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# THREE-DIMENSIONAL CONFORMALLY FLAT PSEUDO-SYMMETRIC SPACES OF CONSTANT TYPE

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*Dedicated to Professor Oldřich Kowalski on the occasion of his birthday*

Abstract. An explicit classification of the spaces in the title is given. The resulting spaces are locally products or locally warped products of the real line and two-dimensional spaces of *constant* curvature.

## Introduction

According to [3], [10], a *pseudo-symmetric space of constant type* is a Riemannian manifold $(M, g)$ whose Riemannian curvature tensor field $R$ satisfies

$$R(X, Y) \cdot R = \tilde{c}\,(X \wedge Y) \cdot R, \tag{0.1}$$

for arbitrary vector fields $X$ and $Y$ on $M$, where $X \wedge Y$ is the endomorphism of the tangent bundle $TM$ defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y,$$

$\tilde{c}$ is a constant, and the dot in the both sides of (0.1) denotes the derivation on the tensor algebra of $TM$ induced by endomorphism of $TM$. A pseudo-symmetric space of constant type with constant $\tilde{c} = 0$ is a *semi-symmetric space.*

We have the following characterization of pseudo-symmetric spaces in dimension three (see [1, Proposition 11.2]):

**Proposition 0.1.** *A three-dimensional Riemannian manifold $(M, g)$ is pseudo-symmetric of constant type with constant $\tilde{c}$ if and only if its principal Ricci curvatures $\rho_1, \rho_2$ and $\rho_3$ locally satisfy the following conditions (up to a numeration):*

(1) $\rho_3 = 2\tilde{c}$,
(2) $\rho_1 = \rho_2$ *everywhere.*

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The pseudo-symmetric spaces of constant type can be seen as a natural generalization of certain homogeneous spaces. For instance, it follows from the paper by J. Milnor [14] that the constant Ricci eigenvalues of a three-dimensional homogeneous Riemannian space must satisfy some algebraic restrictions. But O. Kowalski [6] has shown that even if one prescribes (in dimension three) constant Ricci eigenvalues not satisfying these restrictions then there still exist nonhomogeneous examples. These examples belong to the class of pseudo-symmetric spaces of constant type. We refer to O. Kowalski [6, Section 9] or E. Boeckx, O. Kowalski and L. Vanhecke [1, Chapter 12] for more details. On the other hand, as concerns conformally flat spaces, H. Takagi [16] has classified all conformally flat homogeneous spaces.

Now we understand that a next step of classification of conformally flat spaces is that of conformally flat pseudo-symmetric spaces. Since conformal flatness of a Riemannian manifold is described by the Ricci tensor, we receive a system of partial differential equations of the Ricci eigenvalues $\rho_1 = \rho_2$ for our classification. Solving it, we shall give an explicit classification of all three-dimensional locally conformally flat pseudo-symmetric spaces of constant type. We shall see that the resulting spaces are locally products or locally warped products of the real line and a two-dimensional space of constant curvature. We shall fully use an explicit classification of semi-symmetric spaces given by O. Kowalski [7] and by V. Hájková [5], and that of pseudo-symmetric spaces of constant type given by O. Kowalski [6] and by O. Kowalski and the second author [8]–[11].

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## 1. Pseudo-symmetric space of constant type

Let $(M, g)$ be a three-dimensional pseudo-symmetric space of constant type. Then, by Proposition 0.1, its Ricci operator $Q$ has eigenvalues $\rho_1 = \rho_2 \neq \rho_3$, where $\rho_3 = 2\tilde{c}$ is a constant. We choose a neighborhood $\widetilde{\mathcal{U}}$ of a fixed point $m \in M$ and a smooth vector field $E_3$ of a unit eigenvectors of $Q$ corresponding to $\rho_3$ in $\widetilde{\mathcal{U}}$. Let $S : D^2 \longrightarrow \widetilde{\mathcal{U}}$ be a surface through $m$ which is transversal with respect to all trajectories generated by $E_3$ at all cross-points and not orthogonal to such a trajectory at $m$. (The vector field $E_3$ determines an orientation of $S$.) Then there is a normal neighborhood $\mathcal{U}$ of $m$, $\mathcal{U} \subset \widetilde{\mathcal{U}}$, with the property that each point $p \in \mathcal{U}$ is projected to exactly one point $\pi(p) \in S$ via some trajectory.

O. Kowalski has shown in [7, Theorem 2.1] (see also [10, Proposition 1.1]) that there exists a local coordinate system $(\mathcal{U}; w, x, y)$, which we call the *adapted coordinate system*, such that

$$g = \left(\omega^1\right)^2 + \left(\omega^2\right)^2 + \left(\omega^3\right)^2 , \tag{1.1}$$

where

$$(1.2)\qquad \begin{cases} \omega^1 = f\mathrm{d}w\,, \\ \omega^2 = A\mathrm{d}x + C\mathrm{d}w\,, \\ \omega^3 = \mathrm{d}y + H\mathrm{d}w\,. \end{cases}$$

Here $f$, $A$ and $C$ are smooth functions of the variables $w$, $x$ and $y$, $fA \neq 0$, and $H$ is a smooth function of the variables $w$ and $x$. In particular, for each point $p \in \mathcal{U}$, $y(p)$ is the oriented length $d^{+}(\pi(p), p)$ of the trajectory generated by $E_3$ and joining $p$ with $\pi(p)$.

The conditions that a Riemannian manifold is pseudo-symmetric of constant type (that is, $\rho_1 = \rho_2 \neq \rho_3 = 2\tilde{c}$, $\tilde{c}$ = constant) are described by a system of nine partial differential equations (A1)–(C3) in [7] and that in [10]. Solving this system, O. Kowalski [7] and V. Hájková [5] have given the explicit classification of all pseudo-symmetric spaces with $\tilde{c} = 0$ (that is, of all semi-symmetric spaces). (One can also refer [1, Chapter 6] for the full classification.) O. Kowalski and the second author [10]–[11] have given the explicit classification of all pseudo-symmetric spaces with $\tilde{c} \neq 0$. (See also [12] and [1, Chapter 11].) Among the nine partial differential equations, the equation (A3) is

$$(\mathrm{A3})\qquad (A\alpha)'_w + R'_x + SA'_y - A\beta T = -fA(\rho_1 - \tilde{c})\,,$$

where

$$(1.3)\qquad \begin{cases} \alpha = \dfrac{1}{fA}\left(A'_w - C'_x - HA'_y\right)\,, \\ \beta = \dfrac{1}{2fA}\left(H'_x + AC'_y - CA'_y\right) \end{cases}$$

and

$$(1.4)\qquad \begin{cases} R = \dfrac{f'_x}{A} - C\alpha + H\beta\,, \\ S = f'_y + C\beta\,, \\ T = C'_y - f\beta\,. \end{cases}$$

In the general classification of pseudo-symmetric spaces of constant type, the equation (A3) does not give any additional condition for such a classification because it involves the eigenvalue $\rho_1 = \rho_2$ which is not prescribed in advance. It has been treated just as a formula for calculating $\rho_1$, or equivalently, the scalar curvature. As concerns the classification in the present paper, the equation (A3) gives an additional condition because $\rho_1$ is restricted by conformal flatness of $(M, g)$ as we shall see later.

Let $\{E_1, E_2, E_3\}$ be the local orthonormal frame dual to the coframe $\{\omega^1, \omega^2, \omega^3\}$ from (1.2). Then $E_i, i = 1, 2, 3$, are vector fields of eigenvectors of the Ricci operator $Q$ corresponding to the eigenvalues $\rho_i$, respectively. They are written in the form

$$(1.5)\qquad \begin{cases} E_1 = \dfrac{1}{fA}\left(A\dfrac{\partial}{\partial w} - C\dfrac{\partial}{\partial x} - AH\dfrac{\partial}{\partial y}\right), \\ E_2 = \dfrac{1}{A}\dfrac{\partial}{\partial x}, \\ E_3 = \dfrac{\partial}{\partial y}. \end{cases}$$

The Levi-Civita connection $\nabla$ of $(M,g)$ is given by

$$(1.6)\qquad \begin{cases} \nabla_{E_1}E_1 = -\dfrac{f'_x}{fA}E_2 - aE_3, & \nabla_{E_1}E_2 = \dfrac{f'_x}{fA}E_1 - cE_3, \\ \nabla_{E_2}E_1 = \alpha E_2 - bE_3, & \nabla_{E_2}E_2 = -\alpha E_1 - eE_3, \\ \nabla_{E_1}E_3 = aE_1 + cE_2, & \nabla_{E_2}E_3 = bE_1 + eE_2, \\ \nabla_{E_3}E_1 = -bE_2, & \nabla_{E_3}E_2 = bE_1, \\ \nabla_{E_3}E_3 = 0, \end{cases}$$

where

$$(1.7)\qquad a = \frac{f'_y}{f}, \quad b = \beta, \quad c = \beta - \frac{H'_x}{fA}, \quad e = \frac{A'_y}{A}.$$

The last formula $\nabla_{E_3}E_3 = 0$ in (1.6) implies that the trajectories of the unit vector field $E_3$ (consisting of the eigenvectors of the Ricci operator $Q$ corresponding to $\rho_3 = 2\tilde{c}$ ) are geodesics of $(M,g)$. We call them *principal geodesics* of $(M,g)$. The following notion is crucial in the classification.

**Definition 1.1.** A smooth surface $N \subset (M,g)$ is called an *asymptotic leaf* if it is generated by the principal geodesics and its tangent planes are parallel along these principal geodesics with respect to the Levi-Civita connection $\nabla$ of $(M,g)$.

There are four types of three-dimensional pseudo-symmetric spaces of constant type:

- (E) There is no asymptotic leaf through every point of $M$.
- (H) There are exactly two asymptotic leaves through every point of $M$.
- (P) There is exactly one asymptotic leaf through every point of $M$.
- (P$\ell$) There are infinitely many asymptotic leaves through every point of $M$.

They are called *elliptic, hyperbolic, parabolic*, and *planar* type, respectively. If we put

$$(1.8)\qquad \Delta = (e-a)^2 + 4bc,$$

then they are distinguished by the conditions $\Delta < 0$, $\Delta > 0$, $\Delta = 0$, and $e - a = b = c = 0$, respectively.

## 2. Main result

A *locally conformally flat space* is a Riemannian manifold $(M, g)$ which is locally conformal to a flat Riemannian manifold $(M_0, g_0)$, that is, there exists a local diffeomorphism $\phi$ of $M$ to $M_0$ such that the metric $g$ is proportional to the pull back $\phi^* g_0$. A three-dimensional Riemannian manifold $(M, g)$ is locally conformally flat if and only if the tensor field $Q - (1/4)\,\mathrm{Sc}(g)\,\mathrm{Id}$ is a Codazzi tensor, where $\mathrm{Sc}(g)$ is the scalar curvature of $(M, g)$ and Id is the identity transformation on the tangent bundle $TM$; or equivalently, if and only if

$$(\nabla_X Q)Y - \frac{1}{4}X(\mathrm{Sc}(g))Y = (\nabla_Y Q)X - \frac{1}{4}Y(\mathrm{Sc}(g))X \tag{2.1}$$

holds for every vector fields $X$ and $Y$ on $M$. We refer L. P. Eisenhart [4] for more details.

Now again, let $(M, g)$ be a three-dimensional pseudo-symmetric space of constant type with constant $\tilde{c}$. Then we have $\mathrm{Sc}(g) = 2(\rho_1 + \tilde{c})$. We take $X$ and $Y$ in (2.1) from the local orthonormal frame $\{E_1, E_2, E_3\}$ introduced in the previous section, and use the fact that $QE_i = \rho_i E_i, i = 1, 2, 3$. Then locally conformal flatness for $(M, g)$ gives a system of seven partial differential equations:

$$\rho'_{1y} + 2(\rho_1 - 2\tilde{c})a = 0\,, \tag{2.2}$$

$$\rho'_{1y} + 2(\rho_1 - 2\tilde{c})e = 0\,, \tag{2.3}$$

$$b = 0\,, \tag{2.4}$$

$$c = 0\,, \tag{2.5}$$

$$H'_x = 0\,, \tag{2.6}$$

$$\rho'_{1x} = 0\,, \tag{2.7}$$

$$A\rho'_{1w} - C\rho'_{1x} - AH\rho'_{1y} = 0\,. \tag{2.8}$$

The system of partial differential equations for our problem is that of (2.2)–(2.8) and (A3).

First, we consider the case that $\rho_1 = \rho_2$ is constant.

**Proposition 2.1.** *The three-dimensional locally conformally flat pseudo-symmetric space of constant type with constant Ricci eigenvalues $\rho_1 = \rho_2 \neq \rho_3$ is semi-symmetric.*

**Proof.** Since $\rho_1 = \rho_2$ is constant, we get from (2.3) that $e = 0$ because $\rho_1 - 2\tilde{c} \neq 0$. Hence, by (1.7), $A'_y = 0$. Now we use the partial differential equation (B1) from O. Kowalski [6], namely

$$A''_{yy} - A\beta^2 = -\tilde{c}A. \tag{B1}$$

Substituting $A'_y = 0$ and $\beta = b = 0$ (from (1.7) and (2.4)) into (B1), we obtain $\tilde{c} = 0$ because $A \neq 0$. Thus, the space is semi-symmetric. □

**Corollary 2.2.** *Let $(M, g)$ be a three-dimensional locally conformally flat pseudo-symmetric space of constant type with nonzero constant $\tilde{c}$. Then the Ricci eigenvalue $\rho_1 = \rho_2$ of $(M, g)$ different from $\rho_3 = 2\tilde{c}$ is not constant.*

Now we remark that *if $(M,g)$ is a three-dimensional locally conformally flat semi-symmetric space with constant Ricci eigenvalues $\rho_1 = \rho_2 \neq 0 = \rho_3$, then $(M,g)$ is locally product $\mathbb{R} \times N(\rho_1)$ of the real line $\mathbb{R}$ and a two-dimensional space $N(\rho_1)$ of constant curvature $\rho_1$, and vice versa.* We refer [7, pp.459–460] for more details.

In the locally irreducible case we have

**Theorem 2.3.** *Every three-dimensional locally irreducible conformally flat pseudo-symmetric space of constant type is a space of planar type.*

**Proof.** As remarked above, the Ricci eigenvalue $\rho_1 = \rho_2$ in this case is not constant. Subtracting (2.3) from (2.2), we have $e - a = 0$ since $\rho_1 - 2\tilde{c} \neq 0$. This together with (2.4) and (2.5) implies the assertion. □

Here we refer the classifications given by O. Kowalski [6]–[7], by V. Hájková [5], and by O. Kowalski-M. Sekizawa [8]–[11]. Then, by Theorem 2.3, the spaces in consideration are locally warped products in the sense of B. O'Neill [15]:

**Proposition 2.4.** *The metrics of three-dimensional locally irreducible conformally flat pseudo-symmetric spaces of constant type with constant $\tilde{c}$ are locally warped products. These metrics are described, in terms of the adapted coordinate system, as*

$$g = A^2 \left(\xi^2 (\mathrm{d}w)^2 + (\mathrm{d}x)^2\right) + (\mathrm{d}y)^2 , \tag{2.9}$$

*where $\xi = \xi(w,x)$ is a smooth function of the variables $w$ and $x$, and the warping function $A = A(y)$ is given by*

$$A = \begin{cases} y & \text{if } \tilde{c} = 0 , \\ \sinh(\lambda y) \quad \text{or} \quad \cosh(\lambda y) & \text{if } \tilde{c} = -\lambda^2 , \\ \sin(\lambda y) & \text{if } \tilde{c} = \lambda^2 . \end{cases} \tag{2.10}$$

*In particular, the coefficients $C$ and $H$ from* (1.2) *vanish.*

**Proposition 2.5.** *The Ricci eigenvalue $\rho_1 = \rho_2$ of a three-dimensional locally irreducible conformally flat pseudo-symmetric space of constant type with constant $\tilde{c}$ depends only on one variable. More precisely, $\rho_1 = \rho_2$ is given, in terms of the adapted coordinate system, by*

$$\rho_1 = \pm \frac{\tilde{a}^2}{A^2} + 2\tilde{c} , \tag{2.11}$$

*where $\tilde{a}$ is a positive constant and $A$ is a function given by* (2.10).

**Proof.** Since, by Proposition 2.4, $C = H = 0$ in (1.2), the equation (2.8) reduces to $\rho'_{1w} = 0$ because $A \neq 0$. This together with (2.7) means that $\rho_1$ depends only on the variable $y$. Now, we have $a = e$ because $(M,g)$ is planar (see Theorem 2.3). It means that the partial differential equation (2.2) coincides with (2.3). Substituting for $e$ in (2.3) from (1.7), we can rewrite (2.3) in the form

$$\frac{(\rho_1 - 2\tilde{c})'_y}{\rho_1 - 2\tilde{c}} = -\frac{2A'_y}{A}\,. \tag{2.12}$$

Solving this equation, we obtain (2.11). □

It remains the partial differential equation (A3). Substituting $f = \xi A$ into (1.3) and into (1.4), and using $C = H = 0$ by Proposition 2.4, we obtain (A3) in the form for a partial differential equation of the function $\xi = \xi(w, x)$:

$$\xi''_{xx} + \left((A'_y)^2 + (\rho_1 - \tilde{c})A^2\right)\xi = 0\,. \tag{2.13}$$

This is the basic equation for getting the following main result:

**Theorem 2.6.** *Every three-dimensional locally irreducible conformally flat pseudo-symmetric space $(M, g)$ of constant type with constant $\tilde{c}$ is locally a warped product*

$$\mathbb{R} \times_A N(k)$$

*of the real line $\mathbb{R}$ and a two-dimensional space $N(k)$ of constant curvature, where $A = A(y)$ is a warping function of $\mathbb{R}$ given by (2.10). The constant $k$ can be expressed as follows:*

(1) *If $(M, g)$ is a semi-symmetric space, then $k = 1 \pm \tilde{a}^2$;*
(2) *if $(M, g)$ is a pseudo-symmetric space of constant type with constant $\tilde{c} = -\lambda^2$, then $k = \lambda^2 \pm \tilde{a}^2$ in the case $A(y) = \sinh(\lambda y)$ and $k = -\lambda^2 \pm \tilde{a}^2$ in the case $A(y) = \cosh(\lambda y)$;*
(3) *if $(M, g)$ is a pseudo-symmetric space of constant type with constant $\tilde{c} = \lambda^2$, then $k = \lambda^2 \pm \tilde{a}^2$,*

*where $\tilde{a}$ is a positive constant. Moreover, in each case, the Ricci eigenvalues of $(M, g)$ are $\rho_1 = \rho_2$ given by (2.11) and $\rho_3 = 2\tilde{c}$.*

**Proof.** Due to Propositions 2.4 and 2.5, we only need to calculate the Gaussian curvature $K$ of the metric $\xi^2(\mathrm{d}w)^2 + (\mathrm{d}x)^2$. Since $K = -\xi''/\xi$, we obtain from (2.13) that

$$K = -\frac{\xi''}{\xi} = (A'_y)^2 + (\rho_1 - \tilde{c})A^2\,. \tag{2.14}$$

Substituting here for $A$ and $\rho_1$ from (2.10) and (2.11), respectively, we obtain easily the corresponding constant values of $K = k$. □

**Remark 1.** After we know already (by Proposition 2.4) that our spaces are warped products, the first part of Theorem 2.6 follows from a more general result by J. Mikeš [13, Section 4.3].

**Remark 2.** J. Deprez, R. Deszcz and L. Verstraelen have given in [2, Corollary 3.2] an example of a three-dimensional locally conformally flat pseudo-symmetric space. Their example belongs to a class of spaces of constant type with negative constant.

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