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# CONVERGENCE TESTS FOR ONE SCALAR DIFFERENTIAL EQUATION WITH VANISHING DELAY

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Abstract. In the present paper the differential equation

$$\dot{y}(t) = \alpha(t)[y(t) - y(t - \tau(t))]$$

with positive coefficient $\alpha$ and with positive bounded delay $\tau$ (which can have the property $\tau(+\infty) = 0$) is considered. Explicit tests for convergence of all its solutions (for $t \to +\infty$) are proved.

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## 1. Introduction

We will deal with the linear homogeneous differential equation with delay

$$\dot{y}(t) = \alpha(t)\,[y(t) - y(t - \tau(t))]\,, \tag{1}$$

where $\alpha \in C(I, \mathbb{R}^+)$, $I = [t_0, \infty)$, $t_0 \in \mathbb{R}$, $\mathbb{R}^+ = (0, +\infty)$, $\tau \in C(I, \mathbb{R}^+)$; $\tau(t) \le \tau_0 =$ const and the difference $t - \tau(t)$ is increasing on $I$. Let us denote $I_1 = [t_0 + \tau(t_0), \infty)$.

A function $y$ is called a *solution of Eq.* (1) *corresponding to initial point* $t^* \in I$ if $y$ is defined and is continuous on $[t^* - \tau(t^*), \infty)$, differentiable on $[t^*, \infty)$ and satisfies (1) for $t \geq t^*$. By a *solution* of (1) we mean a solution corresponding to some initial point $t^* \in I$. We denote $y(t^*, \varphi)(t)$ a solution of Eq. (1) which is generated by continuous *initial* function $\varphi : [t^* - \tau(t^*), t^*] \mapsto \mathbb{R}$ and which corresponds to initial point $t^* \in I$.

In the case of the linear Eq. (1) the solution $y(t^*, \varphi)(t)$ is unique on its maximal existence interval $[t^*, \infty)$ ([13]). We say that a solution of Eq. (1) corresponding to initial point $t^*$ is *convergent* or *asymptotically convergent* if it has a finite limit at $+\infty$.

The main goal of this paper is to formulate and prove several explicit tests for convergence of all solutions of Eq. (1).

Problems concerning asymptotic constancy of solutions, asymptotic convergence of solutions or existence of so called asymptotic equilibrium of various classes of retarded functional differential equations were investigated, e.g., by O. Arino, I. Györi and M. Pituk [1], O. Arino and M. Pituk [2], F.V. Atkinson and J.R. Haddock [3], R. Bellman and K.L. Cooke [4], I. Györi and M. Pituk [11], [12], K. Murakami [17], and T. Krisztin [14]–[16].

So called nonconvergence case (i.e. the case when there exists a monotone increasing divergent solution of Eq. (1)) was considered e.g. by S.N. Zhang [18] and by J. Diblík [10]. Some closely connected questions were discussed in the cycle of recent papers by J. Čermák [5]–[8] as well.

In the paper [9] the equation

$$\dot{y}(t) = \sum_{j=1}^{n} \alpha_j(t)[y(t) - y(t - \tau_j(t))], \tag{2}$$

was considered, where $\alpha_j \in C(I, \overline{\mathbb{R}^+})$, $\sum_{j=1}^{n} \alpha_j(t) > 0$ on $I$, $\tau_j \in C(I, \mathbb{R}^+)$, functions $t - \tau_j(t), j = 1, 2, \ldots, n$ are increasing on $I$ and $\tau_j$ are bounded on $I$.

The following theorem is the main result of [9]:

**Theorem 1.** *For the convergence of all solutions of Eq.* (2)*, corresponding to the initial point* $t_0$*, a necessary and sufficient condition is that there exist functions* $k_i \in C(I, \mathbb{R}^+), i = 1, 2, \ldots, n$ *satisfying the system of integral inequalities*

$$1 + k_i(t) \geq \exp\left[\int_{t-\tau_i(t)}^{t} \sum_{j=1}^{n} \alpha_j(s) k_j(s)\, ds\right], \quad i = 1, 2, \ldots, n$$

*on interval* $I_1$.

In this paper the following partial case of this result with respect to Eq. (1) will be used:

**Theorem 2.** *All solutions of Eq.* (1)*, corresponding to the initial point* $t_0$*, converge if and only if there exists a function* $k \in C(I, \mathbb{R}^+)$*, such that*

$$1 + k(t) \geq \exp\left[\int_{t-\tau(t)}^{t} \alpha(s) k(s)\, ds\right] \tag{3}$$

*on the interval* $I_1$.

Theorem 2 serves as a source for several explicit convergence tests. In the sequel we will prove one test which uses the values of the function $\alpha(t)$ itself (point test) and two tests which use an integral weighted average of the function $\alpha(t)$. Note that the case $\tau(+\infty) = 0$ is not excluded from our investigation.

## 2. Point test of convergence

**Theorem 3.** *If for a sufficiently large* $t$

$$\alpha(t) \leq \frac{1}{\tau(t)} - \frac{L}{t}\,, \tag{4}$$

*where* $L > 1/2$ *is a constant, then each solution of Eq.* (1) *is convergent.*

*Proof.* Without loss of generality, let us suppose $t$ sufficiently large. Let us put

$$k(t) \equiv \frac{\varepsilon}{t} \cdot \left(\frac{1}{\tau(t)} - \frac{L}{t}\right)^{-1}, \tag{5}$$

where $\varepsilon$ is a positive number (then $k(t) > 0$ for $t \to +\infty$), and verify inequality (3). Develop the asymptotic expansion $\mathcal{L}(t)$ of the left hand side of (3). We get

$$\mathcal{L}(t) = 1 + k(t) = 1 + \frac{\varepsilon}{t} \cdot \left(\frac{1}{\tau(t)} - \frac{L}{t}\right)^{-1} = 1 + \frac{\varepsilon\tau(t)}{t(1 - L\tau(t)/t)} =$$

$$1 + \frac{\varepsilon\tau(t)}{t} \cdot \left(1 + \frac{L\tau(t)}{t} + \frac{L^2\tau^2(t)}{t^2} \cdot (1 + o(1))\right).$$

Here and throughout this paper "$o$" is the Landau symbol "small" o. The symbol "$O$" used in the sequel is the Landau symbol "big" O. These symbols are used in the neighbourhood of the point $t = \infty$.

Now estimate the right hand side $\mathcal{R}(t)$ of (3). With the aid of (4) and (5) we get

$$\mathcal{R}(t) = \exp\left[\int_{t-\tau(t)}^{t} \alpha(s)k(s)ds\right] \leq$$

$$\exp\left[\int_{t-\tau(t)}^{t} \left(\frac{1}{\tau(s)} - \frac{L}{s}\right)\frac{\varepsilon}{s} \cdot \left(\frac{1}{\tau(s)} - \frac{L}{s}\right)^{-1} ds\right] =$$

$$\exp\left[\varepsilon\int_{t-\tau(t)}^{t} \frac{1}{s}\,ds\right] = \exp\left[\varepsilon \ln\frac{t}{t-\tau(t)}\right] = \left(\frac{t-\tau(t)}{t}\right)^{-\varepsilon} = \left(1 - \frac{\tau(t)}{t}\right)^{-\varepsilon} =$$

$$1 + \binom{-\varepsilon}{1}\left(-\frac{\tau(t)}{t}\right) + \binom{-\varepsilon}{2}\left(-\frac{\tau(t)}{t}\right)^2 \cdot (1 + o(1)) =$$

$$1 + \frac{\varepsilon\tau(t)}{t} + \frac{\varepsilon(\varepsilon+1)}{2} \cdot \frac{\tau^2(t)}{t^2} \cdot (1 + o(1)).$$

We conclude that (3) will hold (supposing $t_0$ sufficiently large) if

$$\mathcal{L}(t) = 1 + \frac{\varepsilon\tau(t)}{t} + \frac{\varepsilon L\tau^2(t)}{t^2} + \frac{\varepsilon L^2\tau^3(t)}{t^3} \cdot (1 + o(1)) \geq$$
$$1 + \frac{\varepsilon\tau(t)}{t} + \frac{\varepsilon(\varepsilon+1)}{2} \cdot \frac{\tau^2(t)}{t^2} \cdot (1 + o(1)) \geq \mathcal{R}(t).$$

Comparing corresponding terms, we can see that this will hold when $L > (\varepsilon+1)/2$. Since $\varepsilon$ is a positive number and may be chosen arbitrarily small, we get $L > 1/2$. Theorem 3 is proved.

## 3. Integral tests of convergence

**Theorem 4.** *If for a sufficiently large $t$*

$$(6)\quad \frac{1}{\tau(t)} \cdot \int_{t-\tau(t)}^{t} \tau(s)\alpha(s)ds \leq 1 - \frac{\tau(t)}{t-\tau(t)} - \frac{\tau(t)}{(t-\tau(t))\ln(t-\tau(t))} - \frac{L\tau(t)}{(t-\tau(t))\ln(t-\tau(t))\ln_2(t-\tau(t))},$$

*with $L > 1, L = \text{const}$ and $\ln_2 t = \ln\ln t$, then each solution of Eq.* (1) *is convergent.*

*Proof.* Without loss of generality, let us suppose $t$ sufficiently large. Let us put

$$k(t) \equiv \frac{\tau(t)}{t\ln t(\ln_2 t)^{\varepsilon}},$$

where $\varepsilon > 1$ is a constant. Obviously, the inequality (3) will be valid if

$$(7)\qquad \mathcal{L}(t) \equiv 1 + k(t) \geq \mathcal{R}(t) \equiv \exp\left[\int_{t-\tau(t)}^{t} \alpha(s)k(s)\,ds\right], \quad t \in I_1.$$

We estimate the expression $\mathcal{R}(t)$. With the aid of the inequality (6), we have

$$\mathcal{R}(t) \leq \exp(\mathcal{R}^\star(t))$$

where

$$\mathcal{R}^\star(t) \equiv \frac{\tau(t)}{(t-\tau(t))\ln(t-\tau(t))(\ln_2(t-\tau(t)))^{\varepsilon}} \cdot \left(1 - \frac{\tau(t)}{t-\tau(t)} - \frac{\tau(t)}{(t-\tau(t))\ln(t-\tau(t))} - \frac{L\tau(t)}{(t-\tau(t))(\ln(t-\tau(t)))(\ln_2(t-\tau(t)))}\right).$$

Let us develop the asymptotic expansion of the expression $\mathcal{R}^\star(t)$. At first, it is trivial to verify that the following asymptotic expansions hold:

$$\frac{1}{t-\tau(t)} = \frac{1}{t}\left(1 + \frac{\tau(t)}{t} + \frac{\tau^2(t)}{t^2} + o\left(\frac{\tau^2(t)}{t^2}\right)\right),$$

$$\frac{1}{\ln(t-\tau(t))} = \frac{1}{\ln t}\left(1+\frac{\tau(t)}{t\ln t}+\frac{\tau^2(t)}{2t^2\ln t}+o\left(\frac{\tau^2(t)}{t^2\ln t}\right)\right)$$

and

$$\frac{1}{\ln_2(t-\tau(t))} = \frac{1}{\ln_2 t}\left(1+\frac{\tau(t)}{t\ln t\ln_2 t}+\frac{\tau^2(t)}{2t^2\ln t\ln_2 t}+o\left(\frac{\tau^2(t)}{t^2\ln t\ln_2 t}\right)\right).$$

Thus

$$\begin{aligned}
\mathcal{R}^\star(t) = \frac{\tau(t)}{t\ln t(\ln_2 t)^\varepsilon}\cdot\left(1+\frac{\tau(t)}{t}+\frac{\tau^2(t)}{t^2}+o\left(\frac{\tau^2(t)}{t^2}\right)\right)\times\\
\left(1+\frac{\tau(t)}{t\ln t}+\frac{\tau^2(t)}{2t^2\ln t}+o\left(\frac{\tau^2(t)}{t^2\ln t}\right)\right)\times\\
\left(1+\frac{\varepsilon\tau(t)}{t\ln t\ln_2 t}+\frac{\varepsilon\tau^2(t)}{2t^2\ln t\ln_2 t}+o\left(\frac{\tau^2(t)}{t^2\ln t\ln_2 t}\right)\right)\times\\
\left[1-\frac{\tau(t)}{t}-\frac{\tau^2(t)}{t^2}+o\left(\frac{\tau^2(t)}{t^2}\right)-\frac{\tau(t)}{t\ln t}\left(1+\frac{\tau(t)}{t}+o\left(\frac{\tau(t)}{t}\right)\right)\times\right.\\
\left(1+\frac{\tau(t)}{t\ln t}+o\left(\frac{\tau(t)}{t\ln t}\right)\right)-\\
\frac{L\tau(t)}{t\ln t\ln_2 t}\left(1+\frac{\tau(t)}{t}+o\left(\frac{\tau(t)}{t}\right)\right)\cdot\left(1+\frac{\tau(t)}{t\ln t}+o\left(\frac{\tau(t)}{t\ln t}\right)\right)\times\\
\left.\left(1+\frac{\tau(t)}{t\ln t\ln_2 t}+o\left(\frac{\tau(t)}{t\ln t\ln_2 t}\right)\right)\right]=\\
\frac{\tau(t)}{t\ln t(\ln_2 t)^\varepsilon}\cdot\left(1+\frac{\varepsilon\tau(t)-L\tau(t)}{t\ln t\ln_2 t}-\frac{\tau^2(t)}{t^2}+o\left(\frac{\tau^2(t)}{t^2}\right)\right).
\end{aligned}$$

At the end we get

$$\begin{aligned}
\exp(\mathcal{R}^\star(t)) = 1+\frac{\tau(t)}{t\ln t(\ln_2 t)^\varepsilon}\cdot\left(1+\frac{\varepsilon\tau(t)-L\tau(t)}{t\ln t\ln_2 t}-\frac{\tau^2(t)}{t^2}+o\left(\frac{\tau^2(t)}{t^2}\right)\right)+\\
\frac{\tau^2(t)}{2t^2\ln^2 t(\ln_2 t)^{2\varepsilon}}\cdot\left(1+\frac{\varepsilon\tau(t)-L\tau(t)}{t\ln t\ln_2 t}-\frac{\tau^2(t)}{t^2}+o\left(\frac{\tau^2(t)}{t^2}\right)\right)^2\cdot(1+o(1))=\\
1+\frac{\tau(t)}{t\ln t(\ln_2 t)^\varepsilon}+\frac{\varepsilon\tau^2(t)-L\tau^2(t)}{t^2\ln^2 t\ln_2^{1+\varepsilon} t}+\frac{\tau^2(t)}{2t^2\ln^2 t\ln_2^{2\varepsilon} t}+o\left(\frac{\tau^3(t)}{t^3}\right).
\end{aligned}$$

For the validity of the inequality (7) it is sufficient to suppose that (for sufficiently large $t$)

$$\mathcal{L}(t) \geq \exp(\mathcal{R}^\star(t)),$$

i.e. that

$$1+\frac{\tau(t)}{t\ln t(\ln_2 t)^\varepsilon} \geq 1+\frac{\tau(t)}{t\ln t(\ln_2 t)^\varepsilon}+\frac{\varepsilon\tau^2(t)-L\tau^2(t)}{t^2\ln^2 t\ln_2^{1+\varepsilon} t}+\frac{\tau^2(t)}{2t^2\ln^2 t\ln_2^{2\varepsilon} t}+o\left(\frac{\tau^3(t)}{t^3}\right).$$

This will hold (we take into account the supposition $\varepsilon > 1$) if $L > \varepsilon$. Since $\varepsilon$ may be chosen arbitrarily close to 1 – this assumption is necessary for the asymptotic dominance of the third term in the right-hand side in above inequality, we obtain $L > 1$. Theorem 4 is proved.

**Theorem 5.** *If for sufficiently large* $t$

$$\frac{1}{\tau(t)}\cdot\int_{t-\tau(t)}^{t}\frac{\tau(s)\alpha(s)}{s^m}ds \le \frac{\tau(t)}{(t-\tau(t))^m}-\frac{L\tau(t)}{(t-\tau(t))^{m+1}}\,,$$

*where* $m, L = \mathrm{const}, m \ge 1, L > m$ *then each solution of Eq.* (1) *is convergent.*

*Proof.* Without loss of generality, let us suppose $t$ sufficiently large. Let us put

$$k(t) \equiv \frac{\tau(t)}{t^{m+p}},$$

where $p$ is a positive constant. Obviously, the of inequality (3) will be valid if, as above, inequality (7) holds. Let us develop (suppposing $t$ sufficiently large) the asymptotic expansion of $\mathcal{R}(t)$. We get

$$\begin{gathered}
\mathcal{R}(t) \equiv \exp\left[\int_{t-\tau(t)}^{t}\frac{\tau(s)\alpha(s)}{s^{m+p}}\,ds\right] \le \exp\left[\frac{1}{(t-\tau(t))^p}\int_{t-\tau(t)}^{t}\frac{\tau(s)\alpha(s)}{s^m}ds\right] \le \\
\exp\left[\frac{\tau(t)}{(t-\tau(t))^{m+p}}-\frac{L\tau^2(t)}{(t-\tau(t))^{m+p+1}}\right] = \\
1+\frac{\tau(t)}{(t-\tau(t))^{m+p}}-\frac{L\tau^2(t)}{(t-\tau(t))^{m+p+1}}+\frac{\tau^2(t)}{2(t-\tau(t))^{2(m+p)}}\,(1+o(1)) = \\
1+\frac{\tau(t)}{t^{m+p}}\left(1-\frac{\tau(t)}{t}\right)^{-(m+p)}-\frac{L\tau^2(t)}{t^{m+p+1}}\left(1-\frac{\tau(t)}{t}\right)^{-(m+p+1)}+O\left(\frac{\tau^2(t)}{t^{2(m+p)}}\right) = \\
1+\frac{\tau(t)}{t^{m+p}}\left(1+\frac{(m+p)\tau(t)}{t}+O\left(\frac{\tau^2(t)}{t^2}\right)\right) - \\
\frac{L\tau^2(t)}{t^{m+p+1}}\left(1+O\left(\frac{\tau(t)}{t}\right)\right)+O\left(\frac{\tau^2(t)}{t^{2(m+p)}}\right) = \\
1+\frac{\tau(t)}{t^{m+p}}+\frac{(m+p)\tau^2(t)-L\tau^2(t)}{t^{m+p+1}}+O\left(\frac{\tau^3(t)}{t^{m+p+2}}\right)+O\left(\frac{\tau^2(t)}{t^{2(m+p)}}\right).
\end{gathered}$$

Now, for

$$\begin{aligned}
&\mathcal{L}(t) \equiv 1+\frac{\tau(t)}{t^{m+p}} \ge \\
&\quad 1+\frac{\tau(t)}{t^{m+p}}+\frac{(m+p)\tau^2(t)-L\tau^2(t)}{t^{m+p+1}}+O\left(\frac{\tau^3(t)}{t^{m+p+2}}\right)+O\left(\frac{\tau^2(t)}{t^{2(m+p)}}\right) \ge \mathcal{R}(t)
\end{aligned}$$

is $m+p > 1$ (this assumption is necessary for the asymptotic dominance of the third term in the right-hand side in above inequality) and $L > (m+p)$ sufficient. Since $p$ may be arbitrarily small positive number, we have $L > m$. Theorem 5 is proved.

## 4. Concluding remarks

### 4.1. Sharpness of Theorem 3.

Let Eq. (1) has the form

$$\dot{y}(t) = \left(t - \frac{a}{t}\right)[y(t) - y(t - 1/t)] \tag{8}$$

with $a > 1/2$, $a$ =const; i.e. $\alpha(t) = t - a/t$ and $\tau(t) = 1/t$. Note that $\tau(+\infty) = 0$. It is easy to see that the inequality (4) holds since the inequality

$$\alpha(t) = t - \frac{a}{t} \leq t - \frac{L}{t} = \frac{1}{\tau(t)} - \frac{L}{t}$$

is valid for $1/2 < L \leq a$. In accordance with Theorem 3, each solution of Eq. (8) is convergent.

In the sequel we will show that the interval $a > 1/2$ is the best possible. Namely, we will show that the property of convergence of all solutions of equation (8) is not valid for $a = 1/2$. In paper [10] (see Theorem 2 in [10]), the following is proved:

**Theorem 6.** *Equation* (1) *has a solution* $y(t)$ *with property* $y(+\infty) = +\infty$ *if and only if the inequality*

$$\dot{\omega}(t) \leq \alpha(t)\left[\omega(t) - \omega(t - \tau(t))\right]$$

*has a solution* $\omega(t)$ *with property* $\omega(+\infty) = +\infty$.

So, it is sufficient to show that, in the case of equation (8) with $a = 1/2$, the inequality

$$\dot{\omega}(t) \leq \left(t - \frac{1}{2t}\right)[\omega(t) - \omega(t - 1/t)] \tag{9}$$

has a solution $\omega(t)$ with the property $\omega(+\infty) = +\infty$. It is easy to verify that the function

$$\omega(t) = \ln t$$

is such solution of inequality (9).

### 4.2. A comparison with Atkinson – Haddock's results.

Let us use the equation (8) as a concrete example of the equation (1) again and let us show with its aid that our convergence results are in some sense more general than the results given in [3].

Really, by Theorem 3.3 in [3] all solutions of equation (1) will converge if for a sufficiently large $t$

$$\int_t^{t+r} \alpha(s)ds \leq 1 - \frac{r}{t} - \frac{K}{t \ln t} \tag{10}$$

with some $K > r$ where $r$ is a positive *constant* which bounds delay. In the case of the equation (8) the left hand side of the inequality (10) equals

$$\int_t^{t+r} \alpha(s)ds = \int_t^{t+r} \left(s - \frac{a}{s}\right) ds = \left[\frac{1}{2}s^2 - a\ln s\right]_t^{t+r} = rt + \frac{r^2}{2} - a\ln\left(1 + \frac{r}{t}\right)$$

and

$$\lim_{t\to+\infty} \int_t^{t+r} \alpha(s)ds = \infty.$$

So inequality (10) does not hold for a positive $r$. Nevertheless in this case our Theorem 4 holds for $a > 1$ since the left hand side of inequality (6)

$$\frac{1}{\tau(t)} \cdot \int_{t-\tau(t)}^{t} \tau(s)\alpha(s)ds = t\int_{t-1/t}^{t} \frac{1}{s}\left(s - \frac{a}{s}\right) ds = 1 - \frac{a}{t^2-1}$$

is not greater than the right hand side of inequality (6) which equals

$$1 - \frac{1}{t^2-1} - \frac{1}{(t^2-1)\ln(t-1/t)} - \frac{L}{(t^2-1)\ln(t-1/t)\ln_2(t-1/t)}.$$

Note that, except this, Theorem 4 generalizes Theorem 3.3 even in the case when the delay is constant, i.e. in the case when $\tau(t) \equiv \tau_0 > 0$.

### 4.3. Comparisons with sufficient conditions of convergence given in [9].

Sufficient conditions of convergence given in [9] (Theorems 8 – 10 in [9]), with respect to the equation (1), are:

**Theorem 7.** *If for a sufficiently large $t$*

$$\alpha(t) \le \frac{1}{\tau_0} - \frac{M_1}{t}$$

*where $M_1 > 1/2$ is a constant, then each solution of Eq.* (1) *is convergent.*

**Theorem 8.** *If for a sufficiently large $t$*

$$\int_t^{t+\tau_0} \alpha(s)ds \le 1 - \frac{\tau_0}{t} - \frac{\tau_0}{t\ln t} - \frac{M_2}{t\ln t\ln_2 t}$$

*where $M_2 > \tau_0, M_2 =$const, then each solution of Eq.* (1) *is convergent.*

**Theorem 9.** *If for a sufficiently large $t$*

$$\int_t^{t+\tau_0} \frac{\alpha(s)}{s^m} ds \le \frac{1}{t^m} - \frac{M_3}{t^{m+1}}$$

*where $m, M_3 =$const, $m > 1, M_3 > \tau_0 m$, then each solution of Eq.* (1) *is convergent.*

The Theorems 7, 8, 9 are at the same time consequences of our Theorems 3, 4, 5 if the delay is constant. Really, putting $\tau(t) \equiv \tau_0 > 0$ and $L = M_1$ in Theorem 3; $L = M_2/\tau_0$ in Theorem 4, and $L = M_3/\tau_0$ in Theorem 5 we get (after the shift $t \to t + \tau_0$), consequently, Theorems 7, 8, 9.

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