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INFINITE ALGEBRAS WITH 3–TRANSITIVE GROUPS OF WEAK AUTOMORPHISMS

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Abstract. The infinite algebras with 3-transitive groups of weak automorphisms are investigated. Among others it is shown that if an infinite algebra with 3-transitive group of weak automorphisms has a nontrivial idempotent polynomial operation then either it is locally functionally complete or it is polynomially equivalent to a vector space over the two element field or it is a simple algebra that is semi-affine with respect to an elementary 2-group. In the second and third cases the group of weak automorphisms cannot be 4-transitive.

Introduction

A. Salomaa in [9] proved that if an at least five element finite algebra with the full symmetric group in its clone has a surjective term operation depending on at least two variables then it is primal. Salomaa's theorem was extended to algebras with 3-transitive permutation groups in their clones in [13]. For finite algebras the most general results in this direction are in [16], where the structure of finite simple surjective algebras with transitive permutation groups in their clones were described. For infinite algebras the most general result in this direction given in [8] is the following: If an infinite algebra with a 3-transitive group in its clone has a nontrivial idempotent polynomial operation then it is either locally complete or semi-affine with respect to an elementary 2-group. This result was slightly improved in [12].

B. Csákány in [1] proved that every nontrivial at least five element finite algebra whose automorphism group is the full symmetric group is functionally complete. Csákány's result was extended to finite algebras with 3-transitive automorphism groups [10], to algebras with 2-transitive automorphism groups [6] and to algebras with primitive automorphism group [7]. The finite simple algebras with transitive automorphism groups were described in [14] and [15]. For finite algebras the most general results in this direction are in [17], where the finite characteristically simple algebras (i.e., algebras that have no nontrivial congruence relation preserved by

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all automorphisms) were classified. For infinite algebras the most general result in this direction proved by H. K. Kaiser and L. Marki in [4] is the following: Every nontrivial infinite algebra with 3-transitive automorphism group is either locally functionally complete or term equivalent to an affine space over the two element field. This result was slightly improved in [11].

Following A. Goetz [3] and E. Marczewski [5], by a weak automorphism of an algebra $\mathbf{A}$ we mean a permutation π on its base set such that for every term operation f of $\mathbf{A}$ we have that f^π and $f^{\pi^{-1}}$ are also term operations of $\mathbf{A}$, where f^π is defined by $f^\pi(x_1,\dots,x_n) = f(x_1\pi^{-1},\dots,x_n\pi^{-1})\pi$. It is easy to see that all automorphisms and if $\mathbf{A}$ is finite then all unary bijective term operations of $\mathbf{A}$ are weak automorphisms. Thus the common property of the algebras mentioned above is that they have "large" sets of weak automorphisms. In [18] we classified the finite algebras that have no nontrivial congruence relations preserved by all weak automorphisms and among others we described the finite algebras with 2-transitive group of weak automorphisms. The aim of the present paper is to investigate and classify the infinite algebras whose groups of weak automorphisms are 3-transitive (Theorem 3.2). As a corollary we have that if an infinite algebra with 3-transitive group of weak automorphisms has a nontrivial idempotent polynomial operation then either it is locally functionally complete or it is polynomially equivalent to a vector space over the two element field or it is an algebra having no nontrivial compatible binary reflexive relations that is semi-affine with respect to an elementary 2-group. In the second and third cases the group of weak automorphisms cannot be 4-transitive.

1. Notions and notations

Let A be a nonempty set. For any positive integer n let $\mathbf{O}_A^{(n)}$ denote the set of all n-ary operations on A and put $\mathbf{O}_A = \bigcup_{n=1}^{\infty} \mathbf{O}_A^{(n)}$. The full symmetric group and the set of all unary constant operations will be denoted by S_A and C_A, respectively. If $m \geq 1$ then we put $\mathbf{m} = \{1,\dots,m\}$, and we write S_m instead of $S_{\mathbf{m}}$. A permutation group $G \leq S_A$ is said to be *k-transitive* ($k \geq 1$) if for any pairwise distinct elements $x_1 \dots, x_k \in A$ and for any pairwise distinct elements $y_1 \dots, y_k \in A$ there exists a permutation $\pi \in G$ such that $x_i\pi = y_i,\ i = 1,\dots,k$; G is termed *highly transitive* if G is k-transitive for any $k \geq 1$. G is said to be *primitive* if $(A; G)$ is simple and $|G| > 1$ if $|A| = 2$. Clearly, primitivity implies transitivity. The stabilizer subgroup of the elements $a_1,\dots,a_n \in A$ in a permutation group $G \leq S_A$ is denoted by $G_{a_1,\dots,a_n}$, i.e., $G_{a_1,\dots,a_n} = \{\pi \in G\colon a_1\pi = a_1,\dots,a_n\pi = a_n\}$ $(n \geq 1)$.

An operation $f \in \mathbf{O}_A$ is *nontrivial* if it is not a projection. By a *clone* we mean a subset of $\mathbf{O}_A$ which is closed under superpositions and contains all projections. A subset $F \subseteq \mathbf{O}_A$ is *locally closed* if it contains every operation $f \in \mathbf{O}_A^{(n)}$ $(n = 1,2,\dots)$ with the following property: for every finite subset $B \subseteq A^n$ there is a $g \in F \cap \mathbf{O}_A^{(n)}$ such that $f|_B = g|_B$. The *local closure* $\operatorname{Loc} F$ of F is the least locally closed clone containing F.

The clone of all term operations and the clone of all polynomial operations of an algebra $\mathbf{A}$ are denoted by $\operatorname{Clo}\mathbf{A}$ and $\operatorname{Pol}\mathbf{A}$, respectively. For every $n \geq 1$ we put $\operatorname{Clo}_n \mathbf{A} = \operatorname{Clo}\mathbf{A} \cap \mathbf{O}_A^{(n)}$ and $\operatorname{Pol}_n \mathbf{A} = \operatorname{Pol}\mathbf{A} \cap \mathbf{O}_A^{(n)}$. Two algebras $\mathbf{A}$ and $\mathbf{B}$ with a common base set are called *term equivalent* (*polynomially equivalent*) if $\operatorname{Clo}\mathbf{A} = \operatorname{Clo}\mathbf{B}$ ($\operatorname{Pol}\mathbf{A} = \operatorname{Pol}\mathbf{B}$). Two algebras $\mathbf{A}$ and $\mathbf{B}$ are also called *term equivalent* (*polynomially equivalent*) if $\mathbf{A}$ is term equivalent (polynomially equivalent) to an algebra isomorphic to $\mathbf{B}$. An algebra $\mathbf{A}$ is *locally primal* or *locally complete* if $\operatorname{Loc} F (= \operatorname{Loc}\operatorname{Clo}\mathbf{A}) = \mathbf{O}_A$. We say that $\mathbf{A}$ is *locally functionally complete* or has the *interpolation property* if $\operatorname{Loc}(F \cup C_A)(= \operatorname{Loc}\operatorname{Pol}\mathbf{A}) = \mathbf{O}_A$.

The automorphism group of an algebra $\mathbf{A} = (A; F)$ is denoted by $\operatorname{Aut}\mathbf{A}$.

We say that an h-ary relation ρ on a set A is *reflexive* if $(a, \dots, a) \in \rho$ for any $a \in A$. For a set of operation F the set of (reflexive) relations preserved by all operations in F will be denoted by $\operatorname{Inv} F$ ($\operatorname{Inv}_{\mathrm{r}} F$). We say that a relation ρ is a *compatible relation* of the algebra $(A; F)$ if $\rho \in \operatorname{Inv} F$. The binary identity relation on A is denoted by ω_A or simply by ω. The converse of a binary relation ρ is the relation $\rho^{-1} = \{(y, x)\colon\ (x, y) \in \rho\}$.

For an equivalence relation Θ on the set $\mathbf{h}$ ($h \geq 1$) put

$$\Delta_\Theta = \{(x_1, \dots, x_h) \in A^h\colon\ x_i = x_j \text{ for any } (i, j) \in \Theta\,.\}$$

The relation Δ_Θ is termed a *diagonal relation* or a *trivial relation*. A relation Θ on $\mathbf{h}$ will be often given by the list $\varepsilon_1, \dots, \varepsilon_l$ of its nonsingleton blocks and so Δ_{12}^h or simply Δ_{12} is the set of h-tuples $(x_1, \dots, x_h)$ with $x_1 = x_2$, $\Delta_{12,34}^h$ or simply $\Delta_{12,34}$ is the set of h-tuples $(x_1, \dots, x_h)$ with $x_1 = x_2$ and $x_3 = x_4$. It is well-known that a nonempty relation is trivial if and only if it is preserved by all operations in $\mathbf{O}_A$.

An h-ary relation ρ on A is called *totally symmetric* if $(a_1, \dots, a_h) \in \rho$ implies $(a_{1\pi}, \dots, a_{h\pi}) \in \rho$ for every $\pi \in S_h$, and ρ is called *totally reflexive* if $(a_1. \dots, a_h) \in \rho$ whenever $a_i = a_j$ for some $i \neq j$ ($1 \leq i, j \leq h$).

An algebra $\mathbf{A}$ is *semi-affine* with respect to an Abelian group $\bar{\mathbf{A}}$, if $\mathbf{A}$ and $\bar{\mathbf{A}}$ have a common base set A and the quaternary relation

$$\{(x, y, z, t) \in A^4\colon\ x - y + z = t\}$$

is a compatible relation of $\mathbf{A}$; if, in addition, $x - y + z$ is a term operation of $\mathbf{A}$ then $\mathbf{A}$ is said to be *affine* with respect to $\bar{\mathbf{A}}$.

2. Weak automorphisms and compatible relations

Let A be a nonempty set. For an n-ary operation f, a set of operations F, a set of relations R, an h-ary relation ρ and a permutation π on A put

$$f^\pi(x_1, \dots, x_n) = f(x_1\pi^{-1}, \dots, x_n\pi^{-1})\pi, \text{ for } x_1, \dots, x_n \in A\,,$$

$$\rho^\pi = \{(x_1\pi, \dots, x_h\pi)\colon\ (x_1, \dots, x_h) \in \rho\}$$

and

$$F^\pi = \{f^\pi\colon\ f \in F\},\ R^\pi = \{\sigma^\pi\colon\ \sigma \in R\}\,.$$

If $B \subseteq A$, i.e., B is a unary relation of A then we often write $B\pi$ instead of B^π.

In the next lemma we summarize some useful facts which are immediate consequences of the definitions and therefore the proofs are left to the reader.

Lemma 2.1. *If f is an operation, ρ is a relation, R is a set of relations, F is a set of operations, π, τ are permutations on A and $\mathbf{A} = (A; F)$ is an algebra then the following statements hold:*

(2.1.1) $(f^\pi)^\tau = f^{\pi\tau}$, $(F^\pi)^\tau = F^{\pi\tau}$, $(\rho^\pi)^\tau = \rho^{\pi\tau}$ *and* $(R^\pi)^\tau = R^{\pi\tau}$.

(2.1.2) $R^\pi = R$ *if and only if* $R^\pi, R^{\pi^{-1}} \subseteq R$.

(2.1.3) $F^\pi = F$ *if and only if* $F^\pi, F^{\pi^{-1}} \subseteq F$.

(2.1.4) $(\operatorname{Inv} F)^\pi = \operatorname{Inv} F^\pi$ *and* $(\operatorname{Inv_r} F)^\pi = \operatorname{Inv_r} F^\pi$.

Following A. Goetz [3] and E. Marczewski [5], by a *weak automorphism* (*pseudo-weak automorphism*) of an algebra $\mathbf{A} = (A; F)$ we mean a permutation $\pi \in S_A$ such that for every $f \in \operatorname{Clo}\mathbf{A}$ ($f \in \operatorname{Pol}\mathbf{A}$) we have that f^π, $f^{\pi^{-1}} \in \operatorname{Clo}\mathbf{A}$ (f^π, $f^{\pi^{-1}} \in \operatorname{Pol}\mathbf{A}$). The set of all weak automorphisms and the set of all pseudo-weak automorphisms of $\mathbf{A}$ will be denoted by $\operatorname{WAut}\mathbf{A}$ and $\operatorname{WAut}^*\mathbf{A}$, respectively. Clearly, they form groups under composition such that $\operatorname{Aut}\mathbf{A} \lhd \operatorname{WAut}\mathbf{A} \le \operatorname{WAut}^*\mathbf{A}$. If A is finite then $\operatorname{Clo}\mathbf{A} \cap S_A$ and $\operatorname{Pol}\mathbf{A} \cap S_A$ form groups under composition such that $\operatorname{Clo}\mathbf{A} \cap S_A \lhd \operatorname{WAut}\mathbf{A}$ and $\operatorname{Pol}\mathbf{A} \cap S_A \lhd \operatorname{WAut}^*\mathbf{A}$.

The next lemma is an immediate consequence of the definition of (pseudo-)weak automorphisms and of (2.1.4). We shall often use it in our arguments without quoting the lemma.

Lemma 2.2. *If $\mathbf{A} = (A; F)$ is an arbitrary algebra, $\rho \in \operatorname{Inv} F$ ($\rho \in \operatorname{Inv_r} F$) and $\pi \in \operatorname{WAut}\mathbf{A}$ ($\pi \in \operatorname{WAut}^*\mathbf{A}$) then $\rho^\pi \in \operatorname{Inv} F$ ($\rho^\pi \in \operatorname{Inv_r} F$).*

Lemma 2.3. *Let $\mathbf{A} = (A; F)$ be an algebra and let G be an arbitrary subgroup of $\operatorname{WAut}\mathbf{A}$ ($\operatorname{WAut}^*\mathbf{A}$). If $\rho \in \operatorname{Inv} F$ ($\rho \in \operatorname{Inv_r} F$) then $\bigcap\{\rho^\pi\colon\ \pi \in G\}$ belongs to $\operatorname{Inv}(F \cup G)$.*

Proof. It is straightforward and is left to the reader. □

Lemma 2.4. *For an algebra $\mathbf{A} = (A; F)$ the following statements hold:*

(a) *If $\operatorname{WAut}\mathbf{A}$ is transitive then either $C_A \subseteq \operatorname{Clo}_1\mathbf{A}$ or $C_A \cap \operatorname{Clo}_1\mathbf{A} = \emptyset$.*

(b) *If $\operatorname{WAut}\mathbf{A}$ is 2-transitive then $\mathbf{A}$ is either idempotent or has no proper subalgebra.*

Proof. Let $\mathbf{A} = (A; F)$ be an algebra. In order to prove (a) suppose that $\operatorname{WAut}\mathbf{A}$ is transitive. If $C_A \cap \operatorname{Clo}_1\mathbf{A} = \emptyset$ then we are done. Assume that for some $a \in A$ the unary constant operation $c_a\colon A \mapsto \{a\}$ is a term operation of $\mathbf{A}$ and let $b \in A$ be an arbitrary element. Since $\operatorname{WAut}\mathbf{A}$ is transitive, there is a $\pi \in \operatorname{WAut}\mathbf{A}$ such that $a\pi = b$. Then, clearly, $c_a^\pi = c_{a\pi} = c_b\colon A \mapsto \{b\}$ is again a unary term operation. Hence we have $C_A \subseteq \operatorname{Clo}_1\mathbf{A}$ completing the proof of (a).

Now in order to prove (b) suppose that WAut **A** is 2-transitive. For an element $a \in A$ let us denote by $[a]$ the subalgebra generated by the singleton $\{a\}$. Since $[a]\pi$ is a subalgebra and $a\pi \in [a]\pi$ therefore $[a\pi] \subseteq [a]\pi$. Replacing a with $a\pi$ and π with π^{-1} we obtain that $[a] \subseteq [a\pi]\pi^{-1}$ and $[a]\pi \subseteq [a\pi]$. Hence $[a]\pi = [a\pi]$ for any $\pi \in \mathrm{WAut}\,\mathbf{A}$. It follows that the binary relation $\rho = \{(a,b)\colon [a] \subseteq [b]\}$ is preserved by all weak automorphisms. Since G is 2-transitive we have that $\rho \in \{\omega, A^2\}$. If **A** has no proper subalgebras then we are done. If **A** has a proper subalgebra then $[a] \neq A$ for some $a \in A$ and $(a,b) \notin \rho$ for every $b \in A \setminus [a]$. It follows that $\rho \neq A^2$ and $\rho = \omega$. If **A** is not idempotent then $|[c]| > 1$ for some $c \in A$, and if $d \in [c]$ with $c \neq d$ then $[d] \subseteq [c]$. Thus $(d,c) \in \rho$ and $\rho \neq \omega$, a contradiction. Hence **A** is idenpotent which completes the proof of (b) and the lemma. □

Lemma 2.5. *If* $\mathbf{A} = (A;F)$ *is a non-simple algebra such that* $\mathrm{WAut}^*\,\mathbf{A}$ *is* 3*-transitive then* **A** *is polynomially equivalent either to* $(A;\mathrm{id}_A)$ *or to* $(A;x+y)$ *or to* $(A;\{x+a\colon\ a \in A\})$ *where* $(A;+)$ *is an elementary* 2*-group. In the second and third case* $\mathrm{WAut}^*\,\mathbf{A} = \{xr + a\colon\ r \in \mathrm{Aut}(A;+)$ *and* $a \in A\}$.

Proof. Let $\mathbf{A} = (A;F)$ be a non-simple algebra and suppose that $\mathrm{WAut}^*\,\mathbf{A}$ is 3-transitive. Put $G = \mathrm{WAut}^*\,\mathbf{A}$. For arbitrary distinct elements $a,b \in A$, as usual, $\Theta(a,b)$ denotes the principal congruence generated by a and b.

Claim 1. $\Theta(a\pi, b\pi) = \Theta(a,b)^\pi$ *and* $\Theta(a,b) \neq A^2$ *for any* $a,b \in A$ *with* $a \neq b$ *and* $\pi \in \mathrm{WAut}^*\,\mathbf{A}$.

In order to prove Claim 1 let us choose two distinct elements $a,b \in A$ and let $\pi \in \mathrm{WAut}^*\,\mathbf{A}$. Consider the principal congruences $\Theta(a,b)$ and $\Theta(a\pi,b\pi)$. Then $(a\pi,b\pi) \in \Theta(a,b)^\pi$ implies that $\Theta(a\pi,b\pi) \subseteq \Theta(a,b)^\pi$. Replacing (a,b) with $(a\pi,b\pi)$ and π with π^{-1} we obtain $\Theta(a,b) \subseteq (\Theta(a\pi,b\pi))^{\pi^{-1}}$ and $(\Theta(a,b))^\pi \subseteq \Theta(a\pi,b\pi)$. Hence $\Theta(a\pi,b\pi) = \Theta(a,b)^\pi$. Since **A** is non-simple, for some distinct elements $x,y \in A$ we have that $\Theta(x,y) \neq A^2$. Since $\mathrm{WAut}^*\,\mathbf{A}$ is 3-transitive there is a $\pi \in \mathrm{WAut}^*\,\mathbf{A}$ such that $x\pi = a$ and $y\pi = b$. Then $\Theta(a,b) = \Theta(x\pi,y\pi) = \Theta(x,y)^\pi$ implies that $\Theta(a,b) \neq A^2$ which completes the proof of Claim 1.

Claim 2. *For the congruence lattice of* **A** *we have one of the following two possibilities:*

(i) *All equivalence relations on* A *are congruence relations of* **A**.

(ii) *Each block of any principal congruence relation of* **A** *has two elements.*

In order to prove Claim 2 let $a,b \in A$ be two distinct elements. If $\pi \in G_{a,b}$ then $\Theta(a,b)^\pi = \Theta(a\pi,b\pi) = \Theta(a,b)$. Hence π preserves $\Theta(a,b)$. Since $\mathrm{WAut}^*\,\mathbf{A}$ is 3-transitive $G_{a,b}$ is transitive on $A \setminus \{a,b\}$. It follows that $a/\Theta(a,b) = \{a,b\}$ and each block of $\Theta(a,b)$ distinct from $\{a,b\}$ has the same cardinality. Indeed, if $c \in a/\Theta(a,b)$ with $c \neq a,b$ then for any $d \neq a,b$ we have $d = c\pi$ for some $\pi \in G_{a,b}$ and $(a,d) = (a\pi,c\pi) \in \Theta(a,b)$ implying that $\Theta(a,b) = A^2$, a contradiction. Hence $a/\Theta(a,b) = \{a,b\}$. If $c,d \notin \Theta(a,b)$ then choose a $\pi \in G_{a,b}$ such that $c\pi = d$. Then, since π and π^{-1} preserves $\Theta(a,b)$ we have $(c/\Theta(a,b))\pi \subseteq c\pi/\Theta(a,b) = d/\Theta(a,b)$, $(d/\Theta(a,b))\pi^{-1} \subseteq d\pi^{-1}/\Theta(a,b) = c/\Theta(a,b)$, $(d/\Theta(a,b)) \subseteq (c/\Theta(a,b))\pi$ and $(c/\Theta(a,b))\pi = d/\Theta(a,b)$. It follows that each block of $\Theta(a,b)$ distinct from

$\{a,b\}$ has the same cardinality, say κ. If $x,y \in A$ with $x \neq y$ then for some $\pi \in \mathrm{WAut}^* \mathbf{A}$ we have $(x,y) = (a\pi, b\pi)$ and $\Theta(x,y) = (\Theta(a,b))^\pi$. It follows that for any $x,y \in A$ with $x \neq y$, $x/\Theta(x,y) = \{x,y\}$ and each block of $\Theta(x,y)$ distinct from $\{x,y\}$ has the same cardinality κ.

If $\kappa = 1$ then for any $x,y \in A$ with $x \neq y$, $\Theta(x,y) = \omega \cup \{(x,y),(y,x)\}$, and if Θ is an arbitrary equivalence relation on A then $\Theta = \bigvee\{\Theta(x,y)\colon\ (x,y) \in \Theta\}$. Hence Θ is a congruence relation of $\mathbf{A}$ and we have (i).

Now suppose that $\kappa \geq 2$. Let $a,b,c,d \in A$ be pairwise distinct elements such that $(c,d) \in \Theta(a,b)$. Then, since $\Theta(c,d) \subseteq \Theta(a,b)$, we have that $2 \leq \kappa = |a/\Theta(c,d)| \leq |a/\Theta(a,b)| = 2$ and $\kappa = 2$. Hence we have (ii). This completes the proof of Claim 2.

It is well-known that if an operation on an at least three element set A preserves all equivalence relations on A then it is either a projection or a constant. Therefore in case (i) $\mathbf{A}$ is polynomially equivalent to $(A; \mathrm{id}_A)$.

Finally in case (ii), taking into consideration the main result of [2] we have that $\mathbf{A}$ is polynomially equivalent to either $(A; x+y)$ or $(A; \{x+a\colon\ a \in A\})$ where $(A:+)$ is an elementary 2-group. Put

$$N = \{x+a\colon\ a \in A\} \text{ and } H = \{xr+a\colon\ r \in \mathrm{Aut}(A;+) \text{ and } a \in A\}\,.$$

In both cases it is easy to check that $H \subseteq G$. Moreover, $\mathrm{Pol}_1\ \mathbf{A} \cap S_A = N$ which implies that $N \lhd G$. Since G is a primitive permutation group, by [19; Theorem 8.2], G_0 is a maximal subgroup of G and thus $G_0 \cup N$ generates G. Therefore we have to show only that $G_0 \subseteq H$. Let $\pi \in G_0$ and let $a,b \in A$ be two arbitrary elements. Then $x+b \in N$ implies that $(x\pi^{-1}+b)\pi \in N$, i.e., $(x\pi^{-1}+b)\pi = x+c$ for some $c \in A$. Then $c = 0+c = (0\pi^{-1}+b)\pi = (0+b)\pi = b\pi$. In case $x = a\pi$ we have $a\pi + b\pi = ((a\pi)\pi^{-1}+b)\pi = (a+b)\pi$. Hence $\pi \in \mathrm{Aut}(A;+)$ and $G_0 \subseteq H$. This completes the proof. □

Lemma 2.6. *If* $\mathbf{A}$ *is an algebra such that* $\mathrm{WAut}^* \mathbf{A}$ *is* k*-transitive for some* $k \geq 3$*, then the following statements hold:*

(a) *If* $\mathbf{A}$ *is simple then* $\mathbf{A}$ *has no nontrivial compatible binary reflexive relations.*

(b) *If* $\mathbf{A}$ *has a compatible* h*-ary* ($3 \leq h \leq k$) *totally reflexive and totally symmetric relation distinct from the full relation then every polynomial operation of* $\mathbf{A}$ *depending on at least two variables takes on at most* $h-1$ *values.*

Proof. Let $\mathbf{A} = (A;F)$ be a simple algebra such that $\mathrm{WAut}^* \mathbf{A}$ is k-transitive ($k \geq 3$). To show (a) suppose that ρ is a nontrivial compatible binary reflexive relation of $\mathbf{A}$. First we show that ρ cannot be symmetric. In orther to show this suppose that ρ is symmetric. If ρ is a central relation, i.e., there is an $a \in A$ such that $\rho_a \subseteq \rho$ where

$$\rho_a = \{(x,y) \in A^2\colon\ x=y \text{ or } x=a \text{ or } y=a\}\,,$$

then, consider the relation $\sigma = \bigcap\{\rho^\pi\colon\ \pi \in G_a\}$ which, by Lemma 2.3, is a compatible relation of $(A; F \cup G_a)$. Clearly, $\rho_a \subseteq \sigma$. If $\rho_a \neq \sigma$ then there are two distinct elements $b, c \in A \setminus \{a\}$ such that $(b, c) \in \sigma$. Since G is 3-transitive therefore G_a is 2-transitive on $A \setminus \{a\}$. Thus for any $x, y \in A \setminus \{a\}$ with $x \neq y$ there is a $\pi \in G_a$ such that $b\pi = x$ and $c\pi = y$. Then $(x, y) = (b\pi, c\pi) \in \sigma$ shows that $\sigma = A^2$ which is impossible since $\sigma \subseteq \rho$. Hence $\sigma = \rho_a$. If $b \in A$ with $a \neq b$ and $\pi \in \mathrm{WAut}^* \mathbf{A}$ with $a\pi = b$ then $\rho_a^\pi = \rho_{a\pi} = \rho_b$. Thus ρ_b is a compatible relation of $\mathbf{A}$ and $\rho_a \cap \rho_b = \{(a, b)\} \cup \{(b, a)\} \cup \omega$ is congruence relation of $\mathbf{A}$, a contradiction.

If ρ is not a central relation and $(a, b) \in \rho$ with $a \neq b$ then consider the relation $\sigma = \bigcap\{\rho^\pi\colon\ \pi \in G_{a,b}\}$. Again, by Lemma 2.3, σ is a compatible relation of $(A; F \cup G_{a,b})$. Then for any $x \in A$, $(a, x) \in \sigma$ if and only if $x = a$ or $x = b$, and $(b, x) \in \sigma$ if and only if $x = b$ or $x = a$. (Indeed, if $(a, x) \in \sigma$ for some $x \neq a, b$ then, since $G_{a,b}$ is transitive on $A \setminus \{a, b\}$, for any $y \in A \setminus \{a, b\}$ there is a $\pi \in G_{a,b}$ such that $x\pi = y$. Therefore $(a, y) = (a\pi, x\pi) \in \sigma$ and $\rho_a \subseteq \sigma \subseteq \rho$ which is a contradiction since ρ is not a central relation.) Therefore the transitive hull of σ is a nontrivial congruence of $\mathbf{A}$, contrary to our assumption on $\mathbf{A}$. Hence $\mathbf{A}$ have no nontrivial compatible binary reflexive and symmetric relations.

If ρ is not symmetric then ρ is antisymmetric since the compatible reflexive and symmetric relation $\rho \cap \rho^{-1}$ is trivial. If ρ is bounded from below, i.e., there is an $a \in A$ such that $(a, x) \in \rho$ for any $x \in A$ then, repeating the corresponding argument for the relation $\sigma = \bigcap\{\rho^\pi\colon\ \pi \in G_a\}$ we used in case of central ρ, we have that $\sigma = \{(a, x)\colon\ x \in A\} \cup \omega$. It follows that $\rho\rho^{-1} = \rho_a$ is a compatible relation of $\mathbf{A}$, a contradiction. If ρ^{-1} is bounded from below then repeating the above argument for ρ^{-1} we obtain again a contradiction.

Finally if neither ρ nor ρ^{-1} is bounded from below and $(a, b) \in \rho$ with $a \neq b$ then consider the relation $\sigma = \bigcap\{\rho^\pi\colon\ \pi \in\ G_{a,b}\}$. Repeating the corresponding argument for σ we used in case of non-central and symmetric ρ, we have that for any $x \in A$, $(a, x) \in \sigma$ if and only if $x = a$ or $x = b$, and $(x, b) \in \sigma$ if and only if $x = b$ or $x = a$. It follows that $\rho\rho^{-1}$ is again a nontrivial compatible binary reflexive and symmetric relation of $\mathbf{A}$. This contradiction completes the proof of (a).

In order to show (b) suppose that ρ is a compatible h-ary $(3 \leq h \leq k)$ totally reflexive and totally symmetric relation of $\mathbf{A}$ distinct from the full relation. Then, by Lemma 2.3, $\sigma = \bigcap\{\rho^\pi\colon\ \pi \in \mathrm{WAut}^* \mathbf{A}\}$ is a compatible h-ary totally reflexive and totally symmetric relation of $(A; F \cup C_A \cup \mathrm{WAut}^* \mathbf{A})$ distinct from A^h. Since $\mathrm{WAut}^* \mathbf{A}$ is h-transitive, it follows that $\tau = \{(x_1 \ldots, x_h) \in A^h\colon\ |\{x_1, \ldots, x_h\}| \leq h\}$. It is well-known that every operation depending on at most two variables and preserving τ takes on at most $h - 1$ values, which completes the proof of (b). $\square$

3. Main results

In [8] we gave a local completeness criterion by means of compatible relations. The next theorem is a direct consequence of this criterion:

Theorem 3.1 ([8])**.** *An algebra $\mathbf{A} = (A; F)$ is locally functionally complete if $\mathbf{A}$ has no compatible relation of one of the following types:*

(3.1.1) *nontrivial binary and reflexive relations,*

(3.1.2) *ternary relations* $\rho = \sigma \cup \Delta_{12}$ *where* $\sigma(\neq \emptyset)$ *consists of triples of pairwise distinct elements and for all* $x, y, z, t \in A$, $(x, y, z) \in \rho$ *implies* $(y, x, z) \in \rho$, $(x, t, z) \in \rho$ *and* $(y, t, z) \in \rho$ *implies* $(x, y, z) \in \rho$, *and for every finite* $B \subseteq A$ *we have* $B^2 \times \{u\} \subseteq \rho$ *for some* $u \in A$,

(3.1.3) *quaternary relations of the form* $\{(x, y, z, t) \in A^4\colon\ x - y + z = t\}$ *where* $(A; +)$ *is an Abelian group which is either an elementary p-group (p prime) or a torsionfree divisible group.*

(3.1.4) *at least ternary totally reflexive and totally symmetric relations distinct from the full relation.*

Now we formulate our main theorem.

Theorem 3.2. *Let* $\mathbf{A} = (A; F)$ *be a nontrivial infinite algebra. If* $\mathrm{WAut}^* \mathbf{A}$ *is* k*-transitive for some* $k \geq 3$ *then one of the following conditions holds:*

(3.2.1) $\mathbf{A}$ *is locally functionally complete.*

(3.2.2) *k=3 and* $\mathbf{A}$ *is polynomially equivalent to either* $(A; \{x + a\colon\ a \in A\})$ *or* $(A; x + y)$ *where* $(A : +)$ *is an elementary* 2*-group. Furthermore* $\mathrm{WAut}^* \mathbf{A} = \{xr + a\colon\ a \in A,\ r \in \mathrm{Aut}(A; +)\}$.

(3.2.3) $\mathbf{A}$ *has neither a nontrivial compatible binary reflexive relation nor a nontrivial idempotent polynomial operation and has a compatible ternary relation* ρ *of the form* $\rho = \sigma \cup \Delta_{12}$ *where* $\sigma(\neq \emptyset)$ *consists of triples with pairwise distinct elements and for all* $x, y, z, t \in A$, $(x, y, z) \in \rho$ *implies* $(y, x, z) \in \rho$, $(x, t, z) \in \rho$ *and* $(y, t, z) \in \rho$ *implies* $(x, y, z) \in \rho$, *and for every finite* $B \subseteq A$ *we have* $B^2 \times \{u\} \subseteq \rho$ *for some* $u \in A$. *Moreover, if* $k \geq 6$ *then* σ *contains all triples of pairwise distinct elements.*

(3.2.4) $k = 3$, $\mathbf{A}$ *has no nontrivial compatible binary reflexive relation and* $(A; F \cup \mathrm{WAut}^* \mathbf{A})$ *is semi-affine with respect to an elementary* 2*-group.*

(3.2.5) $\mathbf{A}$ *has neither a nontrivial compatible binary reflexive relation, nor a surjective polynomial operation depending on at least two variables and* $(A; F \cup \mathrm{WAut}^* \mathbf{A})$ *has an* h*-ary* $(h \geq 3)$ *totally reflexive and totally symmetric relation distinct from the full relation. Moreover, if* $h \leq k$ *then every polynomial operation of* $\mathbf{A}$ *depending on at least two variables takes on at most* $h - 1$ *values.*

Proof. Let $\mathbf{A} = (A; F)$ be an infinite algebra such that $\mathrm{WAut}^* \mathbf{A}$ is k-transitive for some $k \geq 3$. If $\mathbf{A}$ is nonsimple then, by Lemma 2.5, we have (3.2.2). □

From now on in the proof suppose that $\mathbf{A}$ is simple. Then, by Lemma 2.6(a), $\mathbf{A}$ has no nontrivial compatible binary reflexive relations. Apply Theorem 3.1 for $\mathbf{A}$. Then (3.1.1) cannot occur. If $\mathbf{A}$ is locally complete then we have (3.2.1). Suppose that $\mathbf{A}$ has a compatible h-ary totally reflexive and totally symmetric relation ρ distinct from A^h with $h \geq 3$. If $h \leq k$ then, by Lemma 2.6(b), we have that every polynomial operation of $\mathbf{A}$ depending on at least two variables takes on at most $h - 1$ values.

Consider the algebra $\widehat{\mathbf{A}} = (A; \widehat{F})$ where $\widehat{F}$ is the set of all surjective polynomial operations of $\mathbf{A}$. Then, clearly, $\mathrm{WAut}^* \mathbf{A} \subseteq \mathrm{WAut}^* \widehat{\mathbf{A}}$ and ρ is a compatible relation of $\widehat{\mathbf{A}}$. It is known and easy to check that if a surjective operation preserves ρ then it also preserves

$$\sigma = \{(x_1, x_2, x_3)\colon\ (x_1, \dots, x_h) \in \rho \text{ for all } x_4, \dots, x_h \in A\}\,.$$

Thus σ is a ternary totally reflexive and totally symmetric relation of $\widehat{\mathbf{A}}$. Then, by Lemma 2.6(b), every operation in $\widehat{F}$ depending on at most two of its variables takes on at most two values. Hence every operation in $\widehat{F}$ depends on one variable and we have (3.2.5).

From now on in the proof suppose that $\mathbf{A}$ has no nontrivial compatible at least ternary totally reflexive and totally symmetric relations.

Now suppose that $\rho = \sigma \cup \Delta_{12}$ is a ternary compatible relation of $\mathbf{A}$ with the properties given in (3.1.2). We show that $\mathbf{A}$ has no nontrivial idempotent polynomial operations. In order to do this consider the algebra $(A; I)$ where I is the set of all idempotent polynomial operations of $\mathbf{A}$. Then, clearly, $\mathrm{WAut}^* \mathbf{A} \subseteq \mathrm{WAut}^*(A; I)$ and thus $\mathrm{WAut}^*(A; I)$ is 3-transitive. Let $a, b, c \in A$ be pairwise distinct elements such that $(a, b, c) \in \rho$ and consider the binary relation $\rho_c = \{(x, y) \in A^2\colon\ (x, y, c) \in \rho\}$. Then it is easy to check that ρ_c is a compatible relation of $(A; I)$. Taking into consideration the properties of ρ, we have that ρ_c is an equivalence relation with $c/\rho_c = \{c\}$. Therefore, by Lemma 2.5, every operation in I is trivial.

In order to obtain (3.2.3) we have to show that if $k \geq 6$ then σ contains all triples of pairwise distinct elements. Now suppose that $k \geq 6$ and let $u, v, w \in A$ be pairwise distinct elements such that $(u, v, w) \notin \sigma$. Let $a, b \in A \setminus \{u, v, w\}$ be two distinct elements and put $B = \{u, v, w, a, b\}$. Then there is a $c \in A$ such that $B^2 \times \{c\} \subseteq \rho$. It follows that $(a, b, c) \in \rho$. Observe that $c \notin B$. Indeed, if e.g. $c = u$ then we have that $(u, v, u) \in \rho$ which is impossible since $\rho = \sigma \cup \Delta_{12}$. Put $G = \mathrm{WAut}^* \mathbf{A}$ and consider the relation $\tau = \bigcap\{\rho^\pi\colon\ \pi \in G_{a,b,c}\}$. Then, by Lemma 2.3, τ is a compatible relation of $(A; F \cup G_{a,b,c})$. Clearly, $(a, b, c) \in \tau$ and $\tau = \sigma' \cup \Delta_{12}$ where σ' consists of triples of pairwise distinct elements.

For any integer h with $h \geq 2$ consider the compatible h-ary relation

$$\alpha_h = \{(x_1, \dots, x_h) \in A^h\colon\ (x_i, x_j, t) \in \tau \text{ for all } 1 \leq i, j \leq h \text{ for some } t \in A\}$$

of $(A; F \cup G_{a,b,c})$. We show by induction that $\alpha_h = A^h$ for all h. Then $\Delta_{12} \subseteq \tau$ and $(a, b, c) \in \tau$ imply that α_2 is a relfexive relation containing (a, b). Since $\mathbf{A}$ and thus $(A; F \cup G_{a,b,c})$ have no nontrivial compatible binary reflexive relations, we have that $\alpha_2 = A^2$. Now let $h \geq 3$ and assumme that $\alpha_{h-1} = A^{h-1}$. Then, clearly, α_h is a totally symmetric relation and $\alpha_{h-1} = A^{h-1}$ implies that α_h is totally reflexive. Since, by our assumption, $\mathbf{A}$ has no nontrivial compatible totally reflexive and totally symmetric relations therefore $\alpha_h = A^h$. Hence $\alpha_h = A^h$ for all h which implies that for every finite $B \subseteq A$ we have $B^2 \times \{t\} \subseteq \rho$ for some $t \in A$. Now put $B = \{a, b, c, u, v\}$. Then there is a $t \in A$ such that $B^2 \times \{t\} \subseteq \tau$. It follows that $(u, v, t) \in \tau$. Observe again that $t \notin B$. Indeed, if e.g. $t = a$ then we have that $(a, b, a) \in \tau$ which is impossible since $\tau = \sigma' \cup \Delta_{12}$. Since G is

6-transitive, $G_{a,b,c}$ is 3-transitive on $A\setminus\{a,b,c\}$ therefore there is a $\pi \in G_{a,b,c}$ such that $u\pi = u$, $v\pi = v$ and $t\pi = w$. It follows that $(u,v,w) = (u\pi, v\pi, t\pi) \in \tau \subseteq \rho$, which is a contradiction. This contradiction proves that σ contains all triples of pairwise distinct elements. Hence we have (3.2.3).

Finally suppose that $\mathbf{A}$ has no relations of type (3.1.1), (3.1.2) or (3.1.4) and has a quaternary relation $\tau = \{(x,y,z,t) \in A^4\colon\ x - y + z = t\}$ where $(A;+)$ is an Abelian group which is either an elementary p-group (p prime) or a torsionfree divisible group. Consider the relation $\hat{\tau} = \bigcap\{\rho^\pi\colon\ \pi \in \mathrm{WAut}^* \mathbf{A}\}$ and the algebra $\widehat{\mathbf{A}} = (A; F \cup \mathrm{WAut}^* \mathbf{A})$. Then, by Lemma 2.3, $\hat{\tau}$ is a compatible relation on $\widehat{\mathbf{A}}$. It is easy to check that $\Delta_{12,34},\ \Delta_{14,23} \subseteq \hat{\tau}$. It follows that $\hat{\tau}$ cannot be a trivial relation and $\widehat{\mathbf{A}}$ is not locally functionally complete. Apply Theorem 2.1 for $\widehat{\mathbf{A}}$. By our assumptions on $\mathbf{A}$, the algebra $\widehat{\mathbf{A}}$ has no relations of type (3.1.1), (3.1.2) or (3.1.4) and therefore $\widehat{\mathbf{A}}$ has a quaternary relation

$$\rho = \{(x,y,z,t) \in A^4\colon\ x - y + z = t\}$$

where $(A;+)$ is an Abelian group which is either an elementary p-group (p prime) or a torsionfree divisible group. Since every permutation in $\mathrm{WAut}^* \mathbf{A}$ preserves ρ therefore $\mathrm{WAut}^* \mathbf{A}$ cannot be 4-transitive, i.e., $k = 3$. To complete the proof of the theorem we have to show only that $(A;+)$ is an elementary 2-group. If $\pi \in \mathrm{WAut}^* \mathbf{A}$ then for a given $a \in A$, $a \neq 0$ (0 is the neutral element of $(A;+)$), we have $(2a)\pi = (a - 0 + a)\pi = a\pi - 0\pi + a\pi = 2(a\pi) - 0\pi$. Since $\mathrm{WAut}^* \mathbf{A}$ is 3-transitive it follows that $2a = 0$, i.e., $(A;+)$ is elementary 2-group.

Corollary 3.3. *Let* $\mathbf{A} = (A;F)$ *be an infinite algebra with a nontrivial idempotent polynomial operation. If* $\mathrm{WAut}^* \mathbf{A}$ *is* k*-transitive for some* $k \geq 3$ *then one of the following conditions holds:*

- (3.3.1) $\mathbf{A}$ *is locally functionally complete;*
- (3.3.2) k=*3 and* $\mathbf{A}$ *is polynomially equivalent to* $(A; x+y)$ *where* $(A;+)$ *is an elementary* 2*-group. Furthermore* $\mathrm{WAut}^* \mathbf{A} = \{xr + a\colon\ a \in A,\ r \in \mathrm{Aut}(A;+)\}$*;*
- (3.3.3) $k = 3$*,* $\mathbf{A}$ *has no nontrivial compatible binary reflexive relations and* $(A; F \cup \mathrm{WAut}^* \mathbf{A})$ *is semi-affine with respect to an elementary* 2*-group.*

Corollary 3.4. *Let* $\mathbf{A} = (A;F)$ *be an infinite algebra with a nontrivial idempotent polynomial operation. If* $\mathrm{WAut}^* \mathbf{A}$ *is* 4*-transitive then* $\mathbf{A}$ *is locally functionally complete.*

Corollary 3.5. *Let* $\mathbf{A} = (A;F)$ *be an infinite algebra such that for any* k *there is a polynomial operation of* $\mathbf{A}$ *depending on two variables an taking on at least* k *values. If* $\mathrm{WAut}^* \mathbf{A}$ *is highly transitive then one of the following two conditions holds:*

- (3.5.1) $\mathbf{A}$ *is locally functionally complete;*
- (3.5.2) $\mathbf{A}$ *has neither a proper subalgebra nor a nontrivial compatible binary reflexive relation nor a nontrivial idempotent polynomial operation and the ternary relation* $\sigma_3 \cup \Delta_{12}$ *where* σ_3 *consists of all triples with pairwise distinct elements, is a compatible relation of* $\mathbf{A}$*.*

Corollary 3.6. *Let* $\mathbf{A} = (A; F)$ *be a nontrivial infinite algebra with a proper subalgebra. If* $\mathrm{WAut}\,\mathbf{A}$ *is* 3*-transitive then* $\mathbf{A}$ *is idempotent and one of the following conditions holds:*

(3.6.1) $\mathbf{A}$ *is locally functionally complete;*

(3.6.2) $\mathbf{A}$ *is term equivalent to* $(A; x + y + z)$ *where* $(A; +)$ *is an elementary* 2*-group, and* $\mathrm{WAut}\,\mathbf{A} = \mathrm{WAut}^*\,\mathbf{A} = \{xr + a\colon\ a \in A,\ r \in \mathrm{Aut}(A; +)\}$*;*

(3.6.3) $\mathbf{A}$ *has no nontrivial compatible binary reflexive relations and* $(A; F \cup \mathrm{WAut}^*\,\mathbf{A})$ *is semi-affine with respect to an elementary* 2*-group.*

Proof. Let $\mathbf{A} = (A; F)$ be a nontrivial infinite algebra with a proper subalgebra and suppose that $\mathrm{WAut}\,\mathbf{A}$ is 3-transitive. Then, by Lemma 2.4(b), $\mathbf{A}$ is idempotent. Since $\mathrm{WAut}\,\mathbf{A} \subseteq \mathrm{WAut}^*\,\mathbf{A}$ therefore $\mathrm{WAut}^*\,\mathbf{A}$ is also 3-transitive and our statement follows from Corollary 3.3. □

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