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SIMPLICIAL TYPES AND POLYNOMIAL ALGEBRAS

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Abstract. This paper shows that the simplicial type of a finite simplicial complex K is determined by its algebra A of polynomial functions on the baricentric coordinates with coefficients in any integral domain. The link between K and A is done through certain admissible matrix associated to K in a natural way. This result was obtained for the real numbers by I. V. Savel'ev [5], using methods of real algebraic geometry. D. Kan and E. Miller had shown in [2] that A determines the homotopy type of the polyhedron associated to K and not only its rational homotopy type as it was previously proved by D. Sullivan in [6].

§1. Introduction

D. Kan and E. Miller [2] proved that for every finite simplicial complex K and any *unique factorization domain with unit* R the Sullivan's algebra of polynomial 0-forms with coefficients in R, $A^0_R(K)$, determines the homotopy type of the associated polyhedron $|K|$ and not only its rational homotopy type as was previously proven by D. Sullivan [6]. Later I. V. Savel'ev, using methods of real algebraic geometry, proved in a paper published in 1991, [5], that actually one can deduce from $A^0_R(K)$, *for* R *being the real numbers*, the whole structure of the simplicial complex K and not just its homotopy type.

The purpose of this paper is to show *by a different method* that the use of the real field is not essential and any integral domain R could be used to recover from $A^0_R(K)$ the simplicial complex K, up to simplicial equivalence, and therefore contains all the information about the topological type of $|K|$, see Theorem (3.9).

The link between the finite simplicial complex K and its algebra of polynomial 0-forms $A^0_R(K)$ is done here through a certain *admissible matrix* φ_K, associated to K in a natural way.

Corollary (3.7) of this paper gives also a more direct proof of how to obtain Sullivan's de Rham complex from its 0-forms, see [3], and the Example (3.8) shows

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that the cohomology of the algebraic de Rham complex of $A^0_R(K)$ does not give the "correct" cohomology of $|K|$.

In this paper R will be any commutative integral domain with unit and algebras are supposed to be commutative with a unit which is preserved by morphisms.

Exterior power is denoted by Λ and if A is an $R-$algebra, $\Omega_R(A)$ denotes then the $A-$module of Kähler differentials, i.e. ker $\mu/(\text{ker } \mu)^2$ where $\mu : A \otimes_R A \to A$ is the multiplication.

We have $d : A \to \Omega_R(A)$ given by $da =$class of $(a \otimes 1 - 1 \otimes a)$ and the standard extension of d to obtain the *algebraic de Rham complex* on A, $(\Lambda^*_A \Omega_R(A), d)$.

§2. Simplicial complexes and admissible matrices

Let K be a finite simplicial complex with maximal simplices $\sigma_1, \dots, \sigma_r$ and let us denote by $\mathcal{P}_K$ the set of simplices appearing as intersections of maximal simplices of K and so the simplices of K are subsets of members of $\mathcal{P}_K$.

Define a partition Σ_K of K by especifying that two vertices v and w are in the same class if and only if for each maximal simplex σ_i either $\{v, w\} \subset \sigma_i$ or $\{v, w\} \cap \sigma_i = \emptyset$.

It is clear that Σ_K can be regarded as the set of vertices of a simplicial complex with maximal simplices $\bar{\sigma}_i = \{\omega \in \Sigma_K | \omega \subset \sigma_i\}$, $i = 1, \dots, r$. Then $\Sigma_{\Sigma_K} = \Sigma_K$ and we have simplicial maps $f : K \to \Sigma_K$ and $g : \Sigma_K \to K$ such that $f \circ g$ is the identity and $g \circ f$ induces a map homotopic to the identity in the associated polyhedron. In fact, just define $f(v)$ as the member of Σ_K containing v and $g(w) \in w$ for all $w \in \Sigma_K$.

(2.1) Observe that if $\sigma \in \mathcal{P}_K$ and we consider $\tilde{\sigma} = \sigma - \cup_{\sigma_i \not\supset \sigma} \sigma_i$, either $\tilde{\sigma} = \emptyset$ or $\tilde{\sigma} \in \Sigma_K$ and clearly Σ_K coincides with the set of nonempty $\tilde{\sigma}$ for all $\sigma \in \mathcal{P}_K$.

The following formula is deduced easily for the number of elements of $\tilde{\sigma}$

$$|\tilde{\sigma}| = \sum_{\omega} (-1)^{|\omega|} |\omega|$$

where ω in the sum runs through the members of $\mathcal{P}_K$ of the form $\sigma \cap \sigma_i$, for $i = 1, \dots, r$ and $|\quad|$ denotes number of elements.

(2.2) Define then a matrix $\varphi_K = (a_{ij})$ of r rows and s columns by especifying that a_{ij} is either 1 or 0 according to ω_j being or not a subset of σ_i.

Here $\Sigma_K = \{\omega_1, \dots, \omega_s\}$.

We also consider the integer vector $\mathbf{n}_K = (n_1, \dots, n_s)$ where n_i is the number of elements of ω_i, $i = 1, \dots, s$.

We say that $(\mathbf{n}_K, \varphi_K)$ is an *admissible couple for K*.

Remarks.

i) The number of vertices of K is $n_1 + \cdots + n_s$

ii) The number of elements of σ_i is $\sum_{j=1}^{s} a_{ij} n_j$

iii) The simplices $\sigma \in \mathcal{P}_K$ are determined by the sequence $\langle \sigma, \omega_1 \rangle, \dots, \langle \sigma, \omega_s \rangle$, where $\langle \sigma, \omega_j \rangle$ is 1 or 0 depending on whether or not ω_j is a subset of σ.

iv) It is clear the the couple $(\mathbf{n}_K, \varphi_K)$ is determined by K once we have chosen an order $\omega_1, \ldots, \omega_s$ of Σ_K together with an order $\sigma_1, \ldots, \sigma_r$ of the set of maximal simplices of K. Therefore $(\mathbf{n}_K, \varphi_K)$ is determined by K up to an arbitrary permutation of rows of φ_K or any permutation of the components of $\mathbf{n}_K$ and the same permutation of the columns of φ_K.

v) If $(\mathbf{n}_K, \varphi_K)$ is an admissible couple for K, then the corresponding couple for the simplicial complex Σ_K is $(\mathbf{n}_\Sigma, \varphi_\Sigma)$, with $\varphi_\Sigma = \varphi_K$ and $\mathbf{n}_\Sigma = (1, \ldots, 1)$.

vi) The associated polyhedron $|K|$ is not connected if and only if it has an admissible couple of the form $(\mathbf{n}_K, \varphi_K)$ with

$$\varphi_K = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

(2.3) The following properties of an admissible couple $(\mathbf{n}_K, \varphi_K)$ for a simplicial complex K are clear:

(a) n_i are integers greater than 0.

(b) All the numbers a_{ij} are either 0 or 1.

(c) Each column contains at least an 1.

(d) No row is obtained from another row by turning some 1‘s into 0‘s.

(e) Any two columns are different.

(2.4) **Definition.** We say that a couple $(\mathbf{n}, \varphi)$, where $\mathbf{n} = (n_1, \ldots, n_s)$ and φ is an $r \times s$ matrix, is *admissible* if and only if satisfies the above properties and two admissible couples are said to be *equivalent* if and only if one is obtained from the other by permutation of rows or any permutation of the n_i and the same permutation of the columns of φ.

(2.5) **Proposition.** *Admissible couples of equivalent finite simplicial complexes are equivalent and the map that associates to each equivalence class of finite simplicial complexes the equivalence class of its admissible couple is a bijection with the set of equivalence classes of admissible couples.*

In fact, let K and K' be equivalent finite simplicial complexes. Therefore we have a one to one map $f : K \to K'$ sending the maximal simplices $\sigma_1, \ldots, \sigma_r$ of K to the maximal simplices $f(\sigma_1), \ldots, f(\sigma_r)$ of K'.

Let $\Sigma_K = \{\omega_1, \ldots, \omega_s\}$, then $\Sigma_{K'} = \{f(\omega_1), \ldots, f(\omega_s)\}$ is the partition corresponding to K'. It is then obvious that K and K' have equivalent couples.

Suppose now that $(\mathbf{n}, \varphi)$ is admissible and consider the following finite simplicial complex with $n = n_1 + \cdots + n_s$ vertices: $K = \{1, \ldots, n\}$, $\Sigma_K = \{\omega_1, \ldots, \omega_s\}$ with $\omega_1 = \{1, \ldots, n_1\}, \ldots, \omega_s = \{n_1 + \cdots + n_{s-1} + 1, \ldots, n\}$ and maximal simplices $\sigma_i = \cup_{\{j | a_{ij} = 1\}} \omega_j$, $i = 1, \ldots, r$. The number of elements of σ_i being $\sum_{j=1}^{s} a_{ij} n_j$.

It is clear that K is a finite complex whose associated admissible matrix is the given one up to equivalence.

§3. de Rham complexes on a simplicial complex

Let K be a finite simplicial complex with admissible $(\mathbf{n}_K, \varphi_K)$ and let R be any conmutative integral domain with unit.

We may consider $K = \{1, \ldots, n\}$, $n = n_1 + \cdots + n_s$, $\omega_1 = \{1, \ldots, n_1\}, \ldots,$ $\omega_s = \{n_1 + \cdots + n_{s-1} + 1, \ldots, n\}$ and the maximal simplices $\sigma_i = \cup_{\{j|a_{ij}=1\}}\omega_j$, $i = 1, \ldots, r$.

If σ is a simplex of K, its j-th face, $j \in \sigma$, is the simplex $\partial^j\sigma$ obtained by deleting j from σ.

Associated to each simplex σ of K we define a ring

$$R_\sigma = R[X_i]_{i\in\sigma}/(\sum_{i\in\sigma} X_i - 1)$$

and we have face maps $\partial^j : R_\sigma \to R_{\partial^j\sigma}$ given by sending X_j to zero.

It is clear that each R_σ is a polynomial ring, ∂^j is surjective with kernel the ideal generated by the class of X_j and the following relations hold

$$\partial^i\partial^j = \partial^j\partial^i \quad \forall \quad \{i, j\} \subset \sigma$$

Define next $A^0_R(K)$ as follows: an element f of $A^0_R(K)$ associates to each simplex σ of K an element $f(\sigma) \in R_\sigma$ such that $f(\partial^j\sigma) = \partial^j(f(\sigma))$ for all $j \in \sigma$.

Note that $A^0_R(K)$ has an obvious structure of R-algebra. It is the algebra of polynomial functions on the barycentric coordinates of K with coefficients in R.

We have the *algebraic de Rham complex* on $A^0_R(K)$

$$(A^*_R(K), d) = (\Lambda_{A^0_R(K)}\Omega_R(A^0_R(K)), d)$$

and *Sullivan's de Rham complex* $(\tilde{A}^*_R(K), d)$ defined as follows, see chapter 13 of [1]: an element $\Phi \in \tilde{A}^p_R(K), p \geq 0$, is a family $\{\Phi_\sigma\}_{\sigma\in K}$ such that $\Phi_\sigma \in \Lambda^p_{R_\sigma}\Omega_R(R_\sigma)$ and $\Omega(\partial^i)(\Phi_\sigma) = \Phi_{\partial^i\sigma}$ for each face map $\partial^i : R_\sigma \to R_{\partial^i\sigma}$, where $\Omega(\partial^i)$ is the induced map

$$\Omega(\partial^i) : \Lambda_{R_\sigma}\Omega_R(R_\sigma) \to \Lambda_{R_{\partial^i\sigma}}\Omega_R(R_{\partial^i\sigma})\,.$$

It is obvious that $\tilde{A}^0_R(K) = A^0_R(K)$.

We also have a natural homomorphism of commutative graded differential algebras

$$\varphi : A^*_R(K) \to \tilde{A}^*_R(K)$$

given by

$$\varphi(\Phi)(\sigma) = \Omega(\rho^K_\sigma)(\Phi) \in \Lambda^p_{R_\sigma}\Omega_R(R_\sigma)$$

for any simplex σ of K and $\Phi \in A^p_R(K)$.

Here, $\rho^K_\sigma : A^0_R(K) \to R_\sigma$ is the restriction epimorphism, $\rho^K_\sigma(f) = f(\sigma)$, and

$$\Omega(\rho^K_\sigma) : A^p_R(K) \to \Lambda^p_{R_\sigma}\Omega_R(R_\sigma)$$

is the corresponding induced map.

It is easy to find examples showing that $\varphi : A^*_R(K) \to \tilde{A}^*_R(K)$ is not injective in general, but we have the following result.

(3.1) **Proposition.** *If K is a finite simplicial complex $\varphi : A^*_R(K) \to \tilde{A}^*_R(K)$ defined above, is an epimorphism.*

To prove (3.1) we need the following lemma.

(3.2) **Extension lemma** (see Proposition 13.8 of [1]). *Let $\sigma = (1, \ldots, m)$ be an $m-1$-simplex and suppose we are given $\Phi_i \in \Lambda^p_{R_{\partial^i \sigma}} \Omega_R(R_{\partial^i \sigma})$, $i = 1, \ldots, m$, such that*

$$\Omega(\partial^i)(\Phi_j) = \Omega(\partial^j)(\Phi_i)\,, \quad \{i,j\} \subset \{1, \ldots, m\}\,.$$

There exists then $\Phi \in \Lambda^p_{R_\sigma} \Omega_R(R_\sigma)$ such that $\Omega(\partial^i)(\Phi) = \Phi_i$, $i = 1, \ldots, m$.

Proof. The proof is that of Proposition 13.8 given in [1], except that one has to consider for a certain step in the proof the ring of fractions of R_σ by the multiplicatively closed subset $\{(1 - \bar{X}_m)^q\}_{q \geq 0}$ and use the fact that formation of fractions commutes with exterior power and Kähler differentials. □

(3.3) **Proof of Proposition** (3.1). Let Δ^{n-1} be the simplicial complex of all finite subsets of the set of vertices of K Thus K is a subcomplex of Δ^{n-1}.

An abvious step by step procedure and induction on the dimension of the simplices, using the extension lemma (3.2), shows that the restriction map $\bar{\rho}_K : A^*_R(\Delta^{n-1}) \to \tilde{A}^*_R(K)$ is surjective.

This implies that φ is also surjective by the commutativity of the diagram

$$\begin{array}{ccc} A^*_R(\Delta^{n-1}) & \xrightarrow{\quad \rho_K \quad} & A^*_R(K) \\ & {\scriptstyle \bar{\rho}_K} \searrow \quad \swarrow {\scriptstyle \varphi} & \\ & \tilde{A}^*_R(K) & \end{array}$$

where ρ_K is induced by the restriction $A^0_R(\Delta^n) \to A^0_R(K)$. □

(3.4) **Proposition.** *The kernel of $\rho_K : A^0_R(\Delta^{n-1}) \to A^0_R(K)$ is generated by the set of elements $\prod_{i \in \sigma} \bar{X}_i$ for all simplices σ of Δ^{n-1} not being simplices of K.*

Proof. If σ is a simplex of Δ^{n-1} we write $X_\sigma = \prod_{i \in \sigma} X_i$ and denote by $\bar{X}_\sigma = \prod_{i \in \sigma} \bar{X}_i$ its class in $A^0_R(\Delta^{n-1})$.

It is clear that the elements $\bar{X}_\sigma$, for all simplices σ of Δ^{n-1} that are not simplices of K, belong to the kernel of $\rho_K : A^0_R(\Delta^{n-1}) \to A^0_R(K)$.

It remains to be proved that the above elements $\bar{X}_\sigma$ generate ker ρ_K and this is done by induction on the number of maximal simplices of K. □

(3.5) **Corollary.** *The algebra $A^0_R(K)$ is obtained directly from the admissible couple $(\mathbf{n}_K, \varphi_K)$, up to isomorphism, as follows:*

$$A^0_R(K) = R[X_1, \ldots, X_n]/(X_1 + \cdots + X_n - 1, (X^\alpha_t)_{\alpha,t})$$

where $n = n_1 + \cdots + n_s$, $X_t^\alpha = X_{t_1}^{\alpha_1} \ldots X_{t_s}^{\alpha_s}$, $\alpha = \alpha_1, \ldots, \alpha_s$ *is a sequence with each* α_i *being either* 0 *or* 1, $1 \leq t_1 \leq n_1; \ldots ; n_1 + \ldots n_{s-1} + 1 \leq t_s \leq n$, *and* α *and* t *satisfy the following two conditions:*

a) *For all* $i \in \{1, \ldots, r\}$ *there exists* $\rho(i) \in \{1, \ldots, s\}$ *such that* $\alpha_{\rho(i)} = 1$ *but* $a_{i\rho(i)} = 0$

b) *If* $\alpha_j = 1$ *there exists* $i \in \{1, \ldots, r\}$ *such that* $a_{ik} = 0,\ k \neq j \Rightarrow \alpha_k = 0$

Proof. For each α, t as above, define a simplex σ of Δ^{n-1} by

$$\sigma \cap \omega_k = \begin{cases} \{t_k\} & \text{if} \quad \alpha_k = 1 \\ \emptyset & \text{if} \quad \alpha_k = 0 \end{cases}$$

Condition (a) tells us that σ is not contained in any of the maximal simplices σ_i of K, i.e. σ is not a simplex of K, and condition (b) says that σ is minimal among the simplices of Δ^{n-1} that are not simplices of K, i.e. we obtain a simplex of K by deleting any vertex of σ.

The proof of this corollary is now an obvious consequence of our previous proposition. □

(3.6) **Theorem.** *Let* K *be a finite simplicial complex,* $\sigma_1, \ldots, \sigma_r$ *the maximal simplices of* K *and* $\mathbf{p}_i$, $i = 1, \ldots, r$ *the kernels of the restriction epimorphisms* $\rho_{\sigma_i}^K : A_R^0(K) \to R_{\sigma_i}$. *Then :*

a) *For any homomorphism of algebras* $\mu : A_R^0(K) \to P$, P *being any polynomial algebra on* R *with a finite number of variables, there exists* $i \in \{1, \ldots, r\}$ *and a homomorphism of algebras* $\mu_i : A_R^0(K)/\mathbf{p}_i \to P$ *such that* $\mu = \mu_i \circ \pi_i$, *where* $\pi_i : A_R^0(K) \to A_R^0(K)/\mathbf{p}_i$ *is the canonical projection.*

b) $\mathbf{p}_1, \ldots, \mathbf{p}_r$ *is the set of minimal ideals of* $A_R^0(K)$ *having the property that the quotients* $A_R^0(K)/\mathbf{p}_i$ *are polynomial algebras on* R *with a finite number of variables.*

c) *For all* $I \subset \{1, \ldots, r\}$, $\sum_{i \in I} \mathbf{p}_i = \ker\ \rho_{\sigma_I}^K$ *if* $\sigma_I = \cap_{i \in I} \sigma_i \neq \emptyset$ *and* $\sum_{i \in I} \mathbf{p}_i = A_R^0(K)$ *if* $\cap_{i \in I} \sigma_i = \emptyset$.

d) *For any couple of subsets* I, J *of* $\{1, \ldots, r\}$, $\sum_{i \in I} \mathbf{p}_i = \sum_{j \in J} \mathbf{p}_j$ *if and only if* $\cap_{i \in I} \sigma_i = \cap_{j \in J} \sigma_j$.

Observe that c) *and* d) *establishes a one to one correspondence from the set of simplices of* K *appearing as intersections of maximal simplices* $\sigma_1, \ldots, \sigma_r$ *and the set of ideals of* $A_R^0(K)$ *that are sums of minimal ideals* $\mathbf{p}_1, \ldots, \mathbf{p}_r$.

Proof. a) Let Δ^{n-1} be the simplicial complex of all finite subsets of K^0 and consider the algebra homomorphism

$$\mu \circ \rho_K : A_R^0(\Delta^{n-1}) = R[X_1, \ldots, X_n]/(\sum_{i=1}^n X_i - 1) \to P\,.$$

Define a simplex σ of Δ^{n-1} by $i \in \sigma \Leftrightarrow \mu(\rho_K(\bar{X}_i)) \neq 0$

Observe that σ *is not empty* : otherwise, $\mu(\rho_K(\bar{X}_i)) = 0$, $i = 1, \dots, n$ and so

$$0 = \mu(\rho_K(\sum_{i=1}^{n} \bar{X}_i)) = \mu(\rho_K(1)) = 1\,,$$

which is a contradiction.

Moreover, σ *is a simplex of* K. In fact, if σ were not so, Proposition (3.4) would imply that $\rho_K(\prod_{i\in\sigma} \bar{X}_i) = 0$. But then $\prod_{i\in\sigma} \mu(\rho_K(\bar{X}_i)) = 0$ and, since R has no zero-divisors, we would have $\mu(\rho_K(\bar{X}_i)) = 0$ for some $i \in \sigma$, which contradicts the definition of σ.

Consider then the unique algebra homomorphism $\bar{\mu}$ making commutative the following diagram

$$\begin{array}{ccc} A^0_R(\Delta^{n-1}) & \xrightarrow{\rho_\sigma} & R_\sigma \\ {\scriptstyle\rho_K}\downarrow & & \downarrow{\scriptstyle\bar{\mu}} \\ A^0_R(K) & \xrightarrow{\mu} & P \end{array}$$

Note that $\bar{\mu}$ is well defined because, by definition of σ,

$$\mu(\rho_K(\sum_{i\in\sigma} \bar{X}_i)) = \mu(\rho_K(\sum_{i=1}^{n} \bar{X}_i)) = 1\,.$$

On the other hand the following diagram commutes

$$\begin{array}{ccc} A^0_R(K) & \xrightarrow{\rho^K_\sigma} & R_\sigma \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\bar{\mu}} & \\ & P & \end{array}$$

In fact,

$$\bar{\mu} \circ \rho^K_\sigma \circ \rho_K = \bar{\mu} \circ \rho_\sigma = \mu \circ \rho_K$$

and since $\rho_K : A^0_R(\Delta^{n-1}) \to A^0_R(K)$ is surjective, we have $\bar{\mu} \circ \rho^K_\sigma = \mu$.

Choose now any maximal simplex σ_i of K such that $\sigma \subset \sigma_i$ and let μ_i be the composite

$$A^0_R(K)/\mathbf{p}_i \xrightarrow{\cong} R_{\sigma_i} \longrightarrow R_\sigma \xrightarrow{\bar{\mu}} P$$

Clearly we have $\mu_i \circ \pi_i = \mu$ as desired.

b) Let $\mathbf{p}$ be any ideal of $A^0_R(K)$ such that the quotient $A^0_R(K)/\mathbf{p}$ is isomorphic to some polynomial algebra on R with a finite number of variables.

By (a), there exist $j \in \{1, \dots, r\}$ and $\mu_j : A^0_R(K)/\mathbf{p}_j \to A^0_R(K)/\mathbf{p}$, homomorphism of algebras, such that the projection $A^0_R(K) \to A^0_R(K)/\mathbf{p}$ is the composite

$$A^0_R(K) \xrightarrow{\rho^K_{\sigma_j}} A^0_R(K)/\mathbf{p}_j \xrightarrow{\mu_j} A^0_R(K)/\mathbf{p}$$

This shows that $\mathbf{p}_j \subset \mathbf{p}$.

On the other hand if $\mathbf{p}_i \subset \mathbf{p}_j$ for i, j in $\{1, \dots, r\}$, we have then $\sigma_j \subset \sigma_i$. In fact if there is a $k \in \sigma_j - \sigma_i$ we have

$$\rho^K_{\sigma_i}(\rho_K(\bar{X}_k)) = \rho_{\sigma_i}(\bar{X}_k) = 0\,,$$

but

$$\rho^K_{\sigma_j}(\rho_K(\bar{X}_k)) = \rho_{\sigma_j}(\bar{X}_k) \neq 0\,.$$

Therefore $\rho_K(\bar{X}_k) \in \mathbf{p}_i$ and $\rho_K(\bar{X}_k) \notin \mathbf{p}_j$, which contradicts the hypothesis $\mathbf{p}_i \subset \mathbf{p}_j$.

Thus $\sigma_i \subset \sigma_j$, and since σ_i and σ_j are maximal simplices of K, we have $\sigma_i = \sigma_j$ and so $\mathbf{p}_i = \mathbf{p}_j$.

c) For any simplex σ of K we have $\ker\ \rho^K_\sigma = \rho_K(\ker\ \rho_\sigma)$ because of the relation $\rho^K_\sigma \circ \rho_K = \rho_\sigma$ and the surjectivity of ρ_K.

According to Proposition (3.4), the set of elements $\bar{X}_i \in A^0_R(\Delta^{n-1})$, for all $i \notin \sigma$, generate $\ker\ \rho_\sigma$ and so the elements $\rho_K(\bar{X}_i) \in A^0_R(K)$, for all $i \notin \sigma$, generate $\ker\ \rho^K_\sigma$.

If we take now $\sigma = \sigma_I = \cap_{i\in I}\sigma_i$ for some $I \subset \{1, \dots, r\}$ and we assume $\sigma_I \neq \emptyset$, then we have that $\ker\ \rho^K_{\sigma_I}$ is generated by the elements $\rho_K(\bar{X}_i)$ for all $i \notin \sigma_I$, i.e.

$$\ker\ \rho^K_{\sigma_I} = \sum_{i\in I} \ker\ \rho^K_{\sigma_i} = \sum_{i\in I} \mathbf{p}_i\,.$$

Finally, if $\cap_{i\in I}\sigma_i = \emptyset$, for any $i \in \{1, \dots, n\}$ there exists $j \in I$ such that $i \notin \sigma_j$. Therefore $\rho_K(\bar{X}_i) \in \ker\ \rho^K_{\sigma_j} = \mathbf{p}_j$ and so $\sum_{i\in I} \mathbf{p}_i = A^0_R(K)$.

d) If $\sigma_I = \sigma_J$, then $\ker\ \rho^K_{\sigma_I} = \ker\ \rho^K_{\sigma_J}$ and so $\sum_{i\in I} \mathbf{p}_i = \ker\ \rho^K_{\sigma_I} = \ker\ \rho^K_{\sigma_J} = \sum_{j\in J} \mathbf{p}_j$.

Conversely if $\sum_{i\in I} \mathbf{p}_i = \sum_{j\in J} \mathbf{p}_j$, then $\ker\ \rho^K_{\sigma_I} = \ker\ \rho^K_{\sigma_J}$. Assume k is an element of $\sigma_I - \sigma_J$, then $\rho_K(\bar{X}_k) \notin \ker\ \rho^K_{\sigma_I}$. However $\rho_K(\bar{t}_k) \in \ker\ \rho^K_{\sigma_J}$, which is a contradiction. Hence $\sigma_I = \sigma_J$. □

(3.7) **Corollary.** *For any finite simplicial complex K and any ring R the algebra of Sullivan 0-forms $A^0_R(K)$ determines the Sullivan's de Rham complex $\tilde{A}^*_R(K)$.*

Proof. Consider the epimorphism $\varphi : A^*_R(K) \to \tilde{A}^*_R(K)$ and we have to show that its kernel can be deduced directly from $A^0_R(K)$. □

Let $\mathbf{p}_1, \dots, \mathbf{p}_r$ be the set of minimal ideals of $A^0_R(K)$ having the property that the quotients $A^0_R(K)/\mathbf{p}_i$ are polynomial algebras on R with a finite number of variables. This set is completly determined by the algebra $A^0_R(K)$.

But Theorem (3.6) says, in particular, that $\{\mathbf{p}_1, \dots, \mathbf{p}_r\} = \{\ker \rho^K_{\sigma_1}, \dots, \ker \rho^K_{\sigma_r}\}$, where $\sigma_1, \dots, \sigma_r$ are the maximal simplices of K.

Therefore we have

$$\ker\ \varphi = \cap^s_{i=1} \ker\ \Omega(\rho^K_{\sigma_i}) = \cap^s_{i=1} \ker\ \Omega(\pi_i)\,,$$

where $\pi_i : A^0_R(K) \to A^0_R(K)/\mathbf{p}_i$ are the canonical projections and $\Omega(\pi_i)$ the corresponding induced maps for the algebraic de Rham complexes.

It is interesting to observe, as the following example shows, that φ does not induce in general an isomorphism in cohomology. Therefore, to compute the cohomology of $\tilde{A}^*_R(K)$, which for R being a field of characteristic zero is the "correct" cohomology of K, one cannot simply compute the cohomology of the algebraic De Rham complex of $A^0_R(K)$.

(3.8) **Example.** Let K be the simplicial complex having three 0-simplices $1, 2, 3$ and three 1-simplices $12, 23, 31$. The geometric realization of K is, of course, S^1.

$A^0_R(K)$ is the quotient of the polynomial ring $R[X_1, X_2, X_3]$ by the ideal generated by $X_1X_2X_3$, $X_1 + X_2 + X_3 - 1$.

Then

$$\varphi^* : H^1(A^*_R(K)) \to H^1(\tilde{A}^*_R(K))$$

is not an isomorphism.

In fact, one checks easily that $\Phi = \bar{X}_1^2\bar{X}_2^2 d\bar{X}_3$ satisfies: $\Phi \in \ker\ \varphi$, $d\Phi = 0$ and $\Phi \neq 0$. In particular, Φ represents a cohomology class in $H^1(A^*_R(K))$ which applies to 0 in $H^1(\tilde{A}^*_R(K))$.

However, if we had $\Phi = df$ for some $f \in A^0_R(K)$ then one deduces by applying φ that $\tilde{d}f = 0$, where $\tilde{d}$ denotes the differential in $\tilde{A}^*_R(K)$, and so $f \in R$ and therefore $\Phi = df = 0$, which is not true.

(3.9) **Theorem.** *A finite simplicial complex K is determined up to simplicial equivalence either by its associated admissible couple $(\mathbf{n}_K, \varphi_K)$ or by its algebra of polynomial functions with coefficient R on the baricentric coordinates of K.*

Proof. Observe that as a consequence of Theorem (3.6) we deduce from the algebra $A = A^0_R(K)$ the set of minimal ideals $\mathbf{p}_1, \ldots, \mathbf{p}_r$ with respect to the property that the quotients $A/\mathbf{p}_i$ are polynomial algebras and so we have the set $\mathcal{P}_A$ of ideals appearing as sum of some of the ideals $\mathbf{p}_1, \ldots, \mathbf{p}_r$ and in particular for such ideals $\mathbf{p}$ we know the number $n(\mathbf{p}) - 1$ of variables of the polynomial algebra $A/\mathbf{p}$.

For each $\mathbf{p} \in \mathcal{P}_A$ use (2.1) to define the number

$$n_{\mathbf{p}} = \sum_{\mathbf{q}} (-1)^{n(\mathbf{q})} n(\mathbf{q})$$

where $\mathbf{q}$ in the sum runs through the members of $\mathcal{P}_A$ of the form $\mathbf{p} + \mathbf{p}_i$, for $i = 1, \ldots, r$.

Let $\mathbf{q}_1, \ldots, \mathbf{q}_s$ be all the members of $\mathcal{P}_A$ such that $n_{\mathbf{q}_i} > 0$ and define numbers $n_i = n_{\mathbf{q}_i}$, $i = 1, \ldots, s$ and $a_{ij} = 1$ or 0 depending on whether $\mathbf{p}_i \subset \mathbf{q}_j$ or $\mathbf{p}_i \not\subset \mathbf{q}_j$.

Therefore we obtain an admissible couple $(\mathbf{n}, \varphi)$ which clearly coincides with $(\mathbf{n}_K, \varphi_K)$.

Finally we use Proposition (2.5) to complete the proof of our theorem. □

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