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Archivum Mathematicum, Vol. 37 (2001), No. 4, 301--306

Persistent URL: http://dml.cz/dmlcz/107808

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 37 (2001), 301 – 306

SOME EQUALITIES FOR GENERALIZED INVERSES OF MATRIX SUMS AND BLOCK CIRCULANT MATRICES

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Abstract. Let $A_1, A_2, \cdots, A_n$ be complex matrices of the same size. We show in this note that the Moore-Penrose inverse, the Drazin inverse and the weighted Moore-Penrose inverse of the sum $\sum_{t=1}^n A_t$ can all be determined by the block circulant matrix generated by $A_1, A_2, \cdots, A_n$. In addition, some equalities are also presented for the Moore-Penrose inverse and the Drazin inverse of a quaternionic matrix.

Let C be a circulant matrix over the complex number field $\mathbb{C}$ with the form

$$C = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{bmatrix}. \tag{1}$$

Then it is well known (see, e.g., [1] and [3]) that C satisfies the following similarity factorization equality

$$U^*CU = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n), \tag{2}$$

where U is a fixed unitary matrix with the form

$$U = (u_{pq})_{n\times n}, \qquad u_{pq} = \frac{1}{\sqrt{n}}\omega^{(p-1)(q-1)}, \qquad \omega \text{ is the } n\text{th root of unity}, \tag{3}$$

and

$$\lambda_t = a_0 + a_1\omega^{(t-1)} + a_2(\omega^{(t-1)})^2 + \cdots + a_{n-1}(\omega^{(t-1)})^{n-1}, \quad t = 1, \cdots, n. \tag{4}$$

In particular,

$$\lambda_1 = a_0 + a_1 + \cdots + a_{n-1}. \tag{5}$$

2000 *Mathematics Subject Classification*: 15A09, 15A23.
Key words and phrases: block circulant matrix, Moore-Penrose inverse, Drazin inverse, weighted Moore-Penrose inverse, quaternionic matrix.
This work was partially supported by the Natural Sciences and Engineering Research Council of Canada.
Received May 5, 2000.

Observe that U in Eq.(3) has no relation with $a_0, \ldots, a_{n-1}$ in Eq.(1). Thus Eq.(2) can directly be extended to block circulant matrix as follows.

Lemma 1. *Let*

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ A_n & A_1 & \cdots & A_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix} \tag{6}$$

be a block circulant matrix over the complex number field $\mathbb{C}$, *where* $A_t \in \mathbb{C}^{r\times s}$, $t = 1, \cdots, n$. *Then* A *satisfies the following factorization equality*

$$U_r^* A U_s = \mathrm{diag}(J_1, J_2, \cdots, J_n)\,, \tag{7}$$

where U_r *and* U_s *are two fixed block unitary matrices*

$$U_r = (u_{pq} I_r)_{n\times n}\,, \qquad U_s = (u_{pq} I_s)_{n\times n}\,, \tag{8}$$

u_{pq} *is as in Eq.*(3), *meanwhile*

$$J_t = A_1 + A_2\omega^{(t-1)} + A_3(\omega^{(t-1)})^2 + \cdots + A_n(\omega^{(t-1)})^{n-1}\,, \qquad t = 1, \cdots, n\,. \tag{9}$$

Especially, the block entries in the first block rows and first block columns of U_r *and* U_s *are all identity matrices, and* J_1 *is*

$$J_1 = A_1 + A_2 + \cdots + A_n\,. \tag{10}$$

Observe that J_1 in Eq.(7) is the sum of $A_1, A_2, \cdots, A_n$. Thus Eq.(7) implies that the sum $\sum_{t=1}^n A_t$ is closely linked to its corresponding block circulant matrix through a unitary factorization equality. Recall a fundamental fact in the theory of generalized inverses of matrices (see, e.g., [2]) that

$$(PAQ)^\dagger = Q^* A^\dagger P^*, \qquad \text{if } P \text{ and } Q \text{ are unitary.} \tag{11}$$

Then from Eq.(7), we can directly find the following result.

Lemma 2. *Let* A *be given in Eq.*(6), U_r *and* U_s *be given in Eq.*(8). *Then*

(a) *The Moore-Penrose inverse of* A *satisfies*

$$U_s^* A^\dagger U_r = \mathrm{diag}(J_1^\dagger, J_2^\dagger, \cdots, J_n^\dagger)\,. \tag{12}$$

(b) *If* $r = s$, *then the Drazin inverse of* A *satisfies*

$$U_r^* A^D U_r = \mathrm{diag}(J_1^D, J_2^D, \cdots, J_n^D)\,. \tag{13}$$

(c) *Suppose that* $M \in \mathbb{C}^{r\times r}$, *and* $N \in \mathbb{C}^{s\times s}$ *are two positive definite Hermitian matrices. Then the weighted Moore-Penrose inverse of* A *satisfies*

$$U_s^* A^\dagger_{\widehat{M},\widehat{N}} U_r = \mathrm{diag}((J_1)^\dagger_{M,N}, (J_2)^\dagger_{M,N}, \cdots, (J_n)^\dagger_{M,N})\,, \tag{14}$$

where $\widehat{M} = \mathrm{diag}(M, M, \cdots, M)$ *and* $\widehat{N} = \mathrm{diag}(N, N, \cdots, N)$.

Proof. Since U_r and U_s in Eq.(7) are unitary, we have

$$(U_r^* A U_s)^\dagger = U_s^* A^\dagger U_r \tag{15}$$

by Eq.(11). On the other hand, it is easily seen that

$$[\mathrm{diag}(J_1, J_2, \cdots, J_n)]^\dagger = \mathrm{diag}(J_1^\dagger, J_2^\dagger, \cdots, J_n^\dagger)\,.$$

Thus Eq.(12) follows. Secondly, noting

$$(U_r^* A U_r)^D = (U_r^{-1} A U_r)^D = U_r^{-1} A^D U_r = U_r^* A^D U_r\,,$$

and

$$[\mathrm{diag}(J_1, J_2, \cdots, J_n)]^D = \mathrm{diag}(J_1^D, J_2^D, \cdots, J_n^D)\,,$$

we have Eq.(13). To prove Eq.(14), we apply the following well-known identity (see [2])

$$A^\dagger_{M,N} = N^{-\frac{1}{2}}(M^{\frac{1}{2}} A N^{-\frac{1}{2}})^\dagger M^{\frac{1}{2}}\,,$$

and Eq.(11) to the left-hand side of Eq.(7),

$$\begin{aligned}(U_r^* A U_s)^\dagger_{\widehat{M},\widehat{N}} &= \widehat{N}^{-\frac{1}{2}}(\widehat{M}^{\frac{1}{2}} U_r^* A U_s \widehat{N}^{-\frac{1}{2}})^\dagger \widehat{M}^{\frac{1}{2}}\\ &= \widehat{N}^{-\frac{1}{2}}(U_r^* \widehat{M}^{\frac{1}{2}} A \widehat{N}^{-\frac{1}{2}} U_s)^\dagger \widehat{M}^{\frac{1}{2}}\\ &= \widehat{N}^{-\frac{1}{2}} U_s^* (\widehat{M}^{\frac{1}{2}} A \widehat{N}^{-\frac{1}{2}})^\dagger U_r \widehat{M}^{\frac{1}{2}}\\ &= U_s^* \widehat{N}^{-\frac{1}{2}} (\widehat{M}^{\frac{1}{2}} A \widehat{N}^{-\frac{1}{2}})^\dagger \widehat{M}^{\frac{1}{2}} U_r\\ &= U_s^* A^\dagger_{\widehat{M},\widehat{N}} U_r\,,\end{aligned}$$

where two simple facts

$$U_r^* \widehat{M}^{\frac{1}{2}} = U_r^* \widehat{M}^{\frac{1}{2}}\,, \qquad U_s \widehat{N}^{-\frac{1}{2}} = \widehat{N}^{-\frac{1}{2}} U_s$$

are used in the above deduction. On the other hand,

$$\begin{aligned}&[\mathrm{diag}(J_1, \cdots, J_n)]^\dagger_{\widehat{M},\widehat{N}}\\ &\quad= \widehat{N}^{-\frac{1}{2}}[\widehat{M}^{\frac{1}{2}} \mathrm{diag}(J_1, \cdots, J_n) \widehat{N}^{-\frac{1}{2}}]^\dagger \widehat{M}^{\frac{1}{2}}\\ &\quad= \widehat{N}^{-\frac{1}{2}}[\mathrm{diag}((M^{\frac{1}{2}} J_1 N^{-\frac{1}{2}})^\dagger, \cdots, (M^{\frac{1}{2}} J_n N^{-\frac{1}{2}})^\dagger)] \widehat{M}^{\frac{1}{2}}\\ &\quad= \mathrm{diag}(N^{-\frac{1}{2}}(M^{\frac{1}{2}} J_1 N^{-\frac{1}{2}})^\dagger M^{\frac{1}{2}}, \cdots, N^{-\frac{1}{2}}(M^{\frac{1}{2}} J_n N^{-\frac{1}{2}})^\dagger M^{\frac{1}{2}})\\ &\quad= \mathrm{diag}((J_1)^\dagger_{M,N}, \cdots, (J_n)^\dagger_{M,N})\,.\end{aligned}$$

So we have Eq.(14). □

The main results of this note are presented below.

Theorem 3. *Let $A_1, A_2, \cdots, A_n \in \mathbb{C}^{r\times s}$ be given. Then the Moore-Penrose inverse of their sum satisfies the identity*

$$(16)\quad (A_1 + A_2 + \cdots + A_n)^\dagger = \frac{1}{n}[I_s, I_s, \cdots, I_s] \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ A_n & A_1 & \cdots & A_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}^\dagger \begin{bmatrix} I_r \\ I_r \\ \vdots \\ I_r \end{bmatrix}.$$

Proof. Pre-multiplying $[I_s, 0, \cdots, 0]$ and post-multiplying $[I_r, 0, \cdots, 0]^T$ on the both sides of Eq.(12) immediately yield Eq.(16). □

Similarly we can establish the following two theorems.

Theorem 4. *Let $A_1, A_2, \cdots, A_n \in \mathbb{C}^{r\times r}$ be given. Then the Drazin inverse of their sum satisfies the equality*

(17)
$$(A_1 + A_2 + \cdots + A_n)^D = \frac{1}{n}[I_r, I_r, \cdots, I_r] \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ A_n & A_1 & \cdots & A_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}^D \begin{bmatrix} I_r \\ I_r \\ \vdots \\ I_r \end{bmatrix}.$$

In particular, if the block circulant matrix in it is nonsingular, then

(18)
$$(A_1 + A_2 + \cdots + A_n)^{-1} = \frac{1}{n}[I_r, I_r, \cdots, I_r] \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ A_n & A_1 & \cdots & A_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}^{-1} \begin{bmatrix} I_r \\ I_r \\ \vdots \\ I_r \end{bmatrix}.$$

Theorem 5. *Let $A_1, A_2, \cdots, A_n \in \mathbb{C}^{r\times s}$ be given, $M \in \mathbb{C}^{r\times r}$ and $N \in \mathbb{C}^{s\times s}$ be two positive definite Hermitian matrices. Then the weighted Moore-Penrose inverse of their sum satisfies*

(19)
$$(A_1+A_2+\cdots+A_n)^{\dagger}_{M,N} = \frac{1}{n}[I_s, I_s, \cdots, I_s] \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ A_n & A_1 & \cdots & A_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}^{\dagger}_{\widehat{M},\widehat{N}} \begin{bmatrix} I_r \\ I_r \\ \vdots \\ I_r \end{bmatrix},$$

where $\widehat{M} = \mathrm{diag}(M, M, \cdots, M)$ and $\widehat{N} = \mathrm{diag}(N, N, \cdots, N)$.

Eqs.(16)–(18) show that the expressions of the Moore-Penrose inverse, the Drazin inverse, and the weighted Moore-Penrose inverse of the sum$\sum_{t=1}^n A_t$ can all be determined through the block circulant matrix A generated by $A_1, A_2, \cdots, A_n$. Using them one can establish various valuable expressions for generalized inverses of matrices. Some related work was presented in the author's [6].

Note that any complex matrix can be written as $A + iB$. Some interesting equalities can also be derived from Eqs.(16)–(18) for generalized inverses of a complex matrix $A + iB$.

Corollary 6. *Let $A + iB \in \mathbb{C}^{r\times s}$ with $A,\ B \in \mathbb{R}^{r\times s}$. Then the Moore-Penrose inverse of $A + iB$ satisfies the equality*

$$(A + iB)^{\dagger} = \frac{1}{2}[I_s, iI_s] \begin{bmatrix} A & -B \\ B & A \end{bmatrix}^{\dagger} \begin{bmatrix} I_r \\ -iI_r \end{bmatrix}. \tag{20}$$

Proof. According to Eq.(16), we first see that

$$(A + iB)^{\dagger} = \frac{1}{2}[I_s, I_s] \begin{bmatrix} A & iB \\ iB & A \end{bmatrix}^{\dagger} \begin{bmatrix} I_r \\ I_r \end{bmatrix}. \tag{21}$$

Moreover observe that

$$\begin{bmatrix} A & iB \\ iB & A \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & iI_r \end{bmatrix} \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} I_s & 0 \\ 0 & -iI_s \end{bmatrix}.$$

We then get

$$\begin{bmatrix} A & iB \\ iB & A \end{bmatrix}^{\dagger} = \begin{bmatrix} I_s & 0 \\ 0 & iI_s \end{bmatrix} \begin{bmatrix} A & -B \\ B & A \end{bmatrix}^{\dagger} \begin{bmatrix} I_r & 0 \\ 0 & -iI_r \end{bmatrix}.$$

Putting it in Eq.(21) yields Eq.(20). □

Corollary 7. *Let $A + iB \in \mathbb{C}^{r\times r}$ with $A,\ B \in \mathbb{R}^{r\times r}$. Then the Drazin inverse of $A + iB$ satisfies the equality*

$$(A + iB)^D = \frac{1}{2}[I_r, iI_r] \begin{bmatrix} A & -B \\ B & A \end{bmatrix}^D \begin{bmatrix} I_r \\ -iI_r \end{bmatrix}. \tag{22}$$

In particular, if $A + iB$ is nonsingular, then

$$(A + iB)^{-1} = \frac{1}{2}[I_r, iI_r] \begin{bmatrix} A & -B \\ B & A \end{bmatrix}^{-1} \begin{bmatrix} I_r \\ -iI_r \end{bmatrix}. \tag{23}$$

Corollary 8. *Let $A + iB \in \mathbb{C}^{r\times s}$ with $A, B \in \mathbb{R}^{r\times s}$, $M \in \mathbb{C}^{r\times r}$ and $N \in \mathbb{C}^{s\times s}$ be two positive definite Hermitian matrices. Then the weighted Moore-Penrose inverse of $A + iB$ satisfies the equality*

$$(A + iB)^{\dagger}_{M,N} = \frac{1}{2}[I_s, iI_s] \begin{bmatrix} A & -B \\ B & A \end{bmatrix}^{\dagger}_{\widehat{M},\widehat{N}} \begin{bmatrix} I_r \\ -iI_r \end{bmatrix}, \tag{24}$$

where $\widehat{M} = \mathrm{diag}(M, M)$ and $\widehat{N} = \mathrm{diag}(N, N)$.

The results in the above three corollaries on complex matrices motivate us to find the following interesting results on generalized inverses of quaternionic matrices.

Theorem 9. *Let $A = A_0 + iA_1 + jA_2 + kA_3$ be a quaternionic matrix, where $A_0, \ldots, A_3 \in \mathbb{R}^{m\times n}$, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. Then*

(a) *The Moore-Penrose inverse of A satisfies the equality*

$$A^{\dagger} = \frac{1}{4}[I_n, iI_n, jI_n, kI_n] \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix}^{\dagger} \begin{bmatrix} I_m \\ -iI_m \\ -jI_m \\ -kI_m \end{bmatrix}. \tag{25}$$

(b) *If $m = n$, then the Drazin inverse of A satisfies the equality*

$$A^{D} = \frac{1}{4}[I_n, iI_n, jI_n, kI_n] \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix}^{D} \begin{bmatrix} I_n \\ -iI_n \\ -jI_n \\ -kI_n \end{bmatrix}. \tag{26}$$

(c) *In particular, if A is nonsingular, then the inverse of A satisfies the equality*

$$A^{-1} = \frac{1}{4}[I_n, iI_n, jI_n, kI_n] \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix}^{-1} \begin{bmatrix} I_n \\ -iI_n \\ -jI_n \\ -kI_n \end{bmatrix}. \tag{27}$$

The equalities (25)–(27) can be derived from the following universal factorization equality for a quaternionic matrix

$$V_m \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} V_n = \begin{bmatrix} A & & & \\ & A & & \\ & & A & \\ & & & A \end{bmatrix}, \tag{28}$$

where

$$V_t = \frac{1}{2} \begin{bmatrix} I_t & iI_t & jI_t & kI_t \\ -iI_t & I_t & kI_t & -jI_t \\ -jI_t & -kI_t & I_t & iI_t \\ -kI_t & jI_t & -iI_t & I_t \end{bmatrix}, \qquad t = m, n \tag{29}$$

is a unitary quaternionic matrix, that is, $V_t V_t^* = V_t^* V_t = I_t$. The equality was first established by the author in [7]. Based on it, one can easily extend various results in the real and complex matrix theory to the real quaternion algebra.

Acknowledgements. The author wishes to thank Professor Agnes M. Herzberg for her encouragement.

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