Mohammad Ashraf

On left (θ, φ)-derivations of prime rings

Archivum Mathematicum, Vol. 41 (2005), No. 2, 157--166

Persistent URL: `http://dml.cz/dmlcz/107946`

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 41 (2005), 157 – 166

ON LEFT (θ,ϕ)-DERIVATIONS OF PRIME RINGS

MOHAMMAD ASHRAF

Abstract. Let R be a 2-torsion free prime ring. Suppose that θ,ϕ are automorphisms of R. In the present paper it is established that if R admits a nonzero Jordan left (θ,θ)-derivation, then R is commutative. Further, as an application of this resul it is shown that every Jordan left (θ,θ)-derivation on R is a left (θ,θ)-derivation on R. Finally, in case of an arbitrary prime ring it is proved that if R admits a left (θ,ϕ)-derivation which acts also as a homomorphism (resp. anti-homomorphism) on a nonzero ideal of R, then $d=0$ on R.

1. Introduction

Throughout the present paper R will denote an associative ring with centre $Z(R)$. Recall that R is prime if $aRb=\{0\}$ implies that $a=0$ or $b=0$. As usual $[x,y]$ will denote the commutator $xy-yx$. An additive subgroup U of R is said to be a Lie ideal of R if $[u,r]\in U$ for all $u\in U$, $r\in R$. Suppose that θ,ϕ are endomorphisms of R. An additive mapping $d:R\longrightarrow R$ is called a (θ,ϕ)-derivation (resp. Jordan (θ,ϕ)-derivation) if $d(xy)=d(x)\phi(y)+\theta(x)d(y)$, (resp. $d(x^2)=d(x)\phi(x)+\theta(x)d(x)$) holds for all $x,y\in R$. Of course, every $(1,1)$-derivation (resp. Jordan $(1,1)$- derivation), where 1 is the identity mapping on R is a derivation (resp. Jordan derivation) on R. An additive mapping $d:R\to R$ is called a left (θ,ϕ)-derivation (resp. Jordan left (θ,ϕ)-derivation) if $d(xy)=\theta(x)d(y)+\phi(y)d(x)$ (resp. $d(x^2)=\theta(x)d(x)+\phi(x)d(x)$) holds for all $x,y\in R$. Clearly, every left $(1,1)$-derivation (resp. Jordan left $(1,1)$-derivation) is a left derivation (resp. Jordan left derivation) on R. Obviously, every left derivation is a Jordan left derivation but the converse need not be true in general. Recently the author together with Nadeem [1] proved that the converse statement is true in the case when the underlying ring is prime and 2-torsion free. In the present paper we shall show that if a 2-torsion free prime ring R admits an additive mapping satisfying $d(u^2)=2\theta(u)d(u)$ for all $u\in U$, then either $d(U)=\{0\}$ or $U\subseteq Z(R)$ where U is a Lie ideal of R with $u^2\in U$ for all $u\in U$ and θ is an automorphism

1991 *Mathematics Subject Classification*: 16W25, 16N60.

Key words and phrases: Lie ideals, prime rings, derivations, Jordan left derivations, left derivations, torsion free rings.

Received May 26, 2003, revised April 2004.

of R. In fact this result generalizes the main theorem proved in [4]. Further, some more related results are also obtained. Final section of the present paper deals with the study of left (θ, ϕ)-derivation which acts also as a homomorphism of the ring.

2. Preliminaries

We shall make use of the following results, all but one of which are known.

Lemma 2.1 ([9, Lemma 2]). *If $U \not\subseteq Z(R)$ is a Lie ideal of a 2-torsion free prime ring R and $a, b \in R$ such that $aUb = \{0\}$, then $a = 0$ or $b = 0$.*

Lemma 2.2 ([11, Lemma 4]). *Let G and H be additive groups and let R be a 2-torsion free ring. Let $f : G \times G \to H$ and $g : G \times G \to R$ be biadditive mappings. Suppose that for each pair $a, b \in G$ either $f(a, b) = 0$ or $g(a, b)^2 = 0$. In this case either $f = 0$ or $g(a, b)^2 = 0$ for all $a, b \in G$ respectively.*

Lemma 2.3 ([13, Theorem 4]). *Let R be a 2-torsion free prime ring and U a Lie ideal of R. If R admits a derivation d such that $d(u)^n = 0$ for all $u \in U$, where $n \geq 1$ is a fixed integer, then $d(u) = 0$ for all $u \in U$.*

Lemma 2.4 ([16, Lemma 1.3]). *Let R be a 2-torsion free semiprime ring. If U is a commutative Lie ideal of R, then $U \subseteq Z(R)$.*

Now we shall prove the following

Lemma 2.5. *Let R be a 2-torsion free ring and let U be a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Suppose that θ is an endomorphism of R. If $d : R \to R$ is an additive mapping satisfying $d(u^2) = 2\theta(u)d(u)$ for all $u, v \in U$ then*

(i) $d(uvu) = \theta(u^2)d(v) + 3\theta(u)\theta(v)d(u) - \theta(v)\theta(u)d(u)$ *for all* $u, v \in U$.

(ii) $[\theta(u), \theta(v)]\theta(u)d(u) = \theta(u)[\theta(u), \theta(v)]d(u)$ *for all* $u, v \in U$.

(iii) $[\theta(u), \theta(v)]d([u, v]) = 0$ *for all* $u, v \in U$.

(iv) $d(vu^2) = \theta(u^2)d(v) + (3\theta(v)\theta(u) - \theta(u)\theta(v))d(u) - \theta(u)d([u, v])$ *for all* $u, v \in U$.

Proof. (i) Since $uv + vu = (u + v)^2 - u^2 - v^2$, we find that $uv + vu \in U$ for all $u, v \in U$. Hence by linearizing $d(u^2) = 2\theta(u)d(u)$ on u, we get

$$d(uv + vu) = 2\theta(u)d(v) + 2\theta(v)d(u) \quad \text{for all} \quad u, v \in U\,. \tag{2.1}$$

Further, replacing v by $uv + vu$ in (2.1), we get

$$d(u(uv + vu) + (uv + vu)u) = 4\theta(u^2)d(v) + 6\theta(u)\theta(v)d(u) + 2\theta(v)\theta(u)d(u)\,. \tag{2.2}$$

On the other hand,

$$\begin{aligned} d(u(uv + vu) + (uv + vu)u) &= d(u^2v + vu^2) + 2d(uvu) \\ &= 2\theta(u^2)d(v) + 4\theta(v)\theta(u)d(u) + 2d(uvu)\,. \end{aligned}$$

Combining the above equation with (2.2), we get (i).

(ii) By linearizing (i) on u, we get

$$\begin{aligned} d((u+w)v(u+w)) = {} & \theta(u^2)d(v) + \theta(w^2)d(v) + \{\theta(u)\theta(w) + \theta(w)\theta(u)\}d(v) \\ & + 3\theta(u)\theta(v)d(w) + 3\theta(u)\theta(v)d(u) + 3\theta(w)\theta(v)d(w) \\ & + 3\theta(w)\theta(v)d(u) - \theta(v)\theta(u)d(u) - \theta(v)\theta(u)d(w) \\ & - \theta(v)\theta(w)d(u) - \theta(v)\theta(w)d(w)\,. \end{aligned} \tag{2.3}$$

On the other hand,

$$\begin{aligned} d((u+w)v(u+w)) &= d(uvu) + d(wvw) + d(uvw + wvu) \\ &= \theta(u^2)d(v) + 3\theta(u)\theta(v)d(u) - \theta(v)\theta(u)d(u) + \theta(w^2)d(v) \\ &\quad + 3\theta(w)\theta(v)d(w) - \theta(v)\theta(w)d(w) + d(uvw + wvu)\,. \end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), we arrive at

$$\begin{aligned} d(uvw + wvu) = {} & \{\theta(u)\theta(w) + \theta(w)\theta(u)\}d(v) + 3\theta(u)\theta(v)d(w) + 3\theta(w)\theta(v)d(u) \\ & - \theta(v)\theta(u)d(w) - \theta(v)\theta(w)d(u) \quad \text{for all} \quad u, v \in U\,. \end{aligned} \tag{2.5}$$

Since $uv + vu$ and $uv - vu$ both belong to U we find that $2uv \in U$ for all $u, v \in U$. Hence, by our hypothesis we find that $d((2uv)^2) = 2\theta(2uv)d((2uv))$ i.e., $4d(uv)^2 = 8\theta(uv)d(uv)$. Since char$R \neq 2$, we have $d(uv)^2 = 2\theta(u)\theta(v)d(uv)$. Replace w by $2uv$ in (2.5), and use the fact that char $R \neq 2$, to get

$$\begin{aligned} d(uv(uv) + (uv)vu) = {} & \{\theta(u^2)\theta(v) + \theta(u)\theta(v)\theta(u)\}d(v) + 3\theta(u)\theta(v)d(uv) \\ & + 3\theta(u)\theta(v^2)d(u) - \theta(v)\theta(u)d(uv) \\ & - \theta(v)\theta(u)\theta(v)d(u)\,. \end{aligned} \tag{2.6}$$

On the other hand,

$$\begin{aligned} d((uv)^2 + uv^2u) = {} & 2\theta(u)\theta(v)d(uv) + 2\theta(u^2)\theta(v)d(v) \\ & + 3\theta(u)\theta(v^2)d(u) - \theta(v^2)\theta(u)d(u)\,. \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we get

$$[\theta(u), \theta(v)]d(uv) = \theta(u)[\theta(u), \theta(v)]d(v) + \theta(v)[\theta(u), \theta(v)]d(u) \tag{2.8}$$

Replacing $u + v$ for v in (2.8), we have

$$\begin{aligned} 2[\theta(u), \theta(v)]\theta(u)d(u) + [\theta(u), \theta(v)]d(uv) = {} & 2\theta(u)[\theta(u), \theta(v)]d(u) \\ & + \theta(u)[\theta(u), \theta(v)]d(v) + \theta(v)[\theta(u), \theta(v)]d(u)\,. \end{aligned}$$

Now application of (2.8) yields (ii).

(iii) Linearize (ii) on u, to get

$$\begin{aligned}[][\theta(u),\theta(v)]\theta(u)d(u) + [\theta(u),\theta(v)]\theta(v)d(v) + [\theta(u),\theta(v)]\theta(u)d(v) \\ + [\theta(u),\theta(v)]\theta(v)d(u) = \ \theta(u)[\theta(u),\theta(v)]d(u) \\ + \theta(u)[\theta(u),\theta(v)]d(v) + \theta(v)[\theta(u),\theta(v)]d(u) \\ + \theta(v)[\theta(u),\theta(v)]d(v) \quad \text{for all} \quad u,v\in U\,.\end{aligned}$$

Now application of (2.8) and (ii) yields that

$$[\theta(u),\theta(v)]\theta(u)d(v) + [\theta(u),\theta(v)]\theta(v)d(u) = [\theta(u),\theta(v)]d(uv)$$

and hence

(2.9) $$[\theta(u),\theta(v)]\{d(uv) - \theta(u)d(v) - \theta(v)d(u)\} = 0 \quad \text{for all} \quad u,v\in U\,.$$

Combining (2.1) and (2.9) we find that,

(2.10) $$[\theta(u),\theta(v)]\{d(vu) - \theta(u)d(v) - \theta(v)d(u)\} = 0 \quad \text{for all} \quad u,v\in U\,.$$

Further, combining of (2.9) and (2.10) yields the required result.

(iv) Replace v by $2vu$ in (2.1), and use the fact that char$R \neq 2$, to get

(2.11) $$d(uvu + vu^2) = 2(\theta(u)d(uv) + \theta(v)\theta(u)d(u)) \quad \text{for all} \quad u,v\in U\,.$$

Again, replacing v by $2uv$ in (2.1), we get

(2.12) $$d(u^2v + uvu) = 2(\theta(u)d(uv) + \theta(u)\theta(v)d(u)) \quad \text{for all} \quad u,v\in U\,.$$

Now, combining (2.11) and (2.12), we get

(2.13) $$d(u^2v - vu^2) = 2(\theta(u)d([u,v]) + [\theta(u),\theta(v)]d(u)) \quad \text{for all} \quad u,v\in U\,.$$

Replacing u by u^2 in (2.1), we have

(2.14) $$d(u^2v + vu^2) = 2(\theta(u^2)d(v) + 2\theta(v)\theta(u)d(u)) \quad \text{for all} \quad u,v\in U\,.$$

Hence, subtracting (2.13) from (2.14) and using the fact that characteristic of R is different from two we find that

$$d(vu^2) = \theta(u^2)d(v) + \{3\theta(v)\theta(u) - \theta(u)\theta(v)\}d(u) - \theta(u)d([u,v]) \quad \text{for all} \quad u,v\in U\,.$$

3. Left derivation and commutativity of prime ring

A mapping $f : R \to R$ is said to be commuting on R if $f(x)x = xf(x)$ holds for all $x \in R$. Comparing Jordan left derivation with commuting mapping on a ring R, it turns out that notion of Jordan left derivation is in a close connection with the commuting mapping on R. There has been considerable interest for commuting mappings on prime rings. The fundamental result in this direction is due to Posner [18] who proved that if a prime ring R admits a non-zero derivation that is commuting on R, then R is commutative. Using rather weak hypotheses Bresar and Vukman [12] obtained a result which shows that the existence of a non-zero Jordan left derivation on a 2-torsion free and 3-torsion free prime ring R forces R to be commutative. It was also remarked by Bresar and Vukman that the assumption "R is 3-torsion free" in the hypotheses of the above result may be avoided. In this direction we have obtained the following theorem which also includes the main result of [4].

Theorem 3.1. *Let R be a 2-torsion free prime ring and let U be a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Suppose that θ is an automorphism of R. If $d : R \to R$ is an additive mapping satisfying $d(u^2) = 2\theta(u)d(u)$ for all $u \in U$, then either $d(U) = \{0\}$ or $U \subseteq Z(R)$.*

Proof. Suppose that $U \not\subseteq Z(R)$. By Lemma 2.5(ii) we have

$$\{\theta(u^2)\theta(v) - 2\theta(u)\theta(v)\theta(u) + \theta(v)\theta(u^2)\}d(u) = 0 \quad \text{for all} \quad u, v \in U\,. \tag{3.1}$$

Replacing $[u,w]$ for u in (3.1), we get

$$\begin{aligned}[\theta(u),\theta(w)]^2\theta(v)d([u,w]) - 2[\theta(u),\theta(w)]\theta(v)[\theta(u),\theta(w)]d([u,w])\\ + \theta(v)[\theta(u),\theta(w)]^2 d([u,w]) = 0\\ \text{for all} \quad u,v,w \in U\,.\end{aligned}$$

Now, application of Lemma 2.5(iii) yields that $\theta^{-1}([\theta(u),\theta(w)]^2)U\theta^{-1}(d([u,w]) = \{0\}$. Hence by Lemma 2.1 we find that for each pair $u, w \in U$, either $[\theta(u),\theta(w)]^2 = 0$ or $d([u,w]) = 0$. This implies that either $[u,w]^2 = 0$ or $d([u,w]) = 0$. Note that the mappings $(u,w) \mapsto [u,w]$ and $(u,w) \mapsto d([u,w])$ satisfy the requirements of the Lemma 2.2. Hence, either $[u,w]^2 = 0$ for all $u,w \in U$ or $d([u,w]) = 0$ for all $u,w \in U$. If $[u,w]^2 = 0$ for all $u,w \in U$, then for each $u \in U, (I_u(w))^2 = 0$ for all $w \in U$, where I_u is the inner derivation such that $I_u(w) = [u,w]$. Thus by the application of Lemma 2.3 we find that U is a commutative Lie ideal of R, and hence by Lemma 2.4, $U \subseteq Z(R)$, a contradiction. Hence, we consider the remaining case that $d([u,w]) = 0$ for all $u,w \in U$, i.e., $d(uw) = d(wu)$ for all $u,w \in U$. Since $wu - uw$ and $wu + uw$ both belong to U, we find that $2wu \in U$ for all $u,w \in U$. This yields that $d((2wu)u) = d(u(2wu))$. Since (2.1) is valid in the present situation, we find that

$$\begin{aligned}4d((wu)u) &= d((2wu)u + u(2wu))\\ &= 4\theta(w)\theta(u)d(u) + 2\theta(u)d(2wu)\\ &= 4\theta(w)\theta(u)d(u) + 2\theta(u)d(wu+uw)\\ &= 4\{\theta(w)\theta(u)d(u) + \theta(u)\theta(w)d(u) + \theta(u^2)d(w)\}\,.\end{aligned}$$

Since R is a 2-torsion free, we obtain

$$d((wu)u) = \theta(u^2)d(w) + \theta(u)\theta(w)d(u) + \theta(w)\theta(u)d(u) \quad \text{for all} \quad u,w \in U \tag{3.2}$$

Since $d([u,w]) = 0$ for all $u,w \in U$, using Lemma 2.5(iv) and (3.2), we get $2[\theta(u),\theta(w)]d(u) = 0$. This implies that

$$[\theta(u),\theta(w)]d(u) = 0 \quad \text{for all} \quad u,w \in U\,. \tag{3.3}$$

Now, replacing w by $2wv$ in (3.3) and using the fact that $\operatorname{char} R \neq 2$ we get $[\theta(u),\theta(w)]\theta(v)d(u) = 0$ i.e., $\theta^{-1}([\theta(u),\theta(w)])U\theta^{-1}(d(u) = \{0\}$. Thus by Lemma 2.1, we find that for each $u \in U, \theta^{-1}([\theta(u),\theta(w)]) = 0$ or $\theta^{-1}(d(u)) = 0$. This implies that $[u,w] = 0$ or $d(u) = 0$. Now let $U_1 = \{u \in U \mid \ [u,w] = 0$ for all

$w \in U\}$ and $U_2 = \{u \in U \mid d(u) = 0\}$. Clearly, U_1 and U_2 are additive subgroups of U whose union is U. But a group can not be written as a union of two of its proper subgroups and hence by Brauer's trick either $U = U_1$ or $U = U_2$. If $U = U_1$, then $[u, w] = 0$ for all $u, w \in U$ and by using the similar arguments as above we get $U \subseteq Z(R)$, again a contradiction. Hence we have the remaining possibility that $d(u) = 0$ for all $u \in U$ i.e., $d(U) = \{0\}$. This completes the proof of the theorem. □

As an application of the above theorem we get the following result, which generalizes the main theorem of [1].

Theorem 3.2. *Let R be a 2-torsion free prime ring and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Suppose that θ is an automorphism of R. If $d : R \to R$ is an additive mapping satisfying $d(u^2) = 2\theta(u)d(u)$ for all $u \in U$, then $d(uv) = \theta(u)d(v) + \theta(v)d(u)$ for all $u, v \in U$.*

Proof. Suppose that $d = 0$ on U. Since $2uv \in U$, $uv - vu$ and $uv + vu$ both belong to U, we find that $2d(uv) = d(2uv) = 0$. This implies that $d(uv) = 0$ for all $u, v \in U$. Hence, the result is obvious in the present case. Therefore now assume that $d(U) \neq \{0\}$. Then by the above theorem $U \subseteq Z(R)$. Thus R satisfies the property $d(u^2) = d(u)\theta(u) + \theta(u)d(u)$ for all $u \in U$ and hence by Theorem 3.2 of [3] we find that $d(uv) = d(u)\theta(v) + \theta(u)d(v)$ for all $u, v \in U$. Further since $\theta(U) \subseteq Z(R)$, we find that $d(uv) = \theta(u)d(v) + \theta(v)d(u)$ holds for all $u, v \in U$. □

Corollary 3.1. *Let R be a 2-torsion free prime ring. If $d : R \to R$ is a Jordan left derivation, then d is a left derivation.*

If the underlying ring is arbitrary, then we have the following

Theorem 3.3. *Let R be a 2-torsion free ring and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Suppose that θ is an endomorphism of R and R has a commutator which is not a zero divisor. If $d : R \to R$ is an additive mapping satisfying $d(u^2) = 2\theta(u)d(u)$ for all $u \in U$, then $d(uv) = \theta(u)d(v) + \theta(v)d(u)$ for all $u, v \in U$.*

Proof. For any $u, v \in U$, define a map $f : U \times U \to R$ such that $f(u, v) = d(uv) - \theta(u)d(v) - \theta(v)d(u)$. Since θ and d both are additive, f is additive in both the arguments and is zero if d is a left (θ, θ)-derivation. Note that (2.9) is still valid in the present situation and hence we have

$$[\theta(u), \theta(v)]f(u, v) = 0 \quad \text{for all} \quad u, v \in U\,. \tag{3.4}$$

Let a, b be fixed elements of U such that $[\theta(a), \theta(b)]c = 0$ implies that $c = 0$. Application of (3.4) yields that

$$f(a, b) = 0\,. \tag{3.5}$$

Replacing u by $u + a$ in (3.4) and using (3.4), we find that

$$[\theta(u), \theta(v)]f(a, v) + [\theta(a), \theta(v)f(u, v) = 0 \quad \text{for all} \quad u, v \in U\,. \tag{3.6}$$

Replacing v by b in (3.6) and using (3.6), we have

$$f(u, b) = 0 \quad \text{for all} \quad u \in U\,. \tag{3.7}$$

Further, substituting $v + b$ for v in (3.6) and using (3.5) and (3.7), we get

$$[\theta(u), \theta(b)]f(a, v) + [\theta(a), \theta(b)]f(u, v) = 0 \quad \text{for all} \quad u, v \in U\,. \tag{3.8}$$

Now replacing u by a in (3.8) and using the fact that $\operatorname{char} R \neq 2$, we have

$$f(a, v) = 0 \quad \text{for all} \quad v \in U\,. \tag{3.9}$$

Combining of (3.8) and (3.9) yields that $[\theta(a), \theta(b)]f(u, v) = 0$. This implies that $f(u, v) = 0$ for all $u, v \in U$ i.e., d is a left (θ, θ)-derivation. □

In the end of this section it is tempting to conjecture as follows

Conjecture 3.1. *Let R be a 2-torsion free prime ring and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Suppose that θ, ϕ are automorphisms of R. If $d : R \to R$ is an additive mapping satisfying $d(u^2) = \theta(u)d(u) + \phi(u)d(u)$ for all $u \in U$, then either $d(U) = \{0\}$ or $U \subseteq Z(R)$.*

4. Left derivation as a homomorphism or as an anti-homomorphism

Let S be a non-empty subset of R and $d : R \to R$ a derivation of R. If $d(xy) = d(x)d(y)$ (resp. $d(xy) = d(y)d(x)$) holds for all $x, y \in S$, then d is said to act as a homomorphism (resp. anti-homomorphism) on S. Recently, Bell and Kappe [8] proved that if K is a non-zero right ideal of a prime ring R and $d : R \to R$ a derivation of R such that d acts as a homomorphism on K, then $d = 0$ on R. This result was further extended for (θ, ϕ)-derivation in [2] as follows:

Theorem 4.1 ([2, Theorem 3.2])**.** *Let R be a prime ring and K a nonzero ideal of R, and let θ, ϕ be automorphisms of R. Suppose that $d : R \to R$ is a (θ, ϕ)-derivation of R.*

(i) *If d acts as a homomorphism on K, then $d = 0$ on R.*

(ii) *If d acts as an anti-homomorphism on K, then $d = 0$ on R.*

In the present section our objective is to extend the above study to the left derivation of a prime ring R which acts either as a homomorphism or as an anti-homomorphism of R.

Theorem 4.2. *Let R be a prime ring and K a nonzero ideal of R, and let θ, ϕ be automorphisms of R. Suppose $d : R \to R$ is a left (θ, ϕ)-derivation of R.*

(i) *If d acts as an anti-homomorphism on K, then $d = 0$ on R.*

(ii) *If d acts as a homomorphism on K, then $d = 0$ on R.*

Proof. (i) Let d act as an anti-homomorphism on K. By our hypothesis, we have

$$d(xy) = \theta(x)d(y) + \phi(y)d(x) \quad \text{for all} \quad x, y \in K\,. \tag{4.1}$$

In (4.1) replacing y by xy, we get

$$(4.2)\qquad d(xy)d(x) = d(x(xy)) = \theta(x)d(xy) + \phi(xy)d(x) \quad \text{for all} \quad x, y \in K\,.$$

Now multiplying (4.1) in the right by $d(x)$ and using the fact that d is an anti-homomorphism on K, we get

$$(4.3)\qquad d(xy)d(x) = \theta(x)d(xy) + \phi(y)d(x)d(x) \quad \text{for all} \quad x, y \in K\,.$$

Combining (4.2) and (4.3), we get

$$(4.4)\qquad \phi(x)\phi(y)d(x) = \phi(y)d(x)d(x)\,.$$

In (4.4) replace y by ry, to get

$$(4.5)\qquad \phi(x)\phi(r)\phi(y)d(x) = \phi(r)\phi(y)d(x)d(x) \quad \text{for all} \quad x, y \in K \quad \text{and} \quad r \in R\,.$$

Multiplying (4.4) on left by $\phi(r)$ and combining with (4.5), we obtain

$$(4.6)\qquad [\phi(r), \phi(x)]\phi(y)d(x) = 0\,.$$

In (4.6) replacing y by sy, we get

$$[\phi(r), \phi(x)]\phi(s)\phi(y)d(x) = 0 \quad \text{for all} \quad x, y \in K \quad \text{and} \quad r, s \in R\,,$$

and hence, $[r, x]Ry\phi^{-1}(d(x)) = \{0\}$ for all $x, y \in K$ and $r \in R$. Thus for each $x \in K$, the primeness of R forces that either $[r, x] = 0$ or $\phi(y)d(x) = 0$. Let $K_1 = \{x \in K \mid \phi(y)d(x) = 0 \text{ for all } y \in K\}$ and $K_2 = \{x \in K \mid [r, x] = 0 \text{ for all } r \in R\}$. Then clearly K_1 and K_2 are additive subgroups of K whose union is K. By Braur's trick, we have $\phi(y)d(x) = 0$ for all $x, y \in K$ or $[r, x] = 0$ for all $x \in K$ and $r \in R$. If $[r, x] = 0$, replace x by sx, to get $[r, s]x = 0$ for all $x \in K$ and $r, s \in R$, this implies that $[r, s]Rx = \{0\}$. The primeness of R forces that either $x = 0$ or $[r, s] = 0$, but $K \neq \{0\}$, we have $[r, s] = 0$ for all $r, s \in R$, i.e., R is commutative. So, $d(xy) = d(x)\phi(y) + \theta(x)d(y)$ for all $x, y \in K$ i.e., d is a (θ, ϕ)-derivation which acts as an anti-homomorphism on K. Hence by Theorem 4.1(ii), we have $d = 0$ on R. Henceforth, we have remaining possibility that

$$(4.7)\qquad \phi(y)d(x) = 0 \quad \text{for all} \quad x, y \in K\,.$$

Replace y by yr in (4.7), to get $\phi(y)\phi(r)d(x) = 0$ for all $x, y \in K$ and $r \in R$, and hence $yR\phi^{-1}(d(x)) = \{0\}$. This implies that $\phi^{-1}(d(x)) = 0$, that is

$$(4.8)\qquad d(x) = 0 \quad \text{for all} \quad y \in K\,.$$

Replace x by sx in (4.8), to get

$$(4.9)\qquad \phi(x)d(s) = 0 \quad \text{for all} \quad x \in K \quad \text{and} \quad s \in R\,.$$

Replacing x by xr in (4.9), we get $\phi(x)\phi(r)d(s) = 0$ for all $x \in K$ and $r, s \in R$, and hence $xR\phi^{-1}(d(s)) = \{0\}$. Since R is prime, and K a nonzero ideal of R, we find that $d = 0$ on R.

(ii) If d acts as a homomorphism on K, then we have

$$(4.10)\qquad d(x)d(y) = d(xy) = \theta(x)d(y) + \phi(y)d(x) \quad \text{for all} \quad x, y \in K\,.$$

Replacing x by xy in (4.10), we get

$$d(xy)d(y) = \theta(x)\theta(y)d(y) + \phi(y)d(xy) \quad \text{for all} \quad x, y \in K\,.$$

Now, application of (4.10) yields that $\theta(x)d(y)d(y) = \theta(x)\theta(y)d(y)$. This implies that

$$\theta(x)(d(y) - \theta(y))d(y) = 0 \quad \text{for all} \quad x, y \in K\,. \tag{4.11}$$

Replace x by xr in (4.11), to get $\theta(x)\theta(r)(d(y) - \theta(y))d(y) = 0$ for all $x, y \in K$ and $r \in R$, and hence, $xR\theta^{-1}((d(y) - \theta(y))d(y)) = \{0\}$ for all $x, y \in K$. The primeness of R forces that either $x = 0$ or $\theta^{-1}((d(y) - \theta(y))d(y)) = 0$. Since K is a nonzero ideal of R, we have $\theta^{-1}((d(y) - \theta(y))d(y)) = 0$, this yields that $(d(y) - \theta(y))d(y) = 0$ that is $d(y^2) = \theta(y)d(y)$. Since d is a left (θ, ϕ)-derivation, we find that $\phi(y)d(y) = 0$. Linearizing the latter relation, we have

$$\phi(y)d(x) + \phi(x)d(y) = 0 \quad \text{for all} \quad x, y \in K\,. \tag{4.12}$$

Replace x by yx in (4.12), to get

$$\phi(y)\phi(x)d(y) = 0 \quad \text{for all} \quad x, y \in K\,. \tag{4.13}$$

Substituting sx for x in (4.13), we get $\phi(y)\phi(s)\phi(x)d(y) = 0$ for all $x, y \in K$ and $s \in R$, and hence $yRx\phi^{-1}(d(y)) = \{0\}$. Thus for each $y \in K$; the primeness of R forces that either $y = 0$ or $x\phi^{-1}(d(y)) = 0$. But $y = 0$ also implies that $x\phi^{-1}(d(y)) = 0$, that is

$$\phi(x)d(y) = 0 \quad \text{for all} \quad x, y \in K\,. \tag{4.14}$$

Now using similar techniques as used to get (i) from (4.7) we get the required result.

Acknowledgement. The author is greatly indebted to the referee for his valuable comments and several useful suggestions.

References

[1] Ashraf, M. and Rehman, N., *On Lie ideals and Jordan left derivation of prime rings*, Arch. Math.(Brno) **36** (2000), 201–206.

[2] Ashraf, M., Rehman, N. and Quadri, M. A., *On (σ, τ)- derivations in certain class of rings*, Rad. Mat. **9** (1999), 187–192.

[3] Ashraf, M., Rehman, N. and Quadri, M. A., *On Lie ideals and (σ, τ)-Jordan derivations on prime rings*, Tamkang J. Math. **32** (2001), 247–252.

[4] Ashraf, M., Rehman, N. and Shakir Ali, *On Jordan left derivations of Lie ideals in prime rings*, Southeast Asian Bull. Math. **25** (2001), 375–382.

[5] Awtar, R., *Lie and Jordan structures in prime rings with derivations*, Proc. Amer. Math. Soc. **41** (1973), 67–74.

[6] Awtar, R., *Lie ideals and Jordan derivations of prime rings*, Proc. Amer. Math. Soc. **90** (1984), 9–14.

[7] Bell, H. E. and Daif, M. N., *On derivations and commutativity in prime rings*, Acta Math. Hungar. **66** (1995), 337–343.

[8] Bell, H. E. and Kappe, L. C., *Rings in which derivations satisfy certain algebraic conditions*, Acta Math. Hungar. **53** (1989), 339–346.

[9] Bergen, J., Herstein, I. N. and Kerr, J. W., *Lie ideals and derivations of prime rings* , J. Algebra **71** (1981), 259–267.

[10] Bresar, M., *Jordan derivations on semiprime rings*, Proc. Amer. Math. Soc. **104** (1988), 1003–1006.

[11] Bresar, M. and Vukman, J., *Jordan* (θ, ϕ)*-derivations*, Glas. Mat., III. Ser. **26** (1991), 13–17.

[12] Bresar, M. and Vukman, J., *On left derivations and related mappings*, Proc. Amer. Math. Soc. **110** (1990), 7–16.

[13] Cairini, L. and Giambruno, A., *Lie ideals and nil derivations*, Boll. Un. Mat. Ital. **6** (1985), 497–503.

[14] Deng, Q., *Jordan* (θ, ϕ)*-derivations*, Glasnik Mat. **26** (1991), 13–17.

[15] Herstein, I. N., *Jordan derivations of prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1104–1110.

[16] Herstein, I. N., *Topics in ring theory*, Univ. of Chicago Press, Chicago 1969.

[17] Kill-Wong Jun and Byung-Do Kim, *A note on Jordan left derivations*, Bull. Korean Math. Soc. **33** (1996), 221–228.

[18] Posner, E. C., *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100.

[19] Vukman, J.,*Commuting and centralizing mappings in prime rings*, Proc. Amer. Math. Soc. **109** (1990), 47–52.

[20] Vukman, J., *Jordan left derivations on semiprime rings*, Math. J. Okayama Univ. **39** (1997), 1–6.

Department of Mathematics, Faculty of science
King Abdul Aziz University
P. O. Box. 80203, Jeddah 21589, Saudia-Arabia
E-mail: mashraf80@hotmail.com

Current address:
Department of Mathematics, Aligarh Muslim University
Aligarh-202002, India