moreover, $\|x^* - x_n\| \to 0$ as $n \to \infty$ ($\{x_n\}$ is defined by (3)) and the error $\|x^* - x_n\|$ satisfies (6).*

Proof. Is similar to that of theorem 1.

Theorem 4. *Let X, Y, Z be linear normed spaces, $F : X \to Y, f : Y \to Z, g : Z \to X$. Let φ by a linear continuous mapping of Z into Z having a continuous inverse φ^{-1}. Let $z_0 \in Z$ be such that (2) holds for every $u, v \in D$, where $D = \{z \in Z : \|z - z_0\| \leqq r\}$, $r \geqq (1 - \beta)^{-1} \|\varphi^{-1}\| \|z^{(0)}\|$, $\beta = \alpha\|\varphi^{-1}\| < 1$, $z^{(0)} = f F(g(z_0))$ and $m(f) = a > 0$ on the set $F(g(D)) \subset Y$, $f(0) = 0$, $M(g) < +\infty$ on D. If either a) Z is complete, or b) X is complete for g and fF is closed, then the equation (1) has a unique solution x^* in $g(D) \subset X$. The sequence $\{x_n\}$ defined by $x_n = g(z_n)$, where $z_{n+1} = z_n - \varphi^{-1} f F(g(z_n))$, converges in the norm topology of X to x^* and the inequality (4) holds with $\beta = \alpha\|\varphi^{-1}\|$.*

Proof. Define the mapping $\psi(z)$ by $\psi(z) = f F(g(z)) - \varphi(z - z_0)$ for $z \in Z$. It is clear that $\psi : Z \to Z$. The equality $z_{n+1} = z_n - \varphi^{-1} f F(g(z_n))$ is equivalent to $z_{n+1} = z_0 - \varphi^{-1} \psi(z_n)$, $(n \geqq 0)$. We shall prove that $z_n \in D$ for every n $(n = 0, 1, 2, \ldots)$. Since $\|z_1 - z_0\| \leqq \|\varphi^{-1}\| \|\psi(z_0)\| = \|\varphi^{-1}\| \|f F(g(z_0))\| = \|\varphi^{-1}\| \|z^{(0)}\| \leqq (1 - \beta) r < r$, we have $z_1 \in D$. Let Suppose that z_k $(1 \leqq k \leqq n - 1)$ are contained in D. We shall prove that $z_n \in D$. According to (2)

$$\|z_n - z_{n-1}\| \leqq \|\varphi^{-1}\| \|\psi(z_{n-2}) - \psi(z_{n-1})\| \leqq$$
$$\leqq \|\varphi^{-1}\| \|f F(g(z_{n-2})) - f F(g(z_{n-1})) - \varphi(z_{n-2} - z_{n-1})\| \leqq \beta\|z_{n-1} - z_{n-2}\| .$$

Using this inequality we obtain that $\|z_n - z_{n-1}\| \leqq \beta^n\|z_1 - z_0\|$ and

$$\|z_n - z_0\| \leqq \sum_{k=1}^{n} \beta^k\|z_1 - z_0\| < (1 - \beta)^{-1} \|z_1 - z_0\| \leqq (1 - \beta)^{-1} (1 - \beta) r .$$

Hence $z_n \in D$ for every n $(r = 0, 1, 2, \ldots)$. Further

$$\|z_{n+k} - z_n\| \leqq \sum_{i=1}^{k} \|z_{n+i} - z_{n+i-1}\| < \beta(1 - \beta)^{-1} \|z_n - z_{n-1}\| \leqq$$
$$\leqq \beta^n(1 - \beta)^{-1} \|z_1 - z_0\| .$$

The sequence $\{z_n\}$ is Cauchy in Z. Since $M(g) < +\infty$ on D, we see that $\{x_n\}$ is also the Cauchy sequence in X. Assumming a) we denote $\lim_{n\to\infty} z_n = z^*$ and $x^* = g(z^*)$. Then $z^* \in D$ and $\|x_n - x^*\| = \|g(z_n) - g(z^*)\| \leqq M(g) \|z_n - z^*\|$. Thus $x_n \to x^*$ in the norm topology of X and $x^* \in g(D) \subset X$. We shall prove that x^* is a unique solution of (1) in $g(D)$. Note that $\|f F(g(z_n))\| = \|\varphi(z_{n+1} - z_n)\| \leqq \|\varphi\| \cdot \|z_{n+1} - z_n\| \leqq \beta^n\|\varphi\| \|z_1 - z_0\| \to 0$ as $n \to \infty$. According to (2), the mapping $f F(g) : Z \to Z$ is continuous on $D \subset Z$. Since $z_n \to z^*$ we have $f F(g(z_n)) \to f F(g(z^*)) = f F(x^*)$. From $f F(g(z_n) \to 0$ it follows that $f F(x^*) = 0$. From the conditions $m(f) > 0$ on $F(g(D)) \subset Y$, $f(0) = 0$, we obtain that $F(x^*) = 0$. Suppose that $F(x_1^*) = 0$, $F(x_2^*) = 0$, $x_1^*, x_2^* \in g(D)$, $x_1^* \neq x_2^*$. Then there exist $z_1^*, z_2^* \in D$ such that $x_1^* = g(z_1^*)$, $x_2^* = g(z_2^*)$ and $F(g(z_1^*)) = 0$, $F(g(z_2^*)) = 0$. Assuming $f(0) = 0$, we have also $f\,F(g(z_1^*)) = 0$, $f\,F(g(z_2^*)) = 0$. According to (2),

$$\|f\,F(g(z_1^*)) - f\,F(g(z_2^*)) - \varphi(z_1^* - z_2^*)\| = \|\varphi(z_1^* - z_2^*)\| \leqq \alpha\|z_1^* - z_2^*\| .$$

On the other hand,

$$\|z_1^* - z_2^*\| \leqq \|\varphi^{-1}\|\,\|\varphi(z_1^* - z_2^*)\| \leqq \alpha\|\varphi^{-1}\|\,\|z_1^* - z_2^*\| < \|z_1^* - z_2^*\| .$$

Thus $z_1^* = z_2^*$ and therefore $g(z_1^*) = g(z_2^*) = x_1^* = x_2^*$. We have seen that $\{z_n\}$ is a Cauchy sequence in Z. Assuming b) we see that $\{g(z_n)\}$ has a limit in X. Set $\lim_{n\to\infty} g(z_n) = x^*$. Since $f\,Fg(z_n) \to 0$, $g(z_n) \to x^*$ for $n \to \infty$ and fF is closed, we have $f\,F(x^*) = 0$. Similarly as above we obtain that $F(x^*) = 0$, and x^* is the unique solution of (1) in $g(D) \subset X$. This completes the proof.

Theorem 5. *Let F be a mapping defined on the bounded set $D(F) \subset X$, $F : D(F) \to Y$, $f : Y \to Z$, $g : Z \to X$, $\varphi : Z \to Z$, where X, Y, Z are linear normed spaces. Let f, φ be linear mappings, φ continuous, and such that there exist inverses f^{-1}, φ^{-1}; φ^{-1} continuous. Let $z_0 \in Z$ be such that the inequality* (2) *holds for every $u, v \in D$, where $D = \{z \in Z : \|z - z_0\| \leqq r\}$, $r \geqq (1 - \beta)^{-1}\,\|\varphi^{-1}\|\,\|f\,F(g(z_0))\|$, $\beta = \alpha\|\varphi^{-1}\| < 1$. Let $g(D) \subset D(F)$, $M(g) < +\infty$ on D. If either a) Z is complete, or b) X is complete for g and fF is closed, then the conclusions of theorem* 4 *remain valid.*

Proof. Is similar to the proof of theorem 4.

Remark 1. Let X, Y be linear normed spaces. Then Y is complete for $f : X \to Y$ if either a) Y is complete and $M(f) < +\infty$ on X, or b) f is a completely continuous mapping.

The above theorems can be modified so as to obtain some simpler consequences which may be useful for investigation of solutions of nonlinear integral and differential equations.

On taking X, Y Banach spaces, $Z = X$, $g = I$ (I is the identity mapping) we obtain the following

Corollary 1. *Let X, Y be Banach spaces, $F : X \to Y$, φ is a linear continuous mapping from X onto X, $f : Y \to X$ linear having the inverse f^{-1}. Let E be closed subset of X. Furthermore, let the following conditions be fulfilled:* 1) *For every $u, v \in E$ the inequality $\|f(F(u)) - f(F(v)) - \varphi(u - v)\| \leqq \alpha\|u - v\|$ holds.* 2) *The closed ball $D = \{x \in X : \|x - x_1\| \leqq r\}$, where x_1 is defined by the equality $x^{(0)} = \varphi(x_1 - x_0)$, x_0 being an arbitrary element of E, $x^{(0)} = f(F(x_0))$, $r \geqq \beta(1 - \beta)^{-1} \cdot \|x_1 - x_0\|$, $\beta = \alpha M < 1$ (M is a constant from lemma 1), is contained in E. If $m(F) > 0$ on E, then the equation* (1) *has a unique solution x^* in $D \subset X$. The sequence $\{x_n\}$ defined by $x^{(n-1)} = \varphi(x_n - x_{n-1})$, $x^{(n)} = f(F(x_n))$ converges in the* *norm topology of X to* x^* *and*

$$\|x^* - x_n\| \leqq \beta^n(1 - \beta)^{-1} \|x_1 - x_0\| . \tag{8}$$

Remark 2. For φ one may set either $\varphi = I$, $\varphi = \vartheta I$ (ϑ is a positive number), or, $[F'(x_0)]^{-1} = \varphi$, where $F'(x_0)$ denotes the Gâteaux derivative of F at the point x_0, etc.

Corollary 2. *Let* X, Y *be linear normed spaces,* X *complete,* φ *a linear continuous mapping from* X *onto* X, $f : Y \to X$ *linear having the inverse* f^{-1}. *Let* E *be a closed convex subset of* X *such that for every* $x \in E$ *the mapping* $F : X \to Y$ *has the Gâteaux derivative* $F'(x)$. *If the closed ball* $D = \{x \in X : \|x - x_1\| \leqq r\}$ *is contained in* E, *where* x_0 *is an arbitrary element of* E, $x^{(0)} = f(F(x_0))$, $r \geqq \beta(1 - \beta)^{-1} \|x_1 - x_0\|$, $\beta = \alpha M < 1$, $\alpha = \sup_{x \in E} \|\varphi - f F'(x)\|$ (M *is a constant from lemma* 1), *then the equation* (1) *has at least one solution* x^* *in* D. *The sequence* $\{x_n\}$ *defined by* $x^{(n-1)} = \varphi(x_n - x_{n-1})$, $x^{(n)} = f(F(x_n))$ *converges in the norm topology of* X *to the solution* x^* *and the error* $\|x^* - x_n\|$ *satisfies* $\|x^* - x_n\| \leqq \beta^n(1 - \beta)^{-1} \|x_1 - x_0\|$.

Corollary 3. *Let* X *be a real Hilbert space (separable and complete),* $f_1 : X \to X$ *a linear continuous mapping having the inverse* f_1^{-1}. *Let* $F : X \to X$ *be mapping of* X *into* X *having the Gâteaux derivative* $F'(x)$ *such that* $(f_1 F'(x) h, h) \geqq m\|h\|^2$, ($m > 0$) *for every* $x \in D = \{x \in X : \|x - x_0\| \leqq r_\vartheta\}$ *and* $h \in X$, *where* $x_0 \in X$, $r_\vartheta \geqq (1 - \alpha_\vartheta)^{-1} \|f(F(x_0))\|$ $\alpha_\vartheta = \sup_{x \in D} \|I - f F'(x)\|$, $f = \vartheta f_1$, $0 < \vartheta < 2m k^{-1}$, $k = \sup_{x \in D} \|f_1 F'(x)\| < \infty$. ($I$ *is identity mapping of* X). *Then the equation* (1) *has a unique solution* x^* *in* D. *Moreover,* $\|x_n - x^*\| \to 0$ *whenever* $n \to \infty$, $\|x^* - x_n\| = O(\alpha_\vartheta^n)$, *where* $x_{n+1} = x_n - f F(x_n)$.

For a similar result see also [1].

Corollary 4. *Let* X *be a Banach space,* $F : X \to X$, $\varphi : X \to X$, $f : X \to X$, *where linear continuous mappings* φ, f *are such that there exist the inverses* φ^{-1}, f^{-1}; φ^{-1} *continuous. Let* $x_0 \in X$ *be such that the inequality* $\|f F(u) - f F(v) - \varphi(u - v)\| \leqq \alpha\|u - v\|$ *holds for every* $u, v \in D$, *where* $D = \{x \in X : \|x - x_0\| \leqq r\}$ $r \geqq (1 - \beta)^{-1} \|\varphi^{-1}\| \|f(F(x_0))\|$, $\beta = \|\varphi^{-1}\| \alpha < 1$. *Then the equation* (1) *has a unique solution* x^* *in* D. *Furthermore,* $\|x^* - x_n\| \to 0$ *as* $n \to \infty$, $\|x^* - x_n\| = O(\beta^n)$, *where* $x_{n+1} = x_n - \varphi^{-1} f(F(x_n))$.

Consider the mapping F of a Banach space X into X having the Gâteaux derivative $F'(x)$ for every $x \in D = \{x \in X : \|x - x_0\| \leqq r\}$. Let $f : X \to X$ be a linear continuous mapping having the inverse f^{-1}. We introduce the monotone non-negative function $\psi(\varrho)$ defined on $0 \leqq \varrho < r$ by $\psi(\varrho) = \sup_{x \in D} \|f F'(x) - f F'(x_0)\|$. Assume that there exists a bounded operator $[F'(x_0)]^{-1}$. Set $\mu^{-1} = \|[F'(x_0)]^{-1}\|$ and suppose that ϱ_0 is a supremum of all ϱ for which $\psi(\varrho) < \mu$. Then the following corollary is valid.

Corollary 5. *If $x_0 \in X$ is such that $\|F(x_0)\| \leqq \mu(1 - \psi(\varrho_0)\,\mu^{-1})\,\varrho_0$ then the equation* (1) *has a unique solution x^* in D, where $D = \{x \in X : \|x - x_0\| \leqq \varrho_0\}$. The sequence $\{x_n\}$ defined by $x_{n+1} = x_n - [F'(x_0)]^{-1} f(F(x_n))$ converges in the norm topology of X to x^* and the inequality* (8) *holds, where $\beta = \mu^{-1}\,\psi(\varrho_0)$.*

Remark 3. Some of these results were previously published without proofs in [2].

Reference

[1] *J. Kolomý:* Remark to the solution of non-linear functional equations in Banach spaces. Comment. Math. Univ. Carolinae *5* (1964) 97—116.

[2] *J. Kolomý:* Open mapping theorem and solution of nonlinear equations in linear normed spaces. Comment. Math. Univ. Carolinae *6* (1965), 363—366.

[3] *L. Collatz:* Einige Anwendungen functional — analytischer Methoden in der praktischen Analysis. ZAMP, *4* (1953), 327—357.

[4] *H. D. Block:* Construction of solutions and propagation of errors in nonlinear problems. Proc. of Am. Soc. *4* (1953), 715—722.

[5] *L. M. Graves:* Some mapping theorems. Duke Math. Journ. *17* (1950), 111—114.

[6] *М. М. Вайнберг:* Вариоацинные методы проследования нелинейных операторов. Москва 1956.

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Výtah

ŘEŠENÍ NELINEÁRNÍCH FUNKCIONÁLNÍCH ROVNIC V LINEÁRNÍCH NORMOVANÝCH PROSTORECH

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V článku se dokazují některé věty pro řešení nelineárních funkcionálních rovnic v lineárních normovaných prostorech. Tyto věty jsou založeny na lokální aproximaci nelineárních zobrazení lineárními spojitými zobrazeními a na některých větách o otevřeném zobrazení.

Nechť X, Y jsou lineární normované prostory, $f : X \to Y$. Definujme $m(f)$ a $M(f)$ na množině $V \subset X$ takto: $m(f) = \inf \|f(u_1) - f(u_2)\| \,.\, \|u_1 - u_2\|^{-1}$, $M(f) = \sup \|f(u_1) - f(u_2)\| \,.\, \|u_1 - u_2\|^{-1}$ pro všechna $u_1, u_2 \in V$, $u_1 \neq u_2$. Z dokázaných vět uvedeme pouze následující:

Věta 1. *Nechť $F : X \to Y$, kde X, Y jsou lineární normované prostory. Nechť Z je Banachův prostor a f, g jsou zobrazení $f : Y \to Z$, $g : Z \to X$. Nechť φ je lineární spojité zobrazení prostoru Z na Z. Nechť dále jsou splněny následující podmínky:* 1) *pro každé $u, v \in E$, kde E je uzavřená podmnožina Z, je*

$$\|f\,F(g(u)) - f\,F(g(v)) - \varphi(u - v)\| \leqq \alpha\|u - v\|\,,$$

kde zobrazení f je takové, že $m(f) = a > 0$ *na* $F(g(E)) \subset Y$ *a* $f(0) = 0$. 2) *Uzavřená koule* $D = \{z \in Z : \|z - z_1\| \leqq r\} \subset E$ *kde* z_1 *je definováno vztahem* $z^{(0)} = \varphi(z_1 - z_0)$ *a* z_0 *je libovolný prvek z* E, $z^{(0)} = f\,F(g(z_0))$ $r \geqq \beta(1 - \beta)^{-1} \, . \, \|z_1 - z_0\|$, $\beta = \alpha M < 1$ (M *je konstanta z lematu* 1). *Jestliže* $M(g) < +\infty$ *na* E, *potom rovnice* $F(x) = 0$ *má alespoň jedno řešení* x^* *na množině* $g(D) \subset X$. *Posloupnost* $\{x_n\}$, *definovaná rovnicemi* $x_n = g(z_n)$, $z^{(n-1)} = \varphi(z_n - z_{n-1})$, $z^{(n)} = f\,F(g(z_n))$, *konverguje k* x^* *v normě* X *a platí odhad*

$$\|x_n - x^*\| \leqq \beta^n(1 - \beta)^{-1} M(g) \|z_1 - z_0\| \, .$$

Резюме

РЕШЕНИЕ НЕЛИНЕЙНЫХ ФУНКЦИОНАЛЬНЫХ УРАВНЕНИЙ В ЛИНЕЙНЫХ НОРМИРОВАННЫХ ПРОСТРАНСТВАХ

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В статье доказываются некоторые теоремы для решения нелинейных функциональных уравнений в линейных нормированных пространствах. Эти теормы основаны на локальной аппроксимации нелинейных отображений линейными отображениями и на некоторых теоремах об открытом отображении.

Пусть X, Y — линейные нормированные пространства, $f : X \to Y$. Определим $m(f)$ и $M(f)$ на множестве $V \subset X$ следующим образом: $m(f) = \inf \|f(u_1) - f(u_2)\| \|u_1 - u_2\|^{-1}$, $M(f) = \sup \|f(u_2) - f(u_1)\| \|u_1 - u_2\|^{-1}$ для всех $u_1, u_2 \in V$, $u_1 \neq u_2$. Из доказанных теорем приведем только следующую:

Теорема 1. *Пусть* $F : X \to У$, *где* X, $У$ — *линейные нормированные пространства. Пусть* Z — *пространство Банаха и* f, g — *такие отображения, что* $f : У \to Z$, $g : Z \to X$. *Пусть* φ — *линейное непрерывное отображение пространства* Z *на* Z. *Пусть, далее, выполнены следующие условия:* 1) *для каждого* $u, v \in E$, *где* E — *замкнутое множество из* Z, *имеет место неравенство*

$$\|f\,F(g(u)) - f\,F(g(v)) - \varphi(u - v)\| \leqq \alpha \|u - v\| \, ,$$

где отображение f *такое, что* $m(f) = a > 0$ *на* $F(g(E)) \subset У$ *и* $f(0) = 0$. 2) *Замкнутый шар* $D = \{z \in Z : \|z - z_1\| \leqq r\} \subset E$, *причем* z_1 *определено по отношению* $z^{(0)} = \varphi(z_1 - z_0)$, z_0 — *произвольный элемент из* E, $z^{(0)} = f\,F(g(z_0))$, $r \geqq \beta(1 - \beta)^{-1} \|z_1 - z_0\|$, $\beta = \alpha M < 1$, (M — *постоянная из леммы* 1). *Если* $M(g) < +\infty$ *на* E, *то уравнение* $F(x) = 0$ *имеет по крайней мере одно решение* x^* *на множестве* $g(D) \subset X$. *Последовательность* $\{x_n\}$, *определенная равенствами*

$$x_n = g(z_n) \, , \quad z^{(n-1)} = \varphi(z_n - z_{n-1}) \, , \quad z^{(n)} = f\,F(g(z_n)) \, ,$$

сходится к x^* *по норме* X *и имеет место оценка*

$$\|x_n - x^*\| \leqq \beta^n(1 - \beta)^{-1} M(g) \|z_1 - z_0\| \, .$$